Traveling Waves and Functional Differential Equations of Pointwise Type. What Is Common?

Levon A. Beklaryan
Central Economics and Mathematics Institute RAS
Nachimovky prospect 47,
117418 Moscow, Russia.
beklar@cemi.rssi.ru, lbeklaryan@outlook.com
Armen L. Beklaryan
National Research University Higher School of Economics
Kirpichnaya Ulitsa 33/5,
105187 Moscow, Russia.
abeklaryan@hse.ru

Abstract

For equations of mathematical physics, which are the Euler-Lagrange equation of the corresponding variational problem, an important class of solutions are traveling wave solutions (soliton solutions). In turn, soliton solutions for finite-difference analogs of the equations of mathematical physics are in one-to-one correspondence with solutions of induced functional differential equations of pointwise type (FDEPT). The presence of a wide range of numerical methods for constructing FDEPT solutions, as well as the existence of appropriate existence and uniqueness theorems for the solution, a continuous dependence on the initial and boundary conditions, the "rudeness" of such equations, allows us to construct soliton solutions for the initial equations of mathematical physics. Within the framework of the presented work, on the example of a problem from the theory of plastic deformation the mentioned correspondence between solutions of the traveling wave type and the solutions of the induced functional differential equation will be demonstrated.

1 Introduction

In the theory of plastic deformation, the following infinite-dimensional dynamical system is studied

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \quad i \in \mathbb{Z}, \quad y_i \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1}$$

where potential $\phi(\cdot)$ is given by a smooth periodic function. The equation (1) is a system with the Frenkel-Kontorova potential [Frenkel, 1938]. Such a system is a finite difference analog of the nonlinear wave equation.

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It simulates the behavior of a countable number of balls of mass m placed at integer points of the numerical line, where each pair of adjacent balls is connected by an elastic spring, and describes the propagation of longitudinal waves in an infinite homogeneous absolutely elastic rod. The study of such systems with different potentials is one of the intensively developing directions in the theory of dynamical systems. For these systems, the central task is to study solutions of the traveling wave type as one of the observed wave classes.

Definition 1 *We say that the solution $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ of the system (1), defined for all $t \in \mathbb{R}$, has a traveling wave type, if there is $\tau > 0$, independent of t and i, that for all $i \in \mathbb{Z}$ and $t \in \mathbb{R}$ the following equality holds*

$$y_i(t + \tau) = y_{i+1}(t).$$

The constant τ will be called a characteristic of a traveling wave. ■

One of the methods for studying such systems is the construction of solutions using the explicit form of the potential and further, by the methods of perturbation theory, the establishment of the existence of solutions of the traveling wave type for nearby potentials. Another frequently used method is the applying of the presence of symmetries in the original equations.

At the same time, it is important not only the question of the existence of solutions of the traveling wave type, but also the question of their uniqueness. For this, one of the approaches is the localization of solutions in the space of infinitely differentiable functions or analytic functions. As a rule, it is possible to show the existence of a solution in the space of infinitely differentiable functions, and uniqueness in the space of analytic functions [Pustyl'nikov, 1997].

The proposed approach is based on the existence of a one-to-one correspondence of solutions of traveling wave type for infinite-dimensional dynamical systems with solutions of induced FDEPT [Beklaryan, 2007]. To study the existence and uniqueness of solutions of traveling wave type, it is proposed to localize solutions of induced FDEPT in spaces of functions, majorized by functions of a given exponential growth. This approach is particularly successful for systems with Frenkel-Kontorova potentials. In this way, it is possible to obtain a "correct" extension of the concept of a traveling wave in the form of solutions of the quasi-traveling wave type, which is related to the description of processes in inhomogeneous environments for which the set of traveling wave solutions is trivial [Beklaryan, 2010, Beklaryan, 2014].

