Modified Duality Scheme for Solving Elastic Crack Problem

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Abstract

We consider the duality method for solving a model f elastic problem with a crack based on the use of duality methods. An article presents the theorems, allowing to use Uzawa method for search a saddle point of the modified Lagrangian functional. The results of numerical experiments are given.

1 Introduction

Many problems of mechanics and physics are formulated in the form of nonlinear boundary value problems for partial differential equations with additional conditions imposed on the desired solution. In the construction of more complex models, where the boundary conditions are often the system of equations and inequalities, in most cases the exact solution can not be found. Application of variational methods allows to pass from the nonlinear boundary value problems of mathematical physics to minimizing problem, consisting in research a conditional extremal problem [Hlavachek et al., 1988, Kikuchi & Oden, 1988]. So, it is very important to develop efficient methods for solving the original problem.

This paper is devoted to the analysis of model, a wide interest in which is manifested in the recent years. It is a model problem of elastic deformation of the body containing a crack. The formulation of this type of problem can be found in monograph by A.M. Khludnev [Khludnev, 2010]. Model proposed here differs from classical approach to the crack problem because it is characterized by the nonlinear boundary conditions on crack faces. Suitable boundary conditions are written as inequalities which provide mutual nonpenetration between crack faces. From the standpoint of mechanics such models are more preferable than the linear classical models.

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The methodological basis of study is duality scheme, based on replacing the problem of constrained optimization to finding a saddle point of the modified Lagrangian functional. This method was considered to the finite-dimensional problems of linear and convex programming in books by authors E.G. Gol'stein and N.V. Tret'yakov [Gol'stein & Tret'yakov, 1989], A.A. Kaplan and Ch. Grossman [Grossman & Kaplan, 1981]. In works of R.V. Namm, G. Woo and E.M. Vikhtenko [Namm & Vikhtenko, 2011, Vikhtenko et al., 2014] these studies were extended to the infinite-dimensional variational inequalities of mechanics. The regularity of the solution may be arbitrary bad in the neighborhood of the crack edges, and the dual problem may be unsolvable. Therefore the applying a similar scheme for solving the crack problem may be complicated. Despite this problem, it is possible to justify the duality scheme to solve the crack problem, as well as equality of duality for the original and dual problems.

2 Formulation of the Problem

Let $\Omega \subset R^2$ be a bounded domain with Lipschitz boundary Γ and $\gamma \subset \Omega$ be a cut (crack) inside of Ω. For simplicity we assume

$$\gamma = \{ (x_1, x_2) \in \Omega : \ a < x_1 < b, \ x_2 = const \} .$$

and suppose that both end points $(a, 0)$ and $(b, 0)$ do not belong to the boundary Γ. Denote $\Omega_\gamma = \Omega \backslash \bar{\gamma}$.

Consider the minimizing problem

$$\begin{cases} J(v) = \dfrac{1}{2} \displaystyle\int_{\Omega_\gamma} |\nabla v|^2 \, d\Omega - \int_{\Omega_\gamma} f v \, d\Omega - \min, \\ v \in K = \left\{ v \in H^1(\Omega_\gamma) : \ [v] \geq 0 \text{ on } \gamma, \ v = 0 \text{ on } \Gamma \right\}. \end{cases} \tag{1}$$

Here $[v] = v^+ - v^-$ is the jump of v across γ (v^+ is a function value v on upper crack face, v^- is a function value v on lower crack face, marks $\pm$ correspond to positive and negative directs of normal vector on cut γ); $f \in L_2(\Omega)$ is a given function.

Problem (1) has a unique solution $u \in K$, which is, simultaneously, a solution of variational inequality [Khludnev, 2010]

$$\int_{\Omega_\gamma} \nabla u \nabla (v - u) - \int_{\Omega_\gamma} f (v - u) \, d\Omega \geq 0 \quad \forall v \in K. \tag{2}$$

Assuming H^2-regularity of function u it can be shown that the problem (2) is equivalent to the boundary value problem [Khludnev, 2010]

$$\begin{cases} -\Delta u = f & \text{in } \Omega_\gamma, \\ u = 0 & \text{on } \Gamma, \\ [u] \geq 0, \ [u_{x_2}] = 0, \ u_{x_2} \leq 0, \ u_{x_2} [u] = 0 & \text{on } \gamma. \end{cases} \tag{3}$$

The question of solvability of problem (1) is investigated in detail in [Khludnev, 2010]. There is a theorem that establishes the existence of the solution of problem (1).

