Fixed Point Method for Searching of Extremal Controls

Alexander S. Buldaev

Buryat State University

Ulica Smolina 24,

670000 Ulan-Ude, Russia.

buldaev@mail.ru

Abstract

A new approach for searching of extremal controls in nonlinear problems of optimization of control functions and parameters of dynamical systems is considered on basis of the solution of special fixed point problems for constructed operators in the space of controls. Approbation and comparative analysis of the developed approach is carried out within the framework of the known Zermelo problem.

1 Introduction

A common approach to solving optimal control problems is to search for controls that satisfy the necessary [Pontrjagin et al., 1976]-[Vasilev, 1994] or sufficient [Gurman et al., 1987]-[Baturin & Urbanovich, 1997] optimality conditions. In this case, the classical approach consists in constructing a boundary value problem of the maximum principle, the difficulties of solving which are well known [Pontrjagin et al., 1976],[Ashepkov et al., 1984]. Another approach consists in successively solving the problems of local improvement, as a result of which a relaxation sequence of controls is constructed, converging under certain conditions to extremal control. This type includes, for example, known gradient methods [Ashepkov et al., 1984],[Vasilev, 1994]. In [Gurman et al., 1987]-[Srochko, 2000], prospective nonlocal methods for improving control based on non-standard approximations of problem functionals are proposed. The higher quality of the functional approximations, compared with the standard ones, makes the non-local methods more effective.

In [Srochko, 2000], in the classes of linear and quadratic optimal control problems, formulas for the increment of the functionals of problems that do not contain the remainder terms of the expansions are constructed. The complexity of one improvement in control is determined by solving a special Cauchy problem for phase or conjugate variables. The constructed nonlocal methods are characterized by the fact that improvement of control is guaranteed not only in a sufficiently small neighborhood of the improved control, in contrast to local methods such as gradient ones. The methods do not contain a procedure for searching for improving control in a sufficiently small neighborhood of the improved control and have the opportunity to improve non-optimal controls that satisfy the maximum principle. On the basis of the methods, new necessary optimality conditions are obtained, which strengthen the maximum principle in the classes of problems under consideration.

In [Buldaev, 2008]-[Buldaev & Khishektueva, 2013], the nonlocal approach [Srochko, 2000], based on the construction of formulas for the increment of functionals without residual terms of the expansions, was developed and generalized to problems that are polynomial with respect to state and general nonlinear optimal control

problems. In this case, the construction of formulas for the increment of the functionals of problems without residual terms of the expansions is achieved by means of differential-algebraic modifications of the standard conjugate system. The complexity of one improvement in control is determined by solving a special boundary-value problem in the state space or an equivalent operator equation in control space interpreted as a fixed point problem for a special control operator.

Standard methods for the numerical solution of boundary value problems of the maximum principle and special boundary value problems of improving control (the shooting method, the linearization method, the finite-difference method), even in the case of smoothness and uniqueness of the right-hand parts of problems, as a rule, are computationally unstable due to the presence of positive real values of the eigenvalues of the corresponding Jacobi matrix. These problems can be resolved by proceeding to the solution of equivalent operator equations in the space of controls interpreted as the fixed point problems for constructed control operators [Buldaev, 2016],[Buldaev, 2017]. The proposed approach for searching for extremal controls makes it possible to apply the developed theory and fixed-point methods.

In this paper, the nonlocal approach being developed is illustrated in an important class for applications of optimization of non-linear systems with respect to control functions and parameters.

2 Fixed Point Problems

In a generalized form, the class of problems under consideration is represented in the following form:

$$\Phi(\sigma) = \varphi(x(t_1), \omega) + \int_T F(x(t), u(t), \omega, t)dt \to \inf_{\sigma \in \Omega}, \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t), \omega, t), x(t_0) = a, u(t) \in U, \omega \in W, a \in A, t \in T = [t_0, t_1], \tag{2}$$

in which $x(t) = (x_1(t), ..., x_n(t))$ – state vector, $u(t) = (u_1(t), ..., u_m(t))$ – vector of control functions, $\omega = (\omega_1, ..., \omega_l)$ $a = (a_1, ..., a_n)$ - vectors of control parameters. Sets $U \subseteq R^m$, $W \subseteq R^l$, $A \subseteq R^n$ are compact and convex. Interval T is fixed. As admissible control functions, we consider a set V of piecewise-continuous functions on T with values in the set U. $\sigma = (u, \omega, a)$ - admissible control with values in the set $\Omega = V \times W \times A$. The function $\varphi(x, \omega)$ is continuously differentiable on $R^n \times W$. Functions $F(x, u, \omega, t)$, $f(x, u, \omega, t)$ and their partial derivatives with respect to x, u, ω are continuous in the set of arguments on the set $R^n \times U \times W \times T$.