For the infinite-dimensional dynamical system under consideration, the study of solutions of the traveling wave type with the characteristic τ, i.e. solutions of the system

$$\ddot{y}_i = m^{-1}(y_{i+1} - 2y_i + y_{i-1} + \phi(y_i)), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R},$$
$$y_i(t + \tau) = y_{i+1}(t)$$

turns out to be equivalent to the study of a solution space of the induced FDEPT

$$\ddot{x}(t) = m^{-1}(x(t + \tau) - 2x(t) + x(t - \tau) + \phi(x(t))), \quad t \in \mathbb{R}. \tag{2}$$

In this case, the corresponding solutions are related as follows: for any $t \in \mathbb{R}$

$$x(t) = y_{[t\tau^{-1}]}(t - [t\tau^{-1}]),$$

where $[\cdot]$ means the integer part of a number. In fact, the described connection between solutions of the traveling wave type of the infinite-dimensional dynamical system and solutions of the induced functional-differential equation is a fragment of a more general scheme that goes beyond the scope of this article.

2 Existence and Uniqueness Theorem

We assume that the nonlinear potential ϕ satisfies the Lipschitz condition with constant L. Thus, we should study solutions of the functional-differential equation (2) with a quasilinear right-hand side. A solution of the FDEPT with a quasilinear right-hand side will be sought in a one-parameter family of Banach spaces of functions that have at most exponent growth. The exponent is the parameter of the selected family of functions, which is defined as follows

$$\mathcal{L}_\mu^n C^{(k)}(\mathbb{R}) = \left\{ x(\cdot) : x(\cdot) \in C^{(k)}(\mathbb{R}, \mathbb{R}^n), \max_{0 \le r \le k} \sup_{t \in \mathbb{R}} \|x^{(r)}(t)\mu^{|t|}\|_{R^n} < +\infty \right\}. \tag{3}$$

In our approach for the initial infinite-dimensional dynamical system (1) we will study traveling wave solutions having a given growth (exponential) both in time and space. To this end, we define a vector space

$$K^n = \overline{\prod_{i\in\mathbb{Z}} R_i^n}, \quad R_i^n = \mathbb{R}^n, \quad i \in \mathbb{Z}$$
$$(\varkappa \in K^n, \quad \varkappa = \{x_i\}_{-\infty}^{+\infty})$$

with the standard topology of the complete direct product (metrizable space).

In particular, the elements of the space K^2 are infinite sequences

$$\varkappa = \{(u_i, v_i)'\}_{-\infty}^{+\infty}, \quad u_i,\, v_i \in \mathbb{R}$$

(prime means transposition).

In the space K^n we define a family of Hilbert subspaces $K_{2\mu}^n, \mu \in (0,1)$

$$K_{2\mu}^n = \left\{ \varkappa : \varkappa \in K^n; \quad \sum_{i=-\infty}^{+\infty} \|x_i\|_{R^n}^2 \mu^{2|i|} < +\infty \right\}$$

with the norm

$$\|\varkappa\|_{2\mu} = \left(\sum_{i=-\infty}^{+\infty} \|x_i\|_{R^n}^2 \mu^{2|i|} \right)^{\frac{1}{2}}.$$

Here μ is a free parameter, due to which the solution space will be selected.

Let's consider a transcendental equation with respect to two variables $\tau \in (0, +\infty)$ and $\mu \in (0,1)$

$$C\tau \left(2\mu^{-1} + 1\right) = \ln \mu^{-1}, \tag{4}$$

where

$$C = \max\{1; 2m^{-1}\sqrt{L^2+2}\}.$$

The set of solutions of the equation (4) is described by functions $\mu_1(\tau), \mu_2(\tau)$ given in Figure 1.

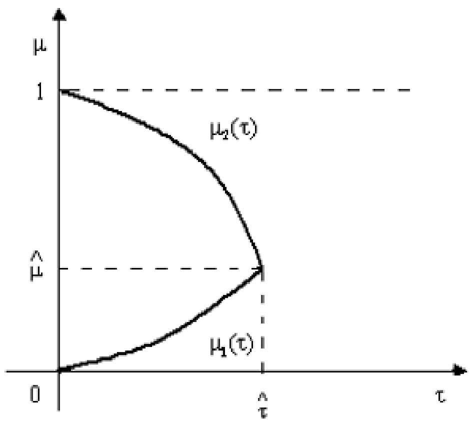


Figure 1: Graphs of Functions $\mu_1(\tau), \mu_2(\tau)$

Let's formulate the theorem of existence and uniqueness of a solution of traveling wave type.