Theorem 1. Let K is a convex and closed set in Hilbert space $H^1(\Omega_\gamma)$, functional $J(v)$ is weakly lower semicontinuous and coercive. Then problem (1) is solvable.

Applying Friedrich's inequality

$$\int_{\Omega_\gamma} v^2 d\Omega \leq C \int_{\Omega_\gamma} |\nabla v|^2 \, d\Omega \quad \forall v \in H^1(\Omega_\gamma), \ v = 0 \text{ on } \Gamma,$$

it can be proved that the problem (1) has unique solution.

3 Duality Scheme for Solving Model Crack Problem

For arbitrary $m \in L_2(\gamma)$ construct the set

$$K_m = \left\{ v \in H^1(\Omega_\gamma) : \ v = 0 \text{ on } \Gamma, \ -[v] \leq m \text{ a.e. on } \gamma \right\}.$$

It is easy to show that K_m is a convex and closed set in $H^1(\Omega_\gamma)$. If the function $[m] \in L_2(\gamma) \backslash H^{1/2}(\gamma)$ the corresponding set K_m may be empty. On space $L_2(\gamma)$ define the sensitivity functional

$$\chi(m) = \begin{cases} \inf\limits_{v \in K_m} J(v), & \text{if } K_m \neq \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

Functional $\chi(m)$ is a proper convex and coercive functional on $L_2(\gamma)$, but it's effective domain $dom\chi = \{m \in L_2(\gamma): \chi(m) < +\infty\}$ does not coincide with $L_2(\gamma)$. Notice that $dom\chi$ is a convex but not closed set. In this case, $\overline{dom\chi} = L_2(\gamma)$.

Theorem 2. The functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\gamma)$.

The proof of this statement is presented in [Vikhtenko & Namm, 2016]. This property of the sensitivity functional is the basis of some theorems allows constructing and justifying the search algorithms of the saddle points of the modified Lagrangian functional.

We define the following functional on the space $H^1(\Omega_\gamma) \times L_2(\gamma) \times L_2(\gamma)$ [Namm & Vikhtenko, 2011, Vikhtenko et al., 2014]

$$K(v,l,m) = \begin{cases} J(v) + \dfrac{1}{2r}\displaystyle\int_\gamma \left((l + r\,m)^2 - l^2\right) d\Gamma, & \text{if } -[v] \le m \text{ a.e. on } \gamma, \\ +\infty, & \text{otherwise,} \end{cases}$$

and modified Lagrangian functional

$$M(v,l) = \inf_{m \in L_2(\gamma)} K(v,l,m) = J(v) + \frac{1}{2r}\int_\gamma \left(\left((l - r\,[v])^+\right)^2 - l^2\right) d\Gamma.$$

Here $r > 0$ is a constant, $(l - r\,[v])^+ = \max\{0, l - r\,[v],\}$.

Let us introduce the modified dual functional

$$\underline{M}(l) = \inf_{v \in H^1(\Omega_\gamma)} M(v,l) = \inf_{v \in H^1(\Omega_\gamma)} \inf_{m \in L_2(\gamma)} K(v,l,m) =$$

$$= \inf_{v \in H^1(\Omega_\gamma)} \left\{ J(v) + \frac{1}{2r}\int_\gamma \left(\left((l - r\,[v])^+\right)^2 - l^2\right) d\Gamma \right\}. \tag{4}$$

Since

$$\inf_{v \in H^1(\Omega_\gamma)} \inf_{m \in L_2(\gamma)} K(v,l,m) = \inf_{m \in L_2(\gamma)} \inf_{v \in H^1(\Omega_\gamma)} K(v,l,m),$$

then functional $\underline{M}(l)$ has the another presentation

$$\underline{M}(l) = \inf_{m \in L_2(\gamma)} \left\{ \chi(m) + \frac{1}{2r}\int_\gamma \left((l + r\,m)^2 - l^2\right) d\Gamma \right\}. \tag{5}$$

For an arbitrary $l \in L_2(\gamma)$, we consider the functional

$$F_l(m) = \chi(m) + \frac{1}{2r}\int_\gamma \left((l + r\,m)^2 - l^2\right) d\Gamma.$$

Then the dual functional $\underline{M}(l)$ has the form

$$\underline{M}(l) = \inf_{m \in L_2(\gamma)} F_l(m). \tag{6}$$

$F_l(m)$ is a weakly lower semicontinuous and coercive functional on $L_2(\gamma)$. Therefore the problem (6) has a solution $m(l)$ for every $l \in L_2(\gamma)$. It is easy to see, that element $m