Decommissioning of Nuclear Facilities: Minimum Accumulated Radiation Dose Routing Problem

Alexander G. Chentsov
Krasovskii Institute of Mathematics and Mechanics UB RAS
S.Kovalevskaya Str. 16,
620990 Yekaterinburg,
Russia.
chentsov@imm.uran.ru

Alexey M. Grigoryev
Krasovskii Institute of Mathematics and Mechanics UB RAS
S.Kovalevskaya Str. 16,
620990 Yekaterinburg,
Russia.
ag@uran.ru

Alexey A. Chentsov
Krasovskii Institute of Mathematics and Mechanics UB RAS
S.Kovalevskaya Str. 16,
620990 Yekaterinburg,
Russia.
chentsov_a_a@mail.ru

Abstract

The problem that we consider has its prototype in an engineering problem that arises in dismantling, one at a time, a system of elements emitting hazardous radiation. Its possible applications can be connected with optimizing the actions of nuclear power plant staff in emergencies or disaster cleanup operations (e.g. those comparable with Chernobyl or Fukushima Daiichi). The main features of the arising routing problem as compared with the well-known Traveling salesman problem (TSP) are connected with the dependence of travel and dismantlement cost functions on the set of pending tasks (not yet dismantled radiation sources) and precedence constraints. For example, a radiation source's influence on the agent is computed by integrating the dependence, which is inversely proportional to the square of the distance; only the sources that are not yet dismantled at the time of a movement are considered. Thus, the agent's exposure to radiation or other hazards depends on his specific trajectory and the order of dismantlement operations.

1 Introduction

This paper is devoted to a study of an important class of discrete optimization problems, namely, the problems concerning multiple movements under constraints. The prototype of the considered problem is the well-known intractable traveling salesman problem (TSP), see [Garey, 1979, Ch.3]; to name just a few works on TSP, [Gutin, 2002, Cook, 2012, Melamed, 1989, Litl, 1965, Bellman, 1962, Held, 1962]. However, in applications, one often has to satisfy certain additional constraints. In particular, constraints arise in the problem of minimizing the exposure of nuclear power plant staff conducting operations related to dismantling radiation sources. A typical characteristic of the latter problem is dependence on the set of pending tasks: only the sources that are not yet dismantled at the time of each operations do "radiate." There are also other applications [Leon, 1996,

In: Yu. G. Evtushenko, M. Yu. Khachay, O. V. Khamisov, Yu. A. Kochetov, V.U. Malkova, M.A. Posypkin (eds.): Proceedings of the OPTIMA-2017 Conference, Petrovac, Montenegro, 02-Oct-2017, published at http://ceur-ws.org

Alkaya, 2010, Kinable, 2016]. Another peculiarity, which arises, for example, in the known problem of dismantling a decommissioned nuclear power generation unit, consists in precedence constraints on the sequence of operations; this is formalized by specifying ordered pairs (OP) of operations where the first component of the pair must be conducted before the second one. Finally, in contrast with "ordinary" TSP, in applications, the "cities" visited may possess some internal structure and are thus rendered not as "cities" but as clusters of cities, or megalopolises, which introduces a certain additional variation into the movements—the problem develops a two-layer hierarchy: layer one is the sequencing of megalopolises through a permutation of their indices—the route, and layer two is the choice of a "track" along the given route, i.e., the exact versions of movements through the megalopolises. The arguments leading to a two-layer optimization problem are stated in [Chentsov, 2008], and the applications of methods developed in [Chentsov, 2008] for problems of nuclear power generation are discussed in [Korobkin, 2012, Tashlykov, 2011]. In this paper, we use a rather general dynamic programming (DP) procedure.

2 Problem Statement and Discussion

Let us start with general definitions, notions, and notations. The symbol $\triangleq$ denotes equality by definition. A set, all elements of which are themselves sets is called a family. The real line is denoted by $\mathbb{R}$; in addition, $\mathbb{R}_+ \triangleq \{\xi \in \mathbb{R} \,|\, 0 \leqslant \xi\}$, $\mathbb{N} \triangleq \{1; 2; \ldots\}$ and $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\} = \{0; 1; 2; \ldots\}$, $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{R}$. For $p \in \mathbb{N}_0$ and $q \in \mathbb{N}_0$, assume $\overline{p,q} \triangleq \{t \in \mathbb{N}_0 \,|\, (p \leqslant t) \,\&\, (t \leqslant q)\}$. By $\mathcal{R}_+[S]$ we denote the set of all functions from a nonempty set S into $\mathbb{R}_+$, i.e., the set of all nonnegative real functions on S. For every ordered pair (OP) $z = (a, b)$ of arbitrary objects a and b, denote by $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$, respectively, its first and second elements: $\mathrm{pr}_1(z) = a$, $\mathrm{pr}_2(z) = b$. For an object x, the corresponding singleton set is denoted by $\{x\}$. For arbitrary objects a, b, and c, by convention [Dieudonne, 1960, p.17], assume $(a, b, c) \triangleq ((a, b), c)$; thus, a triple is a certain ordered pair. For three arbitrary sets A, B, and C, following [Dieudonne, 1960, p.17], we assume $A \times B \times C \triangleq (A \times B) \times C$, thus, for $\mu \in A \times B$ and $\nu \in C$, we have $(\mu, \nu) \in A \times B \times C$. For a set H, denote by $\mathcal{P}(H)$ the set of all subsets of H, i.e., the power set of H; $\mathcal{P}^{'}(H) \triangleq \mathcal{P}(H) \setminus \{\varnothing\}$; denote by $\mathrm{Fin}(H)$ the family of all (nonempty) finite sets of $\mathcal{P}^{'}(H)$. If H is a finite set, then $\mathrm{Fin}(H) = \mathcal{P}^{'}(H)$. To a nonempty finite set K, assign its cardinality $|K| \in \mathbb{N}$ (the number of its elements) and the (nonempty) set $(\mathrm{bi})[K]$ of all bijections [Cormen, 1990, p.87] of the set $\overline{1, |K|} = \{j \in \mathbb{N} \,|\, j \leqslant |K|\}$ onto K; in addition, $|\varnothing| \triangleq 0$. A permutation of a nonempty set T is [Cormen, 1990, p.87] a bijection of T onto itself; for every permutation α of T, there exists an inverse permutation α^{-1} of the set T: $\alpha(\alpha^{-1}(t)) = \alpha^{-1}(\alpha(t)) = t \;\; \forall t \in T$. The permutations of indices used below are interpreted as sequences of visiting the objective sets—the megalopolises.