Theorem 1 *For any initial values $\bar{i} \in \mathbb{Z}, a, b \in \mathbb{R}, \bar{t} \in \mathbb{R}$ and characteristics $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

for the initial system of differential equations (1) there exists a unique solution of the traveling wave type $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ with characteristic τ such that it satisfies the initial conditions $y_{\bar{i}}(\bar{t}) = a, \dot{y}_{\bar{i}}(\bar{t}) = b$. For any parameter $\mu \in (\mu_1(\tau), \mu_2(\tau))$ the vector function $\omega(t) = \{(y_i(t), \dot{y}_i(t))^T\}_{-\infty}^{+\infty}$ belongs to the space $K_{2\mu}^2$ for any $t \in \mathbb{R}$, and the function $\rho(t) = \|\omega(t)\|_{2\mu}$ belongs to the space $\mathcal{L}^1_{\sqrt{\mu}} C^{(1)}(\mathbb{R})$. Such a solution depends continuously on the initial values $a, b \in \mathbb{R}$, as well as on the mass m. ■

Theorem 1 not only guarantees the existence of a solution but also determines the limitation of its possible growth both in time t and in coordinates $i \in \mathbb{Z}$ (over space). It is obvious that for each $0 < \tau < \hat{\tau}$ the space $K^2_{2\mu}$, at $\mu < \mu_2(\tau)$ but close to $\mu_2(\tau)$, is much narrower than the space $K^2_{2\mu}$, at $\mu > \mu_1(\tau)$ but close to $\mu_1(\tau)$. The theorem guarantees the existence of a solution in narrower spaces and uniqueness in wider spaces.

The full text of the proof of theorem 1, as well as a detailed description of the proposed approach, is given in the papers [Beklaryan, 2007, Beklaryan, 2010, Beklaryan, 2014].

3 Numerical Experiments

Next, the results of the computational experiments on the study of boundary value problems for systems of FDEPT using OPTCON-F software will be presented. The software complex OPTCON-F is designed to obtain a numerical solution of boundary value problems, parametric identification problems and optimal control for dynamical systems described by FDEPT [Gornov et al., 2013]. The proposed technology for solving boundary value problems is based on the Ritz method and spline collocation approaches. To solve the problem we discretized system trajectories on the grid with a constant step and formulate the generalized residual functional, including both weighted residuals of the original differential equation and residuals of boundary conditions [Zarodnyuk et al., 2016].

Let's consider the FDEPT of the following form

$$\ddot{x}(t) = m^{-1}(x(t+\tau) - 2x(t) + x(t-\tau) + A \sin Bx(t)), \quad t \in \mathbb{R}, \tag{5}$$

where $A, B \in \mathbb{R}, m, \tau \in \mathbb{R}_+$. Using a time-variable transformation the equation (5) can be rewritten in the form of the following system of equations of the first order:

$$\begin{aligned} \dot{z}_1(t) &= \tau z_2(t), \\ \dot{z}_2(t) &= \tau m^{-1}(z_1(t+1) - 2z_1(t) + z_1(t-1) + A\sin(Bz_1(t))). \end{aligned}$$

Under this system, we have the following real parameters: A, B, τ, m. In the following example, for a given system, we consider such parameter values that conditions of the existence theorem are satisfied.

3.1 Example

We consider dynamical system in the following form:

$$\begin{array}{l} \left\{ \begin{array}{l} \dot{z}_1(t) = 0.15 z_2(t), \\ \dot{z}_2(t) = 0.15 \times 100^{-1}(z_1(t+1) - 2z_1(t) + z_1(t-1) + 500\sin(0.1z_1(t))), \end{array} \right. \quad t \in \mathbb{R}, \\ \textit{initial conditions} \\ \left\{ \begin{array}{l} z_1(0) = 30, \\ z_2(0) = 0. \end{array} \right. \end{array} \tag{6}$$

In this case, the equality (4) takes the form

$$0.003\sqrt{2502}(2\mu^{-1} + 1) = \ln \mu^{-1},$$

that has on the interval $(0, 1)$ two solutions with approximate values 0.22 and 0.424191 (the exact values is expressed in terms of the Lambert W-function and is not written out in quadratures).

