Refinement of the Optima Localization for the Two-Machine Routing Open Shop

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Abstract

We consider the routing open shop problem being a natural combination of two classic discrete optimization problems: metric TSP and Open Shop scheduling problem. Jobs are located at the nodes of a transportation network and have to be processed by mobile machines initially located at the depot. Machines have to complete all the operations and return to the depot minimizing the makespan. The problem is known to be NP-hard even in the simplest case with two machines and only two nodes (including the depot). For this case it is known that optimal makespan doesn't exceed $\frac{6}{5}$ times standard lower bound.

Our goal is to refine that result specifying the maximal ratio of the optimal makespan to the standard lower bound (so called *abnormality*) depending on the jobs' load distribution between two nodes. We propose a new polynomially solvable subcase of the problem under consideration and describe an exact form of the maximal abnormality as a function of the fraction of the total load located outside the depot.

1 Introduction

In the routing open shop problem a set of mobile machines $\mathcal{M} = \{M_1, \ldots, M_m\}$ have to process operations of jobs $\mathcal{J} = \{J_1, \ldots, J_n\}$ in arbitrary order. Jobs are located in the nodes of transportation network described by an edge-weighted graph $G = \langle V, E \rangle$. Machines are initially located at a specific node $v_0 \in V$ referred to as *the depot* and have to return back after completing all the operations. The goal is to minimize the makespan $R_{\max}$ which is defined as a last time moment of returning of a machine to the depot. Following the standard three-field notation for scheduling problems (see for instance [Lawler et al., 1993]) we denote the routing open shop problem as $ROm||R_{\max}$ or $ROm|G = X|R_{\max}$ if we want to specify the graph structure. This model was introduced in [Averbakh et al., 2005]. It is obviously NP-hard as it contains a well-known metric TSP as a special case. Moreover it was proved in [Averbakh et al., 2006] that even a simplest version of the problem with two machines and two nodes (i.e. $RO2|G = K_2|R_{\max}$ there K_p is the complete graph with p vertices) is NP-hard. An FPTAS for $RO2|G = K_2|R_{\max}$ was described in [Kononov, 2012]. A series of approximation algorithms and detailed review for $RO||R_{\max}$ can be found in [Chernykh et al., 2013] and references wherein.

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We use the following notation. The processing time of operation O_{ji} of job J_j by machine M_i is denoted by p_{ji}, τ_{lk} is the distance between nodes v_l and v_k, $\mathcal{J}^k$ is the set of indices of jobs located at v_k. The completion time of operation O_{ji} in schedule S is denoted by $c_{ji}(S)$. The *return time* of machine M_i in schedule S is defined as $R_i(S) \doteq \max_k \left(\max_{j \in \mathcal{J}^k} c_{ji}(S) + \tau_{0k} \right)$ and the makespan $R_{\max}(S) \doteq \max R_i(S)$ is to be minimized over all feasible schedules. Notation $R^*_{\max}(I)$ is used for the optimal makespan of problem instance I. In case of $m = 2$ we also use simplified notation $\mathcal{M} = \{A, B\}$ and a_j (b_j) instead of both p_{j1} (p_{j2}) and O_{j1} (O_{j2}). We also describe jobs in this case by $J_j = (a_j, b_j)$.

Additionally we need the following notation.

- $\ell_i = \sum_{j=1}^{n} p_{ji}$ — load of machine M_i, $\ell_{\max} = \max \ell_i$ — maximal machine load;
- $d_j = \sum_{i=1}^{m} p_{ji}$ — length of job J_j, $d^k_{\max} = \max_{j \in \mathcal{J}_k} d_j$ — maximal length of job from v_k;
- T^* — length of the shortest route over graph G (TSP optimum);
- $\mathcal{I}^X_m$ — class of all *non-trivial* instances (i.e. instances with positive standard lower bound) for problem $ROm|G = X|R_{\max}$.

The following *standard lower bound* was introduced in [Averbakh et al., 2006]:

$$\bar{R} = \max \left\{ \ell_{\max} + T^*, \max_k \left(d^k_{\max} + 2\tau_{0k} \right) \right\} \tag{1}$$

Definition 1 A feasible schedule S for problem instance I is referred to as *normal* if $R_{\max}(S) = \bar{R}(I)$. Instance I is *normal* if it admits construction of a normal schedule.