Fix a nonempty set X, a point $x^\circ \in X$, a number $N \in \mathbb{N}, N \geqslant 2$, (nonempty finite) sets $M_1 \in \mathrm{Fin}(X), \ldots, M_N \in \mathrm{Fin}(X)$, and relations $\mathbb{M}_1 \in \mathcal{P}^{'}(M_1 \times M_1), \ldots, \mathbb{M}_N \in \mathcal{P}^{'}(M_N \times M_N)$. Here and below, assume $(x^\circ \notin M_j \;\; \forall j \in \overline{1,N}) \,\&\, (M_p \cap M_q = \varnothing \;\; \forall p \in \overline{1,N} \;\; \forall q \in \overline{1,N} \setminus \{p\})$. The objects $M_1, \ldots, M_N$ are regarded as the megalopolises to be visited. For each $j \in \overline{1,N}$, OPs from $\mathbb{M}_j$ determine the feasible interior jobs connected with visiting M_j: the first element of such an OP is the entry point, and the second element is the exit point. We consider the processes of the following [Chentsov, 2013, (3.3)] form:

$$x^\circ \to (\mathrm{pr}_1(z_1) \in M_{\alpha(1)} \rightsquigarrow \mathrm{pr}_2(z_1) \in M_{\alpha(1)}) \to \ldots \to (\mathrm{pr}_1(z_N) \in M_{\alpha(N)} \rightsquigarrow \mathrm{pr}_2(z_N) \in M_{\alpha(N)}), \tag{2.1}$$

where $z_1 \in \mathbb{M}_{\alpha(1)}, \ldots, z_N \in \mathbb{M}_{\alpha(N)}$, and α is a permutation of indices of $\overline{1,N}$, hereinafter referred to as route. Assume that in (2.1) the straight arrows denote the exterior movements and the wavy arrows denote the motions connected with (interior) jobs. The objects of our choice are $\alpha, z_1, \ldots, z_N$. Let $\mathbf{M}_j \triangleq \{\mathrm{pr}_2(z) : z \in \mathbb{M}_j\} \in \mathrm{Fin}(M_j) \;\; \forall j \in \overline{1,N}$. In view of this, assume $\mathbb{X} \triangleq \{x^\circ\} \cup \left(\bigcup_{i=1}^{N} M_i\right) \in \mathrm{Fin}(X)$, $\mathbf{X} \triangleq \{x^\circ\} \cup \left(\bigcup_{i=1}^{N} \mathbf{M}_i\right) \in \mathrm{Fin}(\mathbb{X})$. In connection with (2.1), note that both exterior movements and interior jobs conducted as the megalopolises are visited are measured through the given functions, and the results of these measurements are aggregated additively, which is the natural way for many applications. Here and below, assume $\mathbb{P} \triangleq (\mathrm{bi})[\overline{1,N}]$, and also assume that each specific $\alpha \in \mathbb{P}$ (see (2.1)) must satisfy the precedence constraints, which are defined by means of the set $\mathbf{K} \in \mathcal{P}(\overline{1,N} \times \overline{1,N})$; OPs from $\mathbf{K}$ will be called address pairs; the feasibility of $\alpha \in \mathbb{P}$ can be reduced to the following requirement: for $z = (i, j) \in \mathbf{K}$, the set M_i must be visited before M_j. Here and below, assume [Chentsov, 2013, (3.11)] the following: $\forall \mathbf{K}_0 \in \mathcal{P}^{'}(\mathbf{K}) \exists z_0 \in \mathbf{K}_0 : \; \mathrm{pr}_1(z_0) \neq \mathrm{pr}_2(z) \;\; \forall z \in \mathbf{K}_0$. Then, the set $\mathbf{A}$ of all feasible (with respect to precedence) routes from $\mathbb{P}$ has the following form [Chentsov, 2013, (3.12)]:

$$\mathbf{A} \overset{\triangle}{=} \{\alpha \in \mathbb{P} | \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \ \ \forall z \in \mathbf{K}\} \in \mathcal{P}'(\mathbb{P}). \tag{2.2}$$

Note that $\mathrm{pr}_1(z) \neq \mathrm{pr}_2(z) \ \ \forall z \in \mathbf{K}$. As evident from (2.1), the choice of some route $\alpha \in \mathbf{A}$ does not yet determine the whole process, it must be supplemented with some track $z_1, \ldots, z_N$. A track has to agree with the given route, and the solution is constructed in the form of OP consisting of a route and a track, where the route must be an element of $\mathbf{A}$. It is useful to bear in mind that such interpretation allows one to consider not only (2.1) but also the partial processes connected with visiting megalopolises $M_k, k \in K$, where $K \subset \overline{1,N}$. Such interpretation is also useful in connection with the dynamic programming (DP); it was laid out in [Chentsov, 2013, Chentsov, 2015, Chentsov, 2016]. In this paper, we omit its details and devote most of our attention to the computational concerns. Thus, we limit ourselves to a description of "complete" routes and "tracks" (trajectories), similar to (2.1). Thus, if $\alpha \in \mathbb{P}$, denote by $\mathcal{Z}_\alpha$ the set of all tuples $(z_i)_{i\in\overline{0,N}} : \overline{0,N} \to \mathbb{X}\times\mathbb{X}$ such that $(z_0 = (x^0, x^0))\,\&\,(\,z_t \in \mathbb{M}_{\alpha(t)} \ \ \forall t \in \overline{1,N})$; $\mathcal{Z}_\alpha$ is a nonempty finite set. Feasible solutions will be represented by the OPs $(\alpha, (z_i)_{i\in\overline{0,N}})$, where $\alpha \in \mathbf{A}$ and $(z_i)_{i\in\overline{0,N}} \in \mathcal{Z}_\alpha$. Let $\mathfrak{N} \overset{\triangle}{=} \mathcal{P}'(\overline{1,N})$.