Taking into account the impossibility of considering the numerical solution of the system on an infinite interval, we introduce the parameter k and the corresponding family of expanding initial-boundary value problems

$$
\begin{cases}
\dot{z}_1(t) = 0.15 z_2(t), \\
\dot{z}_2(t) = 0.15 \times 100^{-1}(z_1(t+1) - 2z_1(t) + z_1(t-1) + 500\sin(0.1 z_1(t))),
\end{cases}
\quad t \in [-k, k],
$$

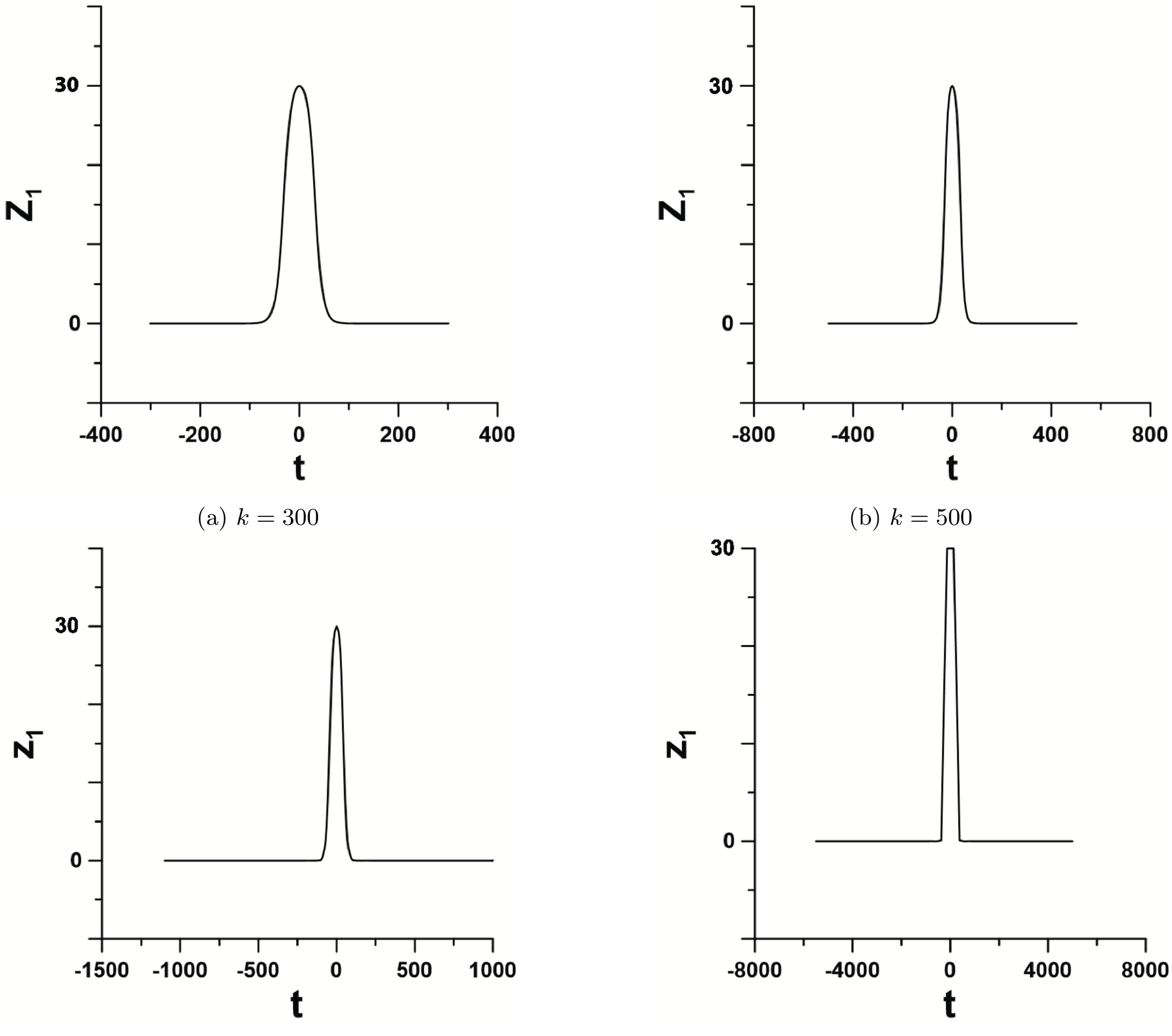
boundary conditions

$$
\begin{cases}
\dot{z}_1(t) = 0, \\
\dot{z}_2(t) = 0,
\end{cases}
\quad t \in (-\infty, -k] \cup [k, +\infty), \tag{7}
$$