The Pontryagin function with the conjugate variable $\psi \in R^n$ and the standard conjugate system have the form:

$$H(\psi, x, u, \omega, t) = \langle \psi, f(x, u, \omega, t) \rangle - F(x, u, \omega, t),$$

$$\dot{\psi}(t) = -H_x(\psi(t), x(t), u(t), \omega, t), t \in T, \psi(t_1) = -\varphi_x(x(t_1), \omega). \tag{3}$$

For an admissible control $\sigma \in \Omega$ we denote $x(t, \sigma)$, $t \in T$ - the solution of the system (2); $\psi(t, \sigma)$, $t \in T$ - the solution of the standard conjugate system (3) for $x(t) = x(t, \sigma)$ and the arguments u, ω, corresponding to the control components σ.

In [Buldaev, 2016],[Buldaev, 2017], the conditions for improving and optimizing control in the form of fixed-point problems are constructed on the basis of operations on the maximum of Pontryagin's control function and on the basis of control design operations on an admissible set of values.

The necessary conditions for optimal control in problem (1), (2) in the form of the maximum principle [Buldaev, 2016] have the form:

$$u(t) = \arg\max_{\tilde{u} \in U} H(\psi(t, \sigma), x(t, \sigma), \tilde{u}, \omega, t), t \in T, \tag{4}$$

$$\omega = \arg\max_{\tilde{\omega} \in W} \left\langle -\varphi_\omega(x(t_1, \sigma), \omega) + \int_T H_\omega(\psi(t, \sigma), x(t, \sigma), u(t), \omega, t)dt, \tilde{\omega} \right\rangle, \tag{5}$$

$$a = \arg\max_{\tilde{a} \in A} \langle \psi(t_0, \sigma), \tilde{a} \rangle. \tag{6}$$

Weakened necessary conditions (the differential maximum principle) by means of the projection operator P_Y on the set $Y \subset R^k$ in the Euclidean norm

$$P_Y(z) = \arg\min_{y \in Y}(\|y - z\|), z \in R^k$$

can be represented [Buldaev, 2017] in the following projection form with the parameter $\alpha > 0$:

$$u(t) = P_U(u(t) + \alpha H_u(\psi(t, \sigma), x(t, \sigma), u(t), \omega, t)), t \in T, \tag{7}$$

$$\omega = P_W(\omega + \alpha(-\varphi_\omega(x(t_1, \sigma), \omega) + \int_T H_\omega(\psi(t, \sigma), x(t, \sigma), u(t), \omega, t)dt)), \tag{8}$$

$$a = P_A(a + \alpha\psi(t_0, \sigma)). \tag{9}$$

Conditions (4) - (6) and (7) - (9) are considered as fixed point problems of the maximum principle in control space for a uniquely determined operator defined by right-hand sides of these conditions. The methods proposed in this paper for the search for extremal controls consist in solving the fixed point problems under consideration.

3   Iterative Methods for Solving Fixed Point Problems

We consider an arbitrary operator $G : Z \to Z$, acting on a set Z in a complete normed space with norm $\|\cdot\|_z$.

To solve the fixed-point problem

$$v = G(v), v \in Z \tag{10}$$

it is possible to apply the method of successive approximations known in computational mathematics and its modifications [Samarskiy & Gulin, 1989]. In particular, the method of simple iteration for $k \geq 0$, having the form:

$$v^{k+1} = G(v^k), v^0 \in Z. \tag{11}$$

To improve the convergence of the iterative process, problem (10) can be transformed to an equivalent fixed-point problem with parameter $\delta \in R$

$$v = v + \delta(v - G(v)), v \in Z, \delta \neq 0,$$

on the basis of which the modification of the iteration process (11) is constructed:

$$v^{k+1} = v^k + \delta(v^k - G(v^k)), v^0 \in Z, \delta \neq 0.$$

When choosing a parameter $\delta \neq 0$, it is possible to regulate the convergence of the modification of the simple iteration method under consideration.