Definition 2 The *abnormality* of instance I is the ratio $\alpha(I) \doteq \dfrac{R^*_{\max}(I)}{\bar{R}(I)}$. For some class $\mathcal{K}$ of instances *abnormality* of class $\mathcal{K}$ is $\alpha(\mathcal{K}) \doteq \sup_{I \in \mathcal{K}} \alpha(I)$.

The problem of finding of the abnormality for some class $\mathcal{K}$ is very similar to the following
Optima Localization Problem. For some class of instances $\mathcal{K}$ find minimal value ρ such that $\forall I \in \mathcal{K}$

$$R^*_{\max}(I) \in [\bar{R}(I), \rho \bar{R}(I)].$$

Obviously such value $\rho = \alpha(\mathcal{K})$. The problem is to find that value and to describe an instance from $\mathcal{K}$ with maximal abnormality (if any).

The optima localization problem was studied for $RO2|G = K_2|R_{\max}$ in [Averbakh et al., 2005]. It was shown that $\alpha\left(\mathcal{I}^{K_2}_2\right) = \frac{6}{5}$. The instance $\tilde{I}$ of $RO2|G = K_2|R_{\max}$ with maximal abnormality contains a single job $J_1 = (4, 0)$ at the depot v_0 and two equivalent jobs $J_2 = J_3 = (2, 4)$ at the *distant* node v_1, distance $\tau \doteq \tau_{01} = 1$. This result was recently generalized on the case of triangular transportation network: it was shown that $\alpha\left(\mathcal{I}^{K_3}_2\right) = \frac{6}{5}$ [Chernykh & Lgotina, 2016].

The goal of this paper is to refine the optima localization result for $RO2|G = K_2|R_{\max}$ depending on the load distribution between nodes v_0 and v_1. Let $\Delta^k \doteq \sum_{j \in \mathcal{J}^k} d_j$ stands for the *load* of node v_k, $\Delta \doteq \sum \Delta^k = \sum \ell_i = \sum d_j$ is the *total load* of instance I (in our case $\Delta = \Delta^0 + \Delta^1$). The *load distribution parameter* for instance I is defined as

$$\delta(I) \doteq \frac{\Delta(I) - \Delta^0(I)}{\Delta(I)} \in [0, 1].$$

How does the maximal abnormality depend on δ? Consider a function

$$F(x) \doteq \alpha\left(\{ I \in \mathcal{I}^{K_2}_2 | \delta(I) = x \} \right), \ x \in [0, 1].$$

Our goal is to describe the behavior of $F(x)$.

The remainder of the paper is organized as follows. Section 2 contains important preliminary results including new sufficient conditions of normality of instance I. The main result on the behavior of the function $F(x)$ is contained in Section 3 (Theorem 10) followed by some conclusions in Section 4.

2 Preliminary Results

2.1 Job Aggregation and Overloaded Nodes

The idea of instance simplifying by means of *job aggregation* procedure has been proved to be a useful tool for solving the Optima Localization Problem as well as for describing efficient approximation algorithms. The detailed description of the procedure in application to $RO2||R_{\max}$ problem can be found in [Chernykh & Lgotina, 2016]. The idea is the following. Let $K \subseteq \mathcal{J}^k$ correspond to some subset of jobs from node v_k. Then we may substitute the jobs with indices from K with a single job J_K with processing times $p_{Ki} \doteq \sum\limits_{j\in K} p_{ji}$. Obviously any feasible schedule for a simplified instance can be treated as a feasible schedule for the initial instance with the same makespan. The goal is to perform such a transformation preserving the standard lower bound $\bar{R}$. In this case the abnormality of an instance would not decrease. We use the following definition from [Chernykh & Lgotina, 2016].

Definition 3 A node v_t of an instance I of problem $RO2||R_{\max}$ is referred to as *overloaded* if $\Delta^t > \bar{R}(I) - 2\tau_{0t}$. Otherwise the node v_t is *underloaded*.

It is easy to see from (1) that job aggregation of the whole set $\mathcal{J}^t$ would not increase the standard lower bound if the node v_t is underloaded.

Two following theorems were proved in [Chernykh & Lgotina, 2016].