initial conditions

$$
\begin{cases}
z_1(0) = 30, \\
z_2(0) = 0.
\end{cases}
$$

According to the [Beklaryan, 2007], the solution of the system (7) converges (according to the metric of the space (3) with $\mu \in (\mu_1(\tau), \mu_2(\tau))$) to the solution of the system (6) as $k \to \infty$. The graphs of the solution of the system (7) at different values of k are shown in Figure 2.



(a) $k = 300$

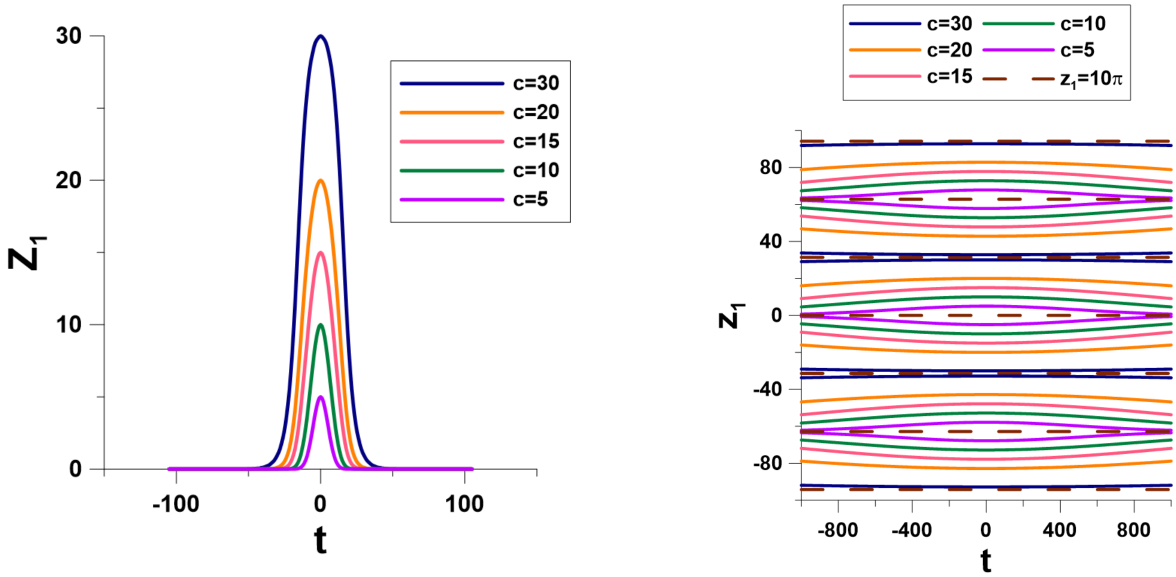
(b) $k = 500$

(c) $k = 1000$

(d) $k = 5000$

Figure 2: Trajectories of the System (7) at Different k.

Since the equation (5) is autonomous, the solution space of such equation is invariant with respect to time-variable shifts. On the other hand, from the periodicity of the right-hand side with respect to the phase variables it follows that the solution space of such equation is invariant with respect to a shift in phase variables for a period equal to $\frac{2\pi}{B}$. Therefore, it suffices to consider a family of solutions of the initial problem (6) with a value of $z_1(0)$ from zero to the value of the period. Nevertheless, the stationary solutions are repeated each half-period $\frac{\pi}{B}$. Since the right-hand side of the equation is an odd function of its arguments, the solution space of such equation can withstand the reflection transformation with respect to the axis t. Hence, it is sufficient to construct trajectories in the strip from zero to the half-period. Figure 3 shows the integral curves for different values of the parameter $c = z_1(0)$ for both the system (7) and the original system (6) (the values of c are reduced to half-period).



(a) Trajectories of the System (7)

(b) Trajectories of the System (6)

Figure 3: Integral Curves at Different c.

Acknowledgements

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