Consider the following cost functions (they are assumed to be known): $\mathbf{c} \in \mathcal{R}_+[\mathbb{X}\times\mathbb{X}\times\mathfrak{N}]$, $c_1 \in \mathcal{R}_+[\mathbb{X}\times\mathbb{X}\times\mathfrak{N}], \ldots, c_N \in \mathcal{R}_+[\mathbb{X}\times\mathbb{X}\times\mathfrak{N}]$, $f \in \mathcal{R}_+[\mathbb{X}]$. In terms of these functions, let us define the additive criterion: for $\alpha \in \mathbf{A}$ and $(z_i)_{i\in\overline{0,N}} \in \mathcal{Z}_\alpha$, set

$$\begin{aligned}\mathfrak{C}_\alpha[(z_i)_{i\in\overline{0,N}}] \overset{\triangle}{=} \sum_{t=1}^{N} [\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \{\alpha(j) : j \in \overline{t,N}\}) + \\ + c_{\alpha(t)}(z_t, \{\alpha(j) : j \in \overline{t,N}\})] + f(\mathrm{pr}_2(z_N)).\end{aligned} \tag{2.3}$$

Our principal problem is as follows:

$$\mathfrak{C}_\alpha[(z_i)_{i\in\overline{0,N}}] \to \min, \ \alpha \in \mathbf{A}, (z_i)_{i\in\overline{0,N}} \in \mathcal{Z}_\alpha; \tag{2.4}$$

for each problem (2.4), there exists a nonempty set of optimal feasible solutions and the problem's value—the following extremum:

$$V \overset{\triangle}{=} \min_{\alpha\in\mathbf{A}} \ \min_{(z_i)_{i\in\overline{0,N}} \in \mathcal{Z}_\alpha} \mathfrak{C}_\alpha[(z_i)_{i\in\overline{0,N}}] \in \mathbb{R}_+. \tag{2.5}$$

Our goal is to find the (global) extremum V (2.5) and some optimal feasible solution $(\alpha^0, (z_i^0)_{i\in\overline{0,N}})$, where $\alpha^0 \in \mathbf{A}$ and $(z_i^0)_{i\in\overline{0,N}} \in \mathcal{Z}_{\alpha^0}$; evidently, they satisfy $\mathfrak{C}_{\alpha^0}[(z_i^0)_{i\in\overline{0,N}}] = V$.

3 Dynamic Programming. Layers of Bellman Function

Problem (2.4) is solved by means of a variety of DP [Chentsov, 2013, Chentsov, 2015, Chentsov, 2016], which we specify in a brief form. We use the construction from [Chentsov, 2013, Chentsov, 2015, Chentsov, 2016], which is based on the layer structure of the Bellman function; the algorithm of their construction is sketched below. Let us consider the construction of state space layers by means of the crossing-out operator (for tasks in the task list) [Chentsov, 2008, Pt. 2]: $\mathbf{I} : \mathfrak{N} \to \mathfrak{N}$, specifically, for $K \in \mathfrak{N}$, set $\Xi(K) \overset{\triangle}{=} \{z \in \mathbf{K}\,|\,(\mathrm{pr}_1(z) \in K)\,\&\,(\mathrm{pr}_2(z) \in K)\}$ and $\mathbf{I}(K) \overset{\triangle}{=} K \setminus \{\mathrm{pr}_2(z) : \ z \in \Xi(K)\}$. This operator is used to construct the layers of the state space. To this end, consider a set $\mathbf{G} \overset{\triangle}{=} \{K \in \mathfrak{N}\,|\,\forall z \in \mathbf{K} \ (\mathrm{pr}_1(z) \in K) \Rightarrow (\mathrm{pr}_2(z) \in K)\}$ clearly, $\overline{1,N} \in \mathbf{G}$. We sort the mentioned lists by their cardinality, $\mathbf{G}_s \overset{\triangle}{=} \{K \in \mathbf{G}\,|\,s = |K|\} \ \forall s \in \overline{1,N}$. Then, the family $\{\mathbf{G}_j : \ j \in \overline{1,N}\}$ is a partition of $\mathbf{G}$. In addition, $\mathbf{G}_1 \overset{\triangle}{=} \{\{t\} : \ t \in \overline{1,N} \setminus \mathbf{K}_1\}$, where $\mathbf{K}_1 \overset{\triangle}{=} \{\mathrm{pr}_1(z) : z \in \mathbf{K}\}$. Also [Chentsov, 2013, Chentsov, 2015, Chentsov, 2016], $\mathbf{G}_{s-1} = \{K \setminus \{j\} : \ K \in \mathbf{G}_s, \ j \in \mathbf{I}(K)\} \ \forall s \in \overline{2,N}$. Thus we obtain a recurrence procedure $\mathbf{G}_N \to \mathbf{G}_{N-1} \to \ldots \to \mathbf{G}_1$. Based on this procedure, we construct the state space layers, which we denote $D_0, D_1, \ldots, \ D_N$. Specifically, $D_0 = \{(x, \varnothing) : \ x \in \mathfrak{M}\}$, where $\mathfrak{M}$ is by definition the union of all the sets $\mathbf{M}_i, \ i \in \overline{1,N} \setminus \mathbf{K}_1$. Next, $D_N \overset{\triangle}{=} \{(x^0, \overline{1,N})\}$ (the singleton that contains the OP $(x^0, \overline{1,N})$). If $s \in \overline{1,N-1}$ and $K \in \mathbf{G}_s$, we have [Chentsov, 2015] the following sequence of definitions:

$$J_s(K) \overset{\triangle}{=} \{t \in \overline{1,N} \setminus K\,|\,\{t\} \cup K \in \mathbf{G}_{s+1}\}, \mathcal{M}_s[K] \overset{\triangle}{=} \bigcup_{j\in J_s(K)} \mathbf{M}_j,$$

$$\mathbb{D}_s[K] \overset{\triangle}{=} \{(x, K) : x \in \mathcal{M}_s[K]\} \in \mathcal{P}'(\mathbf{X} \times \mathbf{G}_s).$$

Thus, for $s \in \overline{1, N-1}$, the layer D_s is defined as the union of all sets $\mathbb{D}_s[K], K \in \mathbf{G}_s$. It is easy to see that $D_0 \neq \varnothing, D_1 \neq \varnothing, \dots, D_N \neq \varnothing$. If $s \in \overline{1, N}$, $(x, K) \in D_s, j \in \mathbf{I}(K)$, and $z \in \mathbb{M}_j$, then $(\mathrm{pr}_2(z), K \setminus \{j\}) \in D_{s-1}$.

Recurrence procedure for construction of the layers. Consider a system of funcions $v_0 \in \mathcal{R}_+[D_0], v_1 \in \mathcal{R}_+[D_1], \dots, v_N \in \mathcal{R}_+[D_N]$. First, define $v_0 \in \mathcal{R}_+[D_0]$ by the condition $v_0(x, \varnothing) \triangleq f(x) \quad \forall x \in \mathfrak{M}$. Further constructions implement the following recurrence scheme: for $s \in \overline{1, N}$, if the function $v_{s-1} \in \mathcal{R}_+[D_{s-1}]$ is already constructed, $v_s \in \mathcal{R}_+[D_s]$ can be determined by the rule

$$v_s(x, K) \triangleq \min_{j \in \mathbf{I}(K)} \min_{z \in \mathbb{M}_j} [\mathbf{c}(x, \mathrm{pr}_1(z), K) + c_j(z, K) + v_{s-1}(\mathrm{pr}_2(z), K \setminus \{j\})| \; \forall (x, K) \in D_s. \tag{3.1}$$

From the general constructions of [Chentsov, 2013, Chentsov, 2015, Chentsov, 2016], there follows the equality

$$V = v_N(x^0, \overline{1, N}), \tag{3.2}$$

The procedure $v_0 \to v_1 \to \dots \to V$, which is completely defined by virtue (3.1), can be regarded as an algorithm for determining V, during which, only one layer of the Bellman function is retained in the computer's RAM; for details, refer to [Chentsov, 2016.1]. Note that a similar idea was also proposed in [Lawler, 1979]. Construction of optimal solution, corresponds to [Chentsov, 2013, Chentsov, 2016].

4 Scheme of Independent Computations for Layers of Bellman Function

Further exposition follows a version of the general construction [Chentsov, 2013, Chentsov, 2012, Chentsov, 2012.1] connected with conducting independent computations. Recall that (see (3.2)) states the "final" formula for determining V through v_{N-1}. Thus, further constructions require one to determine v_{N-1}, which is conducted by the procedure below. The function v_{N-1} is defined on the set D_{N-1}, which is defined through $\mathbf{G}_{N-1}$. Here and below, assume $N \geqslant 3$; we have $\mathbf{G}_{N-1} = \{\overline{1, N} \setminus \{j\} : j \in \mathbf{I}(\overline{1, N})\}$ since $\mathbf{G}_N = \{\overline{1, N}\}$, it is possible to determine D_{N-1}. Assume all the layer-functions will be constructed on $\mathbf{n}$ nodes, where (here and below) $\mathbf{n} \triangleq |\mathbf{G}_{N-1}| = |\mathbf{I}(\overline{1, N})| \in \mathbb{N}$. To facilitate these constructions, each layer $D_0, D_1, \dots, D_N$ will be interpreted as a union of $\mathbf{n}$ subsets; each such subset will be distributed to a specific node, which would construct its fragment of the layer. This construction will be conducted through auxiliary discrete dynamic systems (DDS), the trajectories of which will be determined through systems of inclusions. Thus, assume $K \in \mathbf{G}_{N-1}$, and let $\mathbb{T}[K]$ be the set of all the trajectories $(K_t)_{t \in \overline{0, N-2}} : \overline{0, N-2} \to \mathbf{G}$ such that $(K_0 \triangleq K)$ & $(\forall \tau \in \overline{1, N-2} \;\; \exists s \in \mathbf{I}(K_{\tau-1}) : K_\tau = K_{\tau-1} \setminus \{s\})$. Trajectories of the system are not uniquely determined for each given K; thus, for each "time" $t \in \overline{0, N-2}$, we have a reachability set (RS) $\widetilde{\mathbb{T}}[K; t]$, defined to be the set of all the lists K_t such that $(K_\tau)_{\tau \in \overline{0, N-2}}$ are the trajectories from $\mathbb{T}[K]$. Now, the families of feasible lists of fixed cardinality can be determined through these RSs, specifically,

$$\mathbf{G}_{N-(t+1)} = \bigcup_{K \in \mathbf{G}_{N-1}} \widetilde{\mathbb{T}}[K; t] \quad \forall t \in \overline{0, N-2}. \tag{4.1}$$

Let us also note that the mentioned RSs could [Chentsov, 2012.1, Proposition 16] be determined in a recurrent fashion for a fixed condition: if $K \in \mathbf{G}_{N-1}$ and $t \in \overline{0, N-3}$, then

$$\widetilde{\mathbb{T}}[K; t+1] = \{P \setminus \{h\} : P \in \widetilde{\mathbb{T}}[K; t], \; h \in \mathbf{I}(P)\}. \tag{4.2}$$

Through RSs, we construct the "individual" (to each computation node—its own) state space layers: for $K \in \mathbf{G}_{N-1}$ and $s \in \overline{1, N-1}$, set

$$\mathcal{D}_s[K] \triangleq \bigcup_{P \in \widetilde{\mathbb{T}}[K; N-(s+1)]} \mathbb{D}_s[P] \in \mathcal{P}'(D_s). \tag{4.3}$$

These layers (4.3) possess the property,

$$(\mathrm{pr}_2(z), Q \setminus \{s\}) \in \mathcal{D}_l[K] \;\; \forall (x, Q) \in \mathcal{D}_{l+1}[K] \;\; \forall s \in \mathbf{I}(Q) \;\; \forall z \in \mathbb{M}_s. \tag{4.4}$$

In view of (4.3), for $K \in \mathbf{G}_{N-1}$ and $s \in \overline{1, N-1}$, assume

$$\mathcal{W}_s[K] \triangleq (v_s(x, P))_{(x,P) \in \mathcal{D}_s[K]} = (v(x, P))_{(x,P) \in \mathcal{D}_s[K]} \in \mathcal{R}_+[\mathcal{D}_s[K]]. \tag{4.5}$$

Suppose that $\mathbb{M}_0[K] \overset{\triangle}{=} \{h \in \overline{1,N} \setminus \mathbf{K}_1 | \{h\} \in \widetilde{\mathbb{T}}[K; N-2]\}$. Next, for $K \in \mathbf{G}_{N-1}$ and $h \in \mathbb{M}_0[K]$, we have $\{h\} \in \widetilde{\mathbb{T}}[K; N-2]$, where $h \in \overline{1,N} \setminus \mathbf{K}_1$; thus, $\{h\} \in \mathbf{G}_1$ and $J_1(\{h\}) = \{t \in \overline{1,N} \setminus \{h\} | \{t; h\} \in \mathbf{G}_2\}$, then

$$\mathcal{M}_1[\{h\}] = \bigcup_{j \in J_1(\{h\})} \mathbf{M}_j. \tag{4.6}$$

For $K \in \mathbf{G}_{N-1},\ h \in \mathbb{M}_0[K]$, and $x \in \mathcal{M}_1[\{h\}]$, we have the following property: $(x, \{h\}) \in \mathcal{D}_1[K]$ and

$$\mathcal{W}_1[K](x, \{h\}) = \min_{z \in \mathbb{M}_h} [\mathbf{c}(x, \mathrm{pr}_1(z), \{h\}) + c_h(z, \{h\}) + f(\mathrm{pr}_2(z))]. \tag{4.7}$$

So, through (4.7), we can fully determine the function $\mathcal{W}_1[K]$, while, to use (4.7) we have to, knowing $K \in \mathbf{G}_{N-1}$ and $h \in \mathbb{M}_0[K]$, construct $J_1(\{h\})$ and then determine $\mathcal{M}_1[\{h\}]$ by means of (4.6), after which we can finally use formula (4.7) for $x \in \mathcal{M}_1[\{h\}]$. Thus we determine all the functions $\mathcal{W}_1[K],\ K \in \mathbf{G}_{N-1}$. Further constructions will concern one single computation node.

A transformation of a function $\mathcal{W}_l[K]$ into $\mathcal{W}_{l+1}[K]$, where $K \in \mathbf{G}_{N-1}$ and $l \in \overline{1, N-1}$, is determined by a relation similar to (3.1), which makes use of (4.4), (indeed, by (4.4) and (4.5), for $(x, Q) \in \mathcal{D}_{l+1}[K]$, $s \in \mathbf{I}(Q)$, and $z \in \mathbb{M}_s$, the value $\mathcal{W}_l[K](\mathrm{pr}_2(z), Q \setminus \{s\}) \in \mathbb{R}_+$ is defined and can be used to compute $\mathcal{W}_{l+1}[K](x, Q)$). Thus (see (3.1) and (4.5)), for $K \in \mathbf{G}_{N-1}$, $l \in \overline{1, N-2}$, and $(x, Q) \in \mathcal{D}_{l+1}[K]$, we have [Chentsov, 2013, (10.17)]

$$\mathcal{W}_{l+1}[K](x, Q) = \min_{s \in \mathbf{I}(Q)} \min_{z \in \mathbb{M}_j} [\mathbf{c}(x, \mathrm{pr}_1(z), Q) + c_j(z, Q) + \mathcal{W}_l[K](\mathrm{pr}_2(z),\ Q \setminus \{s\})]. \tag{4.8}$$

Fix $\mathbb{K} \in \mathbf{G}_{N-1}$. For this set, we have the corresponding layers $\mathcal{D}_s[\mathbb{K}]$ (4.3), $s \in \overline{1, N-1}$, which form a nonempty set in the state space. For $s \in \overline{1, N-1}$, the function $\mathcal{W}_s[K] \in \mathcal{R}_+[\mathcal{D}_s[\mathbb{K}]]$ is defined. In particular, the function $\mathcal{W}_1[\mathbb{K}] \in \mathcal{R}_+[\mathcal{D}_1[\mathbb{K}]]$ is defined; to determine its values, (4.7) should be used in view of the representation of $\mathcal{D}_1[\mathbb{K}]$. Further construction of functions $\mathcal{W}_1[\mathbb{K}], \ldots, \mathcal{W}_{N-1}[\mathbb{K}]$ is conducted based on a recurrence procedure based on (4.8). Specifically, for $l \in \overline{1, N-2}$, the transformation of $\mathcal{W}_l[\mathbb{K}]$ into $\mathcal{W}_{l+1}[\mathbb{K}]$ is determined by (4.8) under $K = \mathbb{K}$. Since the choice of l was arbitrary, we obtain the recurrence procedure $\mathcal{W}_1[\mathbb{K}] \to \mathcal{W}_2[\mathbb{K}] \to \ \ldots \ \to \mathcal{W}_{N-1}[\mathbb{K}]$. All these computations are conducted by a single computation node independently of other nodes.

5 Construction of Layers of Bellman Function

Note the positions of [Chentsov, 2012.1, Section 7]. In particular, from [Chentsov, 2012.1, Proposition 17]: we have the equalities

$$D_s = \bigcup_{K \in \mathbf{G}_{N-1}} \mathcal{D}_s[K] \quad \forall s \in \overline{1, N-1}. \tag{5.1}$$

Equalities (5.1) provide for a “union” of the processes conducted by the separate computation nodes. Our intention in this is to construct the layers $v_1, \ldots, v_{N-1}$. For every $s \in \overline{1, N-1}$, we have the functions $\mathcal{W}_s[K],\ K \in \mathbf{G}_{N-1}$, the domains of which (that is, the sets $\mathcal{D}_s[K],\ K \in \mathbf{G}_{N-1}$) form, in view of (5.1), a cover of D_s. The above-mentioned functions agree in view of (4.5): for $K_1 \in \mathbf{G}_{N-1},\ K_2 \in \mathbf{G}_{N-1}, (x, P) \in \mathcal{D}_s[K_1] \cap \mathcal{D}_s[K_2]$,

$$\mathcal{W}_s[K_1](x, P) = v_s(x, P) = \mathcal{W}_s[K_2](x, P). \tag{5.2}$$

In view of (5.1) and (5.2), we obtain the following simple rule of construction of the function v_s for $s \in \overline{1, N-1}$.

Having “particular” functions, to determine $v_s(x_0, K_0)$, where $(x_0, K_0) \in D_s$, first, in view of (5.1), obtain the set $\mathbb{K}_0 \in \mathbf{G}_{N-1}$ such that $(x_0, K_0) \in \mathcal{D}_s[\mathbb{K}_0]$. Then, in view of (4.5), $v_s(x_0, K_0) = \mathcal{W}_s[\mathbb{K}_0](x_0, K_0)$. Property (5.2) means that a specific choice of $\mathbb{K}_0 \in \mathbf{G}_{N-1}$ with the property $(x_0, K_0) \in \mathcal{D}_s[\mathbb{K}_0]$ can be arbitrary. So, all values of the function v_s can be determined and, therefore, the function itself. Thus, being in possession of functions (4.5), we obtain all the layers $v_0, v_1, \ldots, v_N$ of the Bellman function. Based on these layers, through procedures that solve the local problems, we determine the optimal feasible solution. To construct such a solution, the computer should retain all the layers $v_1, \ldots, v_N$ of the Bellman function in its memory.

Algorithm for determining the global extremum. Consider a procedure for determining V (3.2), during the implementation of which, in the memory of each computational node, there is only retained one “particular” function of the form (4.5), i.e., an “individual” Bellman function layer.

Without loss of generality, consider the computational node that processes the set $\mathbb{K} \in \mathbf{G}_{N-1}$. The ultimate goal of the procedure running on this node is to construct the function $\mathcal{W}_{N-1}[\mathbb{K}] \in \mathcal{R}_+[\mathcal{D}_{N-1}[\mathbb{K}]]$.

Principal steps of the iteration procedure.

1)Determine $\mathcal{W}_1[\mathbb{K}] \in \mathcal{R}_+[\mathcal{D}_1[\mathbb{K}]]$ from (4.7).

2)For $s \in \overline{1, N-2}$, assume the "particular" function $\mathcal{W}_s[\mathbb{K}] \in \mathcal{R}_+[\mathcal{D}_s[\mathbb{K}]]$ is already constructed. Then, through (4.8), compute the values of the function $\mathcal{W}_{s+1}[\mathbb{K}]$, which only use the values of the function $\mathcal{W}_s[\mathbb{K}]$ (in (4.8), set $l = s$). This yields the "particular" function $\mathcal{W}_{s+1}[\mathbb{K}] \in \mathcal{R}_+[\mathcal{D}_{s+1}[\mathbb{K}]]$.

Next, the memory holding the values $\mathcal{W}_s[\mathbb{K}]$ is released, and then filled with the values of $\mathcal{W}_{s+1}[\mathbb{K}]$: the "particular" functions are overwritten.

3)After consecutively conducting step 2), we obtain $\mathcal{W}_{N-1}[\mathbb{K}]$.

After steps 1)–3) are completed at each computation node, there are obtained all the functions

$$\mathcal{W}_{N-1}[K],\ K \in \mathbf{G}_{N-1}. \tag{5.3}$$

Now, construct v_{N-1} through the equality $D_{N-1} = \bigcup_{K \in \mathbf{G}_{N-1}} \mathcal{D}_{N-1}[K]$. To this end, for every state $(\hat{x}, \hat{P}) \in D_{N-1}$, determine, the set $\hat{K} \in \mathbf{G}_{N-1}$ such that $(\hat{x}, \hat{P}) \in \mathcal{D}_{N-1}[\hat{K}]$. Then (see (5.3)), we have the value $\mathcal{W}_{N-1}[\hat{K}](\hat{x}, \hat{P}) \in \mathbb{R}_+$ for which, in view of (4.5), there holds the equality $v_{N-1}(\hat{x}, \hat{P}) = \mathcal{W}_{N-1}[\hat{K}](\hat{x}, \hat{P})$. Thus, we obtain all the values of the function v_{N-1}.

Through the function v_{N-1} (the layer of the Bellman function), we determine the global extremum V. In connection with this procedure, note [Chentsov, 2016.1], where it was proved that one can determine the value of V without constructing the optimal feasible solution for problems of greater dimension.

6 Model and Computational Experiment.

Let us consider a specific version of the general problem; assume $X = \mathbb{R} \times \mathbb{R}$ (the problem on a plane). We study a formulation where each megalopolis is a system of entries and exits for a certain zone of heightened intensity of certain harmful factors (in particular, radiation). The necessary activities are conducted by a point agent in the zone. The aim of activities inside a megalopolis is to go from the entry point to the radiation source, dismantle it, and then leave the megalopolis through an exit point. The radiation left after dismantling a source is assumed to be negligible. Thus, a megalopolis is, essentially, a "near zone" of the radiation source. Other (non-dismantled) radiation sources are also assumed to have a nonnegligible effect both during movement between megalopolises and during the near zone activities (interior jobs). The effect of radation is assumed to be cumulative, whence the additive cost aggregation assumed in this paper. Another feature of the problem statement is the fact that when moving to a source to dismantle it, as the agent enters the near zone, the agent is assumed to be affected by it; however, after dismantlement, as the agent moves to exit the zone, the agent is not affected by the (dismantled) source anymore. Consequently, the degree of radiation exposure of the agent during the first stage depends on the length of the stage and, therefore, on the distance between the entry point and the radiation source. During exterior movements, the effect of each single source is not as pronounced, however, the cumulative effect of non-dismantled sources can not be ignored.

Passing by a radiation source s and dismantling it. Let us connect the system of megalopolises with a system of point "radiating" objects $(\mathbf{z}_i)_{i \in \overline{1,N}} : \overline{1, N} \to X$ with the following property: $\mathbf{z}_i \notin M_i\ \forall i \in \overline{1, N}$. Dismantlement of the objects $\mathbf{z}_1, \ldots, \mathbf{z}_N$ is the aim of visiting the megalopolises: specifically, during a visit to $M_{\alpha(i)}$, the agent has to enter at some entry point $\mathrm{pr}_1(z_i)$, move to the point $\mathbf{z}_{\alpha(i)}$, dismantle the radiation source number $\alpha(i)$, and then reach an exit point $\mathrm{pr}_2(z_i)$. This implements scheme (2.1).

Let us sketch a description of the effect of a single source during the movement from the given point of the plane to another point (as mentioned before, the exposure suffered during these movements is summed up) in the "regular" case, when the trajectory does not pass through any "active" radiation sources. Thus, as the agent moves from a point i to a point j, the losses (in our model), or rather, the exposure suffered during the movement due to the effect of the (active) source s is as follows:

$$c_{i,j}[\{s\}] = \int\limits_0^T \frac{\gamma_s}{\rho^2_{s,t}(t)} dt = \ = 2\rho_{i,j} \frac{\gamma_s}{v} \int\limits_0^{\rho_{i,j}} \frac{d\rho}{\left(2\rho_{i,j}\rho + \rho^2_{j,s} - \rho^2_{i,s} - \rho^2_{i,j}\right)^2 + \left(4\rho^2_{i,j}\rho^2_{i,s} - (\rho^2_{j,s} - \rho^2_{i,s} - \rho^2_{i,j})^2\right)}, \tag{6.1}$$

where ρ denotes the Euclidean distance (where necessary, the mentioned distances are subscripted); γ_s is the intensity of the source s, and v is the movement speed. To calculate the integral use one of the following table formulas:

$$\int \frac{dR}{A^2 + R^2} = \frac{1}{A}\mathrm{arctg}\frac{R}{A} + C, \int \frac{dR}{A^2 - R^2} = \frac{1}{2A}\ln\left|\frac{A+R}{A-R}\right| + C,$$

depending on whether the sign is $4\rho^2_{i,j}\rho^2_{i,s} - (\rho^2_{j,s} - \rho^2_{i,s} - \rho^2_{i,j})^2$. If the source s lies on the trajectory of the motion from point i to point j, the cost $c_{i,j}[\{s\}]$ is assumed to be a very large number (roughly speaking, $c_{i,j}[\{s\}] = \infty$; in the actual computation, the number that is several times greater than the most "costly" motion is sufficient).

Approaching radiation source s for dismantlement (determining the cost of interior jobs). In this case, assume that the point j coincides with s and consider the exposure suffered by the agent during the approach to the source that is to be dismantled. Thus, we still follow the model where the exposure is inversely proportional to the square of the distance to the source; however, to avoid the ill-posedness as the agent reaches the source, which could lead to a division by 0, the denominator of the integrand in (6.1) is increased by 1. To account for a more intense radiation in the near zone, we add the factor 3 (which describes the case of radiation intensity not being weakened by obstacles, the latter being possible during exterior movements). So, in this case, the losses (the exposure) are calculated by the following formula: $c_{i,j}[\{s\}] = c_{i,s}[\{s\}] = 3\left(\frac{\gamma_s}{v}\right)\int\limits_0^{\rho_{i,j}} \frac{d\rho}{\rho^2+1} = 3\left(\frac{\gamma_s}{v}\right)\mathrm{arctg}(\rho_{i,j})$.

Computational experiment. Let us consider the model instances of the routing problem for dismantling radiation sources on a plane. The megalopolises, which imitate the possible entry and exit points of the chambers housing the radiation sources, are obtained by discretizing circles: at each circle, there are 12 points, equally spaced starting with the point with the angle coordinate of 0. Each megalopolis is assigned a point object, which imitates the radiation source of the chamber. Let the starting point of the dismantlement process coincide with the point of origin, i.e., $x^0 = (0, 0)$; after dismantling all radiation sources, the agent must return to the depot (the point of origin). Recall that the function ρ is essentially the Euclidean distance. Assume the speed of the agent is 4 times greater outside the chambers than it is within—this models the intrinsic difficulty of moving inside the megalopolises, which is due to the presence of hardware, various structures, or mechanisms, which hamper the rapid movement inside.

Here are some results concerning the calculations on supercomputers URAN. For the model with 30 megalopolises and 30 address pairs (51 pairs in transitive closure [Schmidt, 1993]) that determine the precedence constraints, the following results were obtained: total exposure (total losses): 222.9, total computation time: 17m. 46s. For the model with 31 megalopolis and 34 address pairs (63 pairs in transitive closure[Schmidt, 1993]), the following results were obtained: total exposure (total losses): 226.5, total computation time: 15m. 56s. Let us now consider the results of the accounts of the same tasks on a PC. For the model with 30 megalopolises and 30 address pairs, the following results were obtained: total exposure (total losses): 222.9, total computation time: 7h. 26m. 7s. For the model with 31 megalopolis and 34 address pairs, the following results were obtained: total exposure (total losses): 226.5, total computation time: 6h. 29m. 55s. A decrease in computation time for the problem with the greater number of megalopolises is due to the more "strict" precedence constraints, which significantly decrease both the volume of computations involved in obtaining the values of the Bellman function and the memory usage. For more information on the influence of precedence constraints on computation time and memory usage of dynamic programming solutions, refer to [Steiner, 1990].

Acknowledgements

This work was supported by Russian Science Foundation (project no. 14-11-00109).

References

[Alkaya, 2010] Alkaya, A.F., Duman, E. (2010). A new generalization of the traveling salesman problem. *Appl. Comput. Math.* 9(2), 162–175.

[Bellman, 1962] Bellman, R. (1962). Dynamic programming treatment of the travelling salesman problem. *J. Assoc. Comput. Mach.* 9, 61–63.

[Chentsov, 2008] Chentsov, A.G. (2008). *Extreme Problems of Routing and Tasks Distribution: Regular and Chaotic Dynamics.* Izhevsk Institute of Computer Research, 240 p. (in Russian).

[Chentsov, 2012] Chentsov, A.G. (2012). One parallel procedure for the construction of the Bellman function in the generalized problem of the courier with the inner workings. *Autom. Remote Control* 3, 134–149.

[Chentsov, 2012.1] Chentsov, A.G. (2012). One parallel procedure for the construction of the Bellman function in the generalized problem of the courier with the inner workings. *Vestnik YuUrGU. Gos. Univ. Ser. Mat. Model Program.* 12, 53–76.

[Chentsov, 2013] Chentsov, A.G. (2013). To question of routing of works complexes. *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki,* (1), 59–82.

[Chentsov, 2015] Chentsov, A.G., Chentsov, A.A. (2015). Route problem with constraints depending on a list of tasks. *Doklady Mathematics,* 92(3), 685–688.

[Chentsov, 2016] Chentsov, A.G., Chentsov, P.A. (2016). Routing under constraints: problem of visit to megalopolises. *Autom. Remote Control,* 77(11), 1957–1974.

[Chentsov, 2016.1] Chentsov, A.G., Chentsov, A.A. (2016). On the problem of obtaining the value of routing problem with constraints. *J. Autom. Inf. Sci.* 6, 41–54.

[Chentsov, 2016.2] Chentsov, A.G., Grigoryev A.M. (2016). Dynamic programming method in the route problem: the scheme of independent calculations. *Mekhatronika, Avtomatizatsiya, Upravlenie,* 17(12), 834–846.

[Cook, 2012] Cook, W.J. (2012). *In Pursuit of the Traveling Salesman. Mathematics at the limits of computation.* Princeton University Press, New Jersey, p. 248.

[Cormen, 1990] Cormen, T.H., Leizerson, C.E., and Rivest, R.L. (1990). Introduction to Algorithms. Cambridge: MIT Press.

[Dieudonne, 1960] Dieudonne, J. (1960). *Foundations of Modern Analysis.* New York: Academic.

[Garey, 1979] Garey, M.R., Johnson, D.S. (1979).*Computers and Intractability: A Guide to the Theory of NP-Completeness.* N.Y., W.H. Freeman & , p. 416

[Gutin, 2002] Gutin, G., Punnen, A.P. (2002). *The Traveling Salesman Problem and Its Variations.* Springer, New York.

[Held, 1962] Held, M., Karp, R.M. (1962). A dynamic programming approach to sequencing problems. *J. Soc. Ind. Appl. Math.* 10(1), 196–210.

[Kinable, 2016] Kinable J., Cire A., van Hoeve W. J. (2016). Hybrid optimization methods for time-dependent sequencing problems. *European Journal of Operational Research,* Available online 24 November 2016, ISSN 0377-2217

[Korobkin, 2012] Korobkin, V.V., Sesekin, A.N., Tashlykov, O.L., Chentsov, A.G. (2012). *Routing Methods and Their Applications to the Enhancement of Safety and Efficiency of Nuclear Plant Operation.* Novye Tekhnologii, Moscow.

[Lawler, 1979] Lawler, E.L. (1979). Efficient implementation of dynamic programming algorithms for sequencing problems. *CWI Technical report. Stichting Mathematisch Centrum. Mathematische Besliskunde-BW* 106/79, 1–16.

[Leon, 1996] Leon, V.J., Peters, B.A. (1996). Replanning and analysis of partial setup strategies in printed circuit board assembly systems. *Int J Flex Manuf Syst* 8, 389–411.

[Litl, 1965] Litl, Dzh., Murti, K., Suini, D., Kjerel, K. (1965). Algorithms for Solving the Traveling Salesman Problem. *Economics and Mathematical Methods*, vol.1, 94–107 (in Russian).

[Melamed, 1989] Melamed, I.I., Sergeev, S.I., Sigal, I. (1989). The traveling salesman problem. I. Issues in Theory; II Exact Methods; III Approximate Algorithms. *Automation and Remote Control.* 50(9), 1147–1173; 50(10), 1303–1324; 50(11), 1459–1479.

[Schmidt, 1993] Schmidt, G., Strohlein, T. (1993). *Relations and Graphs: Discrete Mathematics for Computer Scientists.* EATCS Monographs on Theoretical Computer Science, Springer-Verlag.

[Steiner, 1990] Steiner, G. (1990). On the complexity of dynamic programming for sequencing problems with precedence constraints. *Annals of Operations Research,* 26(1), 103–123.

[Tashlykov, 2011] Tashlykov, O.L. (2011). *Personnel Dose Costs in the Nuclear Industry. Analysis. Ways to Decrease. Optimization.* LAP LAMBERT Academic Publishing GmbH & Co. RG., Saarbruke.