The fixed-point problems of the maximum principle (4) - (6), (7) - (9) generate the corresponding simple iteration methods. In particular, for the solution of the fixed point problem of the differential maximum principle (7) - (9), the following iterative process for $k \geq 0$ with initial control $\sigma^0 \in \Omega$ can be used:

$$u^{k+1}(t) = P_U(u^k(t) + \alpha H_u(\psi(t, \sigma^k), x(t, \sigma^k), u^k(t), \omega^k, t)), t \in T,$$

$$\omega^{k+1} = P_W(\omega^k + \alpha(-\varphi_\omega(x(t_1, \sigma^k), \omega^k) + \int_T H_\omega(\psi(t, \sigma^k), x(t, \sigma^k), u^k(t), \omega^k, t)dt)),$$

$$a^{k+1} = P_A(a^k + \alpha\psi(t_0, \sigma^k)).$$

To analyze the convergence of the iterative processes under consideration to solutions of fixed point problems for sufficiently small design parameters, the known principle of contracting mappings is applied [Buldaev, 2008],[Buldaev, 2016].

4   The Problem with Non-Fixed Time

Many problems of optimal control with non-fixed time control of changing the time variable can be reduced to a class of optimization problems for control functions and parameters with a fixed time to which the proposed fixed point methods can be applied. Let us demonstrate this approach on a specific model problem with non-fixed time.

In [Gornov, 2009], the well-known optimal speed problem (Zermelo problem) is reduced by the penalty method to the following optimal control problem with non-fixed control time:

$$\begin{cases} \dot{x}_1(t) = \cos x_3(t), & x_1(0) = 0, \\ \dot{x}_2(t) = \sin x_3(t), & x_2(0) = 0, \\ \dot{x}_3(t) = u(t), & x_3(0) = 0, \end{cases}$$

$$t \in [0, t_1], \quad |u(t)| \leq 0,5.$$

$$G(u) = t_1 + 1000\left((x_1(t_1) - 4)^2 + (x_2(t_1) - 3)^2\right) \to \min.$$

Making the replacement of time by rule

$$t(\tau) = t_0 + (t_1 - t_0)\tau = \omega\tau, \quad \tau \in [0, 1], \quad \omega = t_1 \geq 0,$$

$$v(\tau) = u(t(\tau)), \quad y(\tau) = x(t(\tau)), \quad \sigma = (v, w),$$

this problem reduces to a problem with a fixed time:

$$\begin{cases} \dot{y}_1(\tau) = \omega \cos y_3(\tau), & y_1(0) = 0, \\ \dot{y}_2(\tau) = \omega \sin y_3(\tau), & y_2(0) = 0, \\ \dot{y}_3(\tau) = \omega v(\tau), & y_3(0) = 0, \end{cases}$$

$$\Phi(\sigma) = \omega + 1000\left((y_1(1) - 4)^2 + (y_2(1) - 3)^2\right) \to \min.$$

The Pontryagin function with derivatives and the standard conjugate system in the transformed problem have the form:

$$H(\psi, y, v, \omega, \tau) = \omega(\psi_1 \cos y_3 + \psi_2 \sin y_3 + \psi_3 v), H_{y_1} = 0, H_{y_2} = 0,$$

$$H_{y_3} = -\omega\psi_1 \sin y_3 + \omega\psi_2 \cos y_3, H_v = \psi_3\omega, H_\omega = \psi_1 \cos y_3 + \psi_2 \sin y_3 + \psi_3 v,$$

$$\begin{cases} \dot{\psi}_1(\tau) = 0, & \psi_1(1) = -2000(y_1(1) - 4), \\ \dot{\psi}_2(\tau) = 0, & \psi_2(1) = -2000(y_2(1) - 3), \\ \dot{\psi}_3(\tau) = \omega\psi_1 \sin y_3(\tau) - \omega\psi_2 \cos y_3(\tau), & \psi_3(1) = 0. \end{cases}$$

For an admissible control $\sigma = (v, \omega) \in \Omega$, we denote $\psi(\tau, \sigma)$, $\tau \in [0, 1]$ - the solution of the conjugate system for $y(\tau) = y(\tau, \sigma)$ and with the corresponding components v, ω.

The fixed point problem of the differential maximum principle in the projection form according to the rule (7) - (9) takes the form:

$$v(\tau) = P_U(v(\tau) + \alpha\omega\psi_3(\tau, \sigma)), \tau \in [0, 1],$$

$$\omega = P_W\left(\omega + \alpha\left(-1 + \int_0^1 (\psi_1(\tau, \sigma)\cos y_3(\tau, \sigma) + \right.\right.$$

$$\left.\left. + \psi_2(\tau, \sigma)\sin y_3(\tau, \sigma) + \psi_3(\tau, \sigma)v(\tau))d\tau\right)\right), \alpha > 0.$$