Theorem 1 ([Chernykh & Lgotina, 2016]) *Any instance of $RO2||R_{\max}$ has at most one overloaded node.*

Theorem 2 ([Chernykh & Lgotina, 2016]) *Any instance I of the problem $RO2||R_{\max}$ can be transformed by using job aggregations into instance I' such that*

1. $\bar{R}(I') = \bar{R}(I)$,
2. *I' has at most 3 jobs in the overloaded node (if any) and single job in every other node.*

Note that a transformation described in Theorem 2 can be done in $O(n)$ time. Both proofs are based on the following inequality

$$\Delta^k \leqslant \Delta \leqslant \ell_1 + \ell_2 \leqslant 2(\bar{R} - T^*) \leqslant 2(\bar{R} - 2\tau_{0k}) \tag{2}$$

that holds for any node v_k.

We can consider a job aggregation procedure at node v_k preserving the standard lower bound as a variant of a Bin Packing problem. Indeed the jobs' lengths serve as objects sizes and value $\bar{R} - 2\tau_{0k}$ as a bin capacity. According to Theorem 2 for any overloaded node v_k optimal bin packing takes 2 or 3 bins. In the next subsection we will describe a special case of bin packing problem which has important applications for the optima localization problem of $RO2||R_{\max}$.

2.2 Irreducible Bin Packing

Consider the following decision problem.

Irreducible Bin Packing (IBP).

Input. Bin capacity B, a set $\mathcal{E} = \{e_1, \dots, e_N\}$ of integer object sizes with additional condition $B < \sum e_j \leqslant 2B$.

Question. Does there exist a feasible packing of $\mathcal{E}$ into exactly three bins of size B such that total contents of any pair of bins strictly exceeds B?

Definition 4 An input I is referred to as *irreducible* if the answer to the Irreducible Bin Packing question is positive. Instance I is *efficiently irreducible* if such a packing can be found in polynomial time.

Theorem 3 *The **IBP** problem is NP-complete.*

Proof. We will show a reduction from the well-known **PARTITION** problem.

PARTITION.

Input. A set of integers $\mathcal{T} = \{t_1, \dots, t_k\}$, $\sum t_j = 2T$.

Question. Does there exist a subset $\mathcal{T}' \subset \mathcal{T}$ such that $\sum\limits_{\mathcal{T}'} t_j = T$?

Let $\mathcal{T} = \{t_1, \dots, t_k\}$ be the input of the **PARTITION**. Transform it into the following input of the **IBP**: $B = 2T - 2$, $N = k + 1$, $e_j = t_j$, $1 \leqslant j \leqslant k$, $e_{k+1} = T - 1$. Lets prove that the former instance is irreducible iff the partition for initial instance exists.

Suppose the partition exists. Then we may pack object e_{k+1} into the first bin and separate the rest equally into second and third bins. This packing is obviously irreducible.

From the other hand, suppose that irreducible packing exists. If some bin contains only object e_{k+1} then that bin has $T - 1$ of free space and in order for packing to be irreducible both other bins have to contain at least T total amount of objects, hence the partition exists. In other case there can be only single object of size 1 in the same bin with e_{k+1} (otherwise two other bins would violate the irreducibility). Thus that bin contains T total amount and has $T - 2$ of free space, therefore each of other bins has to contain at least $T - 1$ amount. That means that one of them contains exactly T and other $T - 1$, therefore the partition exists. $\square$

Note that for every irreducible instance of the **IBP** total size of all objects strictly exceeds $\frac{3}{2}B$ although this condition is obviously not sufficient. We present the following sufficient condition of efficient irreducibility.

Theorem 4 *Let I be an instance of **IBP** and*

$$\sum_{j=1}^{N} e_j > \frac{3}{2}B + \max_j e_j.$$

Then I is efficiently irreducible.

Proof. Consider arbitrary enumeration of $\mathcal{E} = \{e_1, \dots, e_N\}$. Note that due to the condition $\sum\limits_{j=1}^{N} e_j \leqslant 2B$ we have $\max\limits_j e_j < \frac{1}{2}B$. We can pack those objects into three bins using the following algorithm.