To realize the obtained fixed point problem, the following explicit iterative process is considered for $k \geq 0$ with a given initial approximation $\sigma^0 = (v^0, \omega^0) \in \Omega$:

$$v^{k+1}(\tau) = P_U\left(v^k(\tau) + \alpha\omega\psi_3(\tau, \sigma^k)\right), \tau \in [0, 1],$$

$$\omega^{k+1} = P_W\left(\omega^k + \alpha\left(-1 + \int_0^1 (\psi_1(\tau, \sigma^k)\cos y_3(\tau, \sigma^k) + \right.\right.$$

$$\left.\left. + \psi_2(\tau, \sigma^k)\sin y_3(\tau, \sigma^k) + \psi_3(\tau, \sigma^k)v^k(\tau))d\tau\right)\right), \alpha > 0.$$

The numerical solution of the phase and conjugate Cauchy problems was carried out by the fifth-order or sixth-order Runge-Kutta-Werner method using the Fortran PowerStation 4.0 IMSL library. The values of controllable, phase and conjugate variables were memorized at the nodes of a fixed uniform grid with a discretization step $\Delta\tau = 10^{-3}$. In the intervals between neighboring grid nodes, the value of the control function was assumed constant and equal to the value in the left node. A numerical calculation of the fixed point problem was carried out until the condition was satisfied

$$\left|\Phi\left(\sigma^{k+1}\right) - \Phi\left(\sigma^k\right)\right| \leq \left|\Phi\left(\sigma^k\right)\right| \cdot \varepsilon,$$

where $\varepsilon > 0$ is the specified accuracy of the calculation. In the quality $\varepsilon > 0$, the value of the "machine epsilon" calculated according to [Forsyth et al., 1980] was chosen, which turned out to be equal 10^{-16} when calculating with double-precision variables.

Table 1 gives the comparative results of the calculation of the Zermelo problem under consideration by the proposed fixed-point method (FPM) with methods [Gornov, 2009], which in the indicated work are conventionally designated by numbers from 1 to 8.

<div align="center">Table 1.</div>

Method	Φ^*	N
1	5,85	100010
2	5,26	100125
3	5,13	100002
4	5,18	100019
5	5,26	100036
6	6,41	100008
7	5,33	100015
8	5,11	100012
FPM	5,117	70619

In the table Φ^* - the calculated value of the functional, N - the number of calculated phase and conjugate Cauchy problems.

Table 2 shows the results of the calculation of the FPM algorithm for various values of the projection parameter and the initial permissible approximation $\sigma^0 = (v^0, \omega^0) \in \Omega$.

<div align="center">Table 2.</div>

α	v^0	ω^0	Φ^*	N
10^{-5}	**0,5**	**5**	**5,117**	**70619**
$>10^{-5}$	0,5	5	does not converge	
10^{-6}	0,5	5	5,117	> 200000
10^{-5}	0	5	5,117	103999
$>10^{-5}$	0	5	does not converge	
10^{-6}	0	5	5,117	> 200000
10^{-5}	-0,5	5	5,117	141853
$>10^{-5}$	-0,5	5	does not converge	
10^{-5}	0,5	10	5,117	> 200000
$>10^{-5}$	0,5	10	does not converge	
10^{-6}	0,5	10	5,117	> 200000
10^{-5}	0	10	5,117	> 200000
$>10^{-5}$	0	10	does not converge	
10^{-5}	-0,5	10	does not converge	
$>10^{-5}$	-0,5	10	does not converge	
10^{-5}	0,5	1	5,117	> 200000
$>10^{-5}$	0,5	1	does not converge	
10^{-6}	0,5	1	5,117	> 200000
10^{-5}	0	1	does not converge	
$>10^{-5}$	0	1	does not converge	
10^{-6}	0	1	5,117	> 200000
10^{-5}	-0,5	1	does not converge	
$>10^{-5}$	-0,5	1	does not converge	

The calculations showed that the algorithm does not converge for $\alpha > 10^{-5}$. When the value α is lower 10^{-5}, the convergence of the algorithm slows down significantly and the number of computed Cauchy problems increases accordingly. The computational efficiency of the algorithm, estimated by the number of computed Cauchy problems, depends significantly on the choice of the initial approximation.

The projection method of fixed points for the calculation of the Zermelo model problem is characterized by a relatively high computational efficiency, a sufficiently wide range of convergence of the iterative algorithm with respect to the initial approximation and the ease of adjusting the projection parameter determining the rate of convergence of the iterative process.

5 Conclusion

The proposed fixed-point methods for searching for extremal controls do not guarantee relaxation by the objective functional, in contrast to the gradient methods, but compensate for this property by the nonlocality of successive

control approximations due to the fixed design parameter; the absence of an operation of varying the control in the neighborhood of the current approximation; obtaining the necessary conditions for the optimality of control in a new constructive form of the fixed point problems of the control operators defined in the class of problems under consideration, which makes it possible to effectively apply the well-known apparatus of fixed-point theory and methods to search for extremal controls.

The projection method of fixed points is characterized by the fact that the extremal controls are determined by the solutions of the corresponding fixed point problem for any value of the design parameter $\alpha > 0$. In particular, for sufficiently small $\alpha > 0$, for which convergence of the constructed iterative processes of successive approximations to solutions of fixed point problems is provided.

The main difference between the projection method of fixed points and the standard method of gradient projection is that the design parameter $\alpha > 0$ is fixed in the iterative process of successive approximations. In the gradient projection method, this parameter varies at each iteration to provide improvement of control.

In general, the optimization of controls based on the calculation of the constructed fixed point problems by iterative methods of successive approximations reduces to computationally stable calculation of alternating Cauchy problems for phase and conjugate variables.

These properties of the proposed fixed point methods are important factors in improving the computational and qualitative efficiency of solving optimal control problems and determine the perspective direction of the development of optimization methods for nonlinear dynamical systems.

Acknowledgements

References

[Pontrjagin et al., 1976] Pontrjagin L. S. i dr. *Matematicheskaja teorija optimal'nyh processov.* Moscow, Nauka Publ., 1976.

[Ashepkov et al., 1984] Ashepkov L.T. i dr. *Metody reshenija zadach matematicheskogo programmirovanija i optimal'nogo upravlenija.* Novosibirsk, Nauka Publ., 1984.

[Vasilev, 1994] Vasilev O.V. *Lektsii po metodam optimisatsii.* Irkutsk, Izd-vo Irkut. un-ta, 1994.

[Gurman et al., 1987] Gurman V.I. i dr. *Novie metody uluchsheniya upravlyaemih protsessov.* Novosibirsk, Nauka Publ., 1987.

[Gurman et al., 1988] Gurman V.I. i dr. *Metody uluchsheniya v vichislitelnom eksperimente.* Novosibirsk, Nauka Publ., 1988.

[Baturin & Urbanovich, 1997] Baturin V.A., Urbanovich D. E. *Priblizhenniye metody upravlenya, osnovannie na printsipe rashireniya.* Novosibirsk, Nauka Publ., 1997.

[Srochko, 2000] Srochko V.A. *Iteratsionnyye metody resheniya zadach optimal'nogo upravleniya.* Moscow, Fizmatlit Publ., 2000.

[Buldaev, 2008] Buldaev A.S. *Metody vozmusheniy v zadachah uluchsheniya i optimizatsii upravlyaemih system.* Ulan-Ude, Buryat State University Publishing Department, 2008.

[Buldaev & Morzhin, 2009] Buldaev A.S., Morzhin O. V. Improvement of Controls in Nonlinear Systems on Basis of Boundary Value Problems. *Izvestiya Irkutskogo gosudarstvennogo universiteta. Series "Mathematics", 2009, 2(1)*, 94–106. (in Russian).

[Buldaev & Khishektueva, 2013] Buldaev A.S., Khishektueva I.-Kh. D. The Fixed Point Method in Parametric Optimization Problems for Systems. *Automation and Remote Control, 2013, 74(12)*, 1927–1934. doi:10.1134/S0005117913120011.

[Buldaev, 2016] Buldaev A.S. *Metody nepodvizhnih tochek v zadachah optimalnogo upravleniya.* Ulan-Ude, Buryat State University Publishing Department, 2016.

[Buldaev, 2017] Buldaev A.S. Fixed Point Methods on the Basis of Design Operations in Optimization Problems of Control Functions and Parameters of Dynamical Systems. *Bulletin of the Buryat State University. Mathematics, Informatics, 2017, 1*, 38 -54. doi:10.18101/2304-5728-2017-1-38-54.

[Samarskiy & Gulin, 1989] Samarskiy A.A., Gulin A.V. *Chislennie metody.* Moscow, Nauka Publ., 1989.

[Gornov, 2009] Gornov A.Yu. *Vychislitel'nyye tekhnologii resheniya zadach optimal'nogo upravleniya.* Novosibirsk, Nauka Publ., 2009.

[Forsyth et al., 1980] Forsyth J., Malcolm M., Molar K. *Machine methods of mathematical computations.* Moscow, Mir Publ., 1980.