1. Find the smallest index p such that $\sum\limits_{j=1}^{p} e_j > \frac{1}{2}B$. Pack objects $e_1, \dots, e_p$ into the first bin. Let $X \doteq \sum\limits_{j=1}^{p} e_j = \frac{1}{2}B + x \leqslant \frac{1}{2}B + \max\limits_j e_j < B$.
2. Find the smallest index $q > p$ such that $\sum\limits_{j=p+1}^{q} e_j > \frac{1}{2}B - x$. Pack objects $e_{p+1}, \dots, e_q$ into the second bin. Let $Y \doteq \sum\limits_{j=p+1}^{q} e_j = \frac{1}{2}B - x + y < B$.
3. Pack the rest objects into the third bin. Let $Z \doteq \sum\limits_{j=q+1}^{N} e_j$.

Lets prove that we obtained an irreducible packing. Note that

- $X + Y = B + y > B$, $y \leqslant \max\limits_j e_j$, therefore $Z = \sum\limits_{j=1}^{N} e_j - B - y > 0$;
- $Y + Z = \sum\limits_{j=1}^{N} e_j - \frac{1}{2}B - x \geqslant \sum\limits_{j=1}^{N} e_j - \frac{1}{2}B - \max\limits_j e_j > B$;
- $X + Z = \sum\limits_{j=1}^{N} e_j - \frac{1}{2}B - y + x \geqslant \sum\limits_{j=1}^{N} e_j - \frac{1}{2}B - \max\limits_j e_j > B$.

Therefore the packing is irreducible and obtained in linear time. $\square$

Note that the condition in Theorem 4 is given in its strongest form: the claim would be wrong with non-strict inequality.

2.3 Sufficient Conditions of Normality

The goal of this subsection is to describe several sufficient conditions of normality in order to discover a necessary condition of abnormality, as strict as possible. We will apply the former condition to our search of the most abnormal instances from various subclasses. We will focus on $RO2|G = K_2|R_{\max}$ problem from now on. We have two sets of jobs ($\mathcal{J}^0$ and $\mathcal{J}^1$) located at the depot and at the distant node respectively. Distance between nodes is denoted by τ. The standard lower bound (1) can be rewritten in the following form:

$$\bar{R} = \max\{\ell_{\max} + 2\tau, d^0_{\max}, d^1_{\max} + 2\tau\}.$$

Note that if $\ell_{\max} + 2\tau < \bar{R}$ then it is easy to construct a normal schedule for the instance given.

There are several sufficient conditions of normality described in [Chernykh & Lgotina, 2017] for $RO2|G = tree|R_{\max}$. The one that is of interest for us is the following.

Theorem 5 ([Chernykh & Lgotina, 2017]) *Let I be an instance of $RO2|G = star|R_{\max}$ with depot at the center and either the depot is overloaded or there are no overloaded nodes in I. Then I is normal and an optimal schedule can be constructed in $O(n)$.*

A useful corollary for our case would be the following: any instance of $RO2|G = K_2|R_{\max}$ with underloaded distant node is normal.

Now consider an instance with overloaded distant node.

Definition 5 Let I be an instance of $RO2||R_{\max}$, $v_k \in V$. Node v_k is referred to as *superoverloaded* if the following instance of **IBP** is irreducible:

$$B = \bar{R} - 2\tau_{0k},\ \mathcal{E} = \{d_j | j \in \mathcal{J}^k\}.$$

The meaning is the following: it is possible to perform the job aggregation in v_k preserving $\bar{R}$ into three jobs in such manner that any following job aggregation would with necessity lead to increase of the standard lower bound. It turned out that the existence of superoverloaded node in $RO2|G = K_2|R_{\max}$ implies the normality.

Theorem 6 *Let I be an instance of $RO2|G = K_2|R_{\max}$ with superoverloaded distant node. Then I is normal.*

PROOF. Lets apply the job aggregation procedure to the depot (we'll get single job due to Theorem 2) and to the distant node to obtain exactly three jobs there according to the irreducible packing of the underlying **IBP**. We'll obtain an instance I' with the same $\bar{R}$ (see Theorem 2). Let J_0 be the single job from the depot and J_1, J_2, J_3 — jobs from v_1. Note that $\forall p \neq q \in \{1, 2, 3\}$

$$d_p + d_q > \bar{R} - 2\tau. \tag{3}$$

Without lost of generality let a_1 be the smallest processing time among all of operations from v_1.

Construct a schedule S_1 in the following manner. Let machine A process jobs in order J_1, J_2, J_3, J_0, while machine B processes jobs in order J_0, J_3, J_1, J_2. Jobs J_1 and J_2 are processed first by machine A then B, while operations of other jobs are processed in the opposite order. As soon as each machine travels only once from the depot to the distant node and back, if S_1 has no idles then S_1 is normal. Let it has some idle intervals. Note that (2) and (3) imply $b_0 + \tau + b_3 + a_3 + \tau + a_0 < \bar{R}$, and operation b_3 completes not earlier than a_1. That means that the only idle interval in S_1 is possible between operations b_1 and b_2 and in this case

$$R_{\max}(S_1) = \tau + a_1 + a_2 + b_2 + \tau. \tag{4}$$

Construct another schedule S_2 in which machine B processes jobs in the same order as in S_1, and A processes in sequence J_2, J_1, J_3, J_0. Job J_1 now is processed first by B then by A, other jobs are processed in the same order as in S_1. Again suppose S_2 has some idle intervals. In this case

$$R_{\max}(S_2) = \tau + b_0 + b_3 + \max\{a_3, b_1\} + a_1 + a_0 + \tau. \tag{5}$$

Note that $a_1 \leqslant \min\{a_3, b_1\}$ implies

$$R_{\max}(S_1) + R_{\max}(S_2) \leqslant 4\tau + \ell_1 + \ell_2 \leqslant 2\bar{R},$$

therefore the best of those schedules is normal. □

Note that the conditions of Theorem 6 don't give us a polynomially solvable subcase because the aggregation of jobs from distant nodes into three irreducible jobs can be hard (see Theorem 3). Although combining them together with some additional conditions (e.g. from Theorem 4) we can obtain such a new polynomially solvable subclass of instances.

3 Abnormality as a Function of Total Load Distribution

The goal of this section is to describe function $F(x) = \alpha\left(\{I \in \mathcal{I}_2^{K_2} | \delta(I) = x\}\right)$, that is to find an instance with maximal abnormality from class $\{I \in \mathcal{I}_2^{K_2} | \delta(I) = x\}$ for each $x \in [0,1]$. According to Theorems 5 and 6 distant node of any abnormal instance of $RO2|G = K_2|R_{\max}$ has to be overloaded but not superoverloaded. From Theorem 2 we can safely apply job aggregation procedure to any instance as its abnormality wouldn't decrease. Moreover we can assume that $\ell_1 = \ell_2$ as soon as we can easily add dummy jobs to make this property hold preserving both $\bar{R}$ and δ. Summarizing, for each $x \in [0,1]$ there exists an instance I_x of $RO2|G = K_2|R_{\max}$ such that

1. $\delta(I_x) = x$;
2. I_x contains single job J_0 at the depot and two jobs J_1 and J_2 at the distant node;
3. $\ell_1 = \ell_2$;
4. $\bar{R}(I_x) = \ell_{\max} + 2\tau$;
5. $\alpha(I_x) = \alpha\left(\{I \in \mathcal{I}_2^{K_2} | \delta(I) = x\}\right)$.

In this section we consider only instances with properties 2–4 mentioned above.

Note that if $\delta(I) \leqslant 0.5$ then the distant node is underloaded and therefore (see Theorem 5) I is normal. Indeed $\delta(I) \leqslant 0.5$ implies $\Delta^1 \leqslant \Delta^0$. Assumption $\Delta^1 > \bar{R} - 2\tau$ would imply $\Delta > 2(\bar{R} - 2\tau)$ contradicting (2). Therefore $F(x) = 1$ for each $x \in [0, \frac{1}{2}]$.

The following lemma shows the lower bound on the function $F(x)$ for $x \in \left[\frac{1}{2}, 1\right]$.

Lemma 7 *1. For $x \in \left[\frac{1}{2}, \frac{3}{4}\right]$, $F(x) \geqslant \dfrac{4x}{2x+1}$;*

2. for $x \in \left[\frac{3}{4}, 1\right]$, $F(x) \geqslant \dfrac{3-2x}{2-x}$.

Proof. We will prove the claim by presenting series of instances with desired δ and abnormality.

1. Consider the series of instances $I(\tau)$ for $0 \leqslant \tau \leqslant 0.1$ with the following processing times:

$$J_0 = (0, 1-6\tau), J_1 = (1-6\tau, 2\tau), J_2 = (4\tau, 2\tau).$$

Note that for $I(\tau)$ $\bar{R} = 1$ and $\delta = \frac{1+2\tau}{2-4\tau} \in \left[\frac{1}{2}, \frac{3}{4}\right]$ therefore $\tau = \frac{2\delta-1}{4\delta+2}$. It is easy to show by consideration of all possible schedules that $R^*_{\max}(I(\tau)) = 1 + 2\tau$ and $\alpha(I(\tau)) = \dfrac{4\delta}{2\delta+1}$.

2. Consider the following series of instances $I(\varepsilon)$ with $0 \leqslant \varepsilon \leqslant 2$:

$$J_0 = (4-\varepsilon, \varepsilon), \tau = 1, J_1 = J_2 = (2+\varepsilon, 4).$$

For $I(\varepsilon)$ $\bar{R} = 10 + \varepsilon$, $\delta = \frac{6+\varepsilon}{8+\varepsilon} \in \left[\frac{3}{4}, \frac{4}{5}\right]$ therefore $\varepsilon = \frac{6-8\delta}{\delta-1}$. Optimal schedule for this instance has a makespan $R^*_{\max}(I(\varepsilon)) = 12 + \varepsilon$ therefore $\alpha(I(\varepsilon)) = \dfrac{3-2\delta}{2-\delta}$.

Now consider $\delta \in \left[\frac{4}{5}, 1\right)$ and the following instances $I(M)$, $M \geqslant 4$:

$$J_0 = (2, 2), \tau = 1, J_1 = J_2 = (M, M).$$

Here we have $\bar{R} = 2M + 4$, $\delta = \frac{M}{M+1} \in \left[\frac{4}{5}, 1\right)$ and $M = \frac{\delta}{1-\delta}$. It is easy to observe that $R^*_{\max}(I(M)) = 2M + 6$ therefore $\alpha(I(M)) = \dfrac{3-2\delta}{2-\delta}$.

The remaining case of $\delta = 1$ obviously contains only normal instances which concludes the proof of Lemma. □

We have just proved that

$$F(x) \geqslant \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right], \\ \frac{4x}{2x+1}, & x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \frac{3-2x}{2-x}, & x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

To prove that this lower bound is in fact tight we'll need two additional lemmas.

Lemma 8 ([Kononov, 2012]) *For any instance of $RO2|G = K_2|R_{\max}$ there exists a feasible schedule of makespan $\bar{R} + 2\tau$. Such a schedule can be constructed in linear time $O(n)$.*

Lemma 9 *Let S be a schedule for instance $I \in \mathcal{I}_2^{K_2}$ in which each machine travels to the distant node exactly once and total idle time of each machine doesn't exceed t. Then $\alpha(I) \leqslant 2 - \dfrac{\ell_{\max}}{\ell_{\max} + t}$.*

Proof. Each machine in schedule S spends exactly 2τ time on travel therefore $R_{\max}(S) \leqslant \ell_{\max} + 2\tau + t$. Therefore by Lemma 8 $R^*_{\max}(I) \leqslant \ell_{\max} + 2\tau + \min\{t, 2\tau\}$.

Case 1: $t \leqslant 2\tau$.

In this case

$$\alpha(I) \leqslant \frac{\ell_{\max} + 2\tau + t}{\ell_{\max} + 2\tau} = 1 + \frac{t}{\ell_{\max} + 2\tau} \leqslant 1 + \frac{t}{\ell_{\max} + t} = 2 - \frac{\ell_{\max}}{\ell_{\max} + t}.$$

Case 2: $t > 2\tau$.

In this case we have

$$\alpha(I) \leqslant \frac{\ell_{\max} + 4\tau}{\ell_{\max} + 2\tau} = 2 - \frac{\ell_{\max}}{\ell_{\max} + 2\tau} \leqslant 2 - \frac{\ell_{\max}}{\ell_{\max} + t}.$$

Lemma is proved. □

Now we are ready to prove the main result. The proof is constructive and describes a way to build a schedule with desired abnormality of at most $F(\delta(I))$ in linear time.

Theorem 10 *Let $F(x) = \alpha\left(\left\{I \in \mathcal{I}_2^{K_2} | \delta(I) = x\right\}\right)$. Then*

$$F(x) = \begin{cases} 1, & x \in [0, 0.5], \\ \dfrac{4x}{2x+1}, & x \in [0.5, 0.75], \\ \dfrac{3-2x}{2-x}, & x \in [0.75, 1]. \end{cases}$$

Proof. From Lemma 7 we just need to prove that

$$F(x) \leqslant \begin{cases} \dfrac{4x}{2x+1}, & x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \dfrac{3-2x}{2-x}, & x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Apply job aggregation procedure to I according to Theorem 2. If we'll obtain three irreducible jobs at distant node then it is superoverloaded and I is normal by Theorem 6. Note that in this case we can construct an optimal schedule in constant time as described in the proof of Theorem 6.

Now we have an instance I' with single job J_0 at the depot, two jobs J_1 and J_2 at the distant node, $\ell_1 = \ell_2 = \bar{R} - 2\tau$.

First we'll prove that $\alpha(I') \leqslant 2 - \dfrac{1}{2-\delta} = \dfrac{3-2\delta}{2-\delta}$. Without lost of generality let $d_1 \geqslant d_2$.

Construct a schedule S in the following manner. Let machine A process jobs in order J_1, J_2, J_0 and machine B in order J_0, J_2, J_1. Job J_1 is processed first by A then by B, other jobs are processed in opposite sequence. Note that each machine travels in S exactly once and machine B doesn't idle. Machine A in S idles for at most $t = b_0 + b_2 - a_1 = \ell_{\max} - d_1 = \dfrac{\Delta}{2} - d_1 \leqslant \dfrac{\Delta_0}{2} = (1-\delta)\ell_{\max}$ time units. By Lemma 9 we have $\alpha(I') \leqslant 2 - \dfrac{\ell_{\max}}{\ell_{\max} + t} = 2 - \dfrac{1}{2-\delta}$.

Now let's prove that $\alpha(I') \leqslant 2 - \dfrac{2}{2\delta+1} = \dfrac{4\delta}{2\delta+1}$. Without lost of generality let $b_1 \geqslant a_2$ (we do not demand $d_1 \geqslant d_2$ this time).

Construct a schedule S_1 according to the following partial order of the operations. Let A process jobs in order J_1, J_2, J_0 and B — in order J_0, J_1, J_2. All jobs except J_0 are processed first by A then by B. Note that each machine travels exactly once and A doesn't idle. The possible idle of B doesn't exceed $t = a_1 - b_0$ due to the assumption $b_1 \geqslant a_2$.

Construct another schedule S_2. Let A process jobs in order J_0, J_1, J_2 and B in order J_1, J_2, J_0. All jobs except J_0 are processed first by B then by A. Again, each machine travels exactly once. Machine B doesn't idle. The total idle of machine A doesn't exceed either $t' = b_1 - a_0$ or $t'' = b_1 + b_2 - a_0 - a_1$ (the former takes place if there is an idle between operations a_1 and a_2).

The total idle of at least one of those schedules S_1 and S_2 doesn't exceed the following

$$\frac{t + \max\{t', t''\}}{2} = \frac{\max\{d_1, b_1 + b_2\} - d_0}{2} \leqslant \frac{\ell_{\max} - \Delta^0}{2} = \frac{\Delta^1 - \ell_{\max}}{2} = \ell_{\max}\left(\delta - \frac{1}{2}\right).$$

Therefore by Lemma 9, $\alpha(I') \leqslant 2 - \frac{2}{2\delta+1}$ which concludes the proof of the theorem. □.

4 Conclusions

Theorem 10 gives a more specific result comparing with [Averbakh et al., 2005] (with significantly simpler proof as well). As soon as the proof is constructive we basically described a $F(\delta)$-approximation algorithm for $RO2|G = K_2|R_{\max}$. Note that $F(\delta)$ reaches the extremal value of $\frac{6}{5}$ only at a single point $\delta = \frac{3}{4}$. That means that although the worst-case performance guarantee of our algorithm is the same as in [Averbakh et al., 2005], the actual worst case is now extremely specific.

Theorem 6 can be easily generalized to the case of $G = K_3$ and even to general graph with additional conditions on the superoverloaded node's location. It would be of interest to extend this research on some more general case of two-machine routing open shop. Note that the optima localization problem is currently solved only for the cases of $G = K_2$ and $G = K_3$. Technique used in this paper can actually help to solve that problem for some more general cases with constant number of nodes, which in turn would lead to description of better approximation algorithms and probably new polynomially solvable subclasses for those cases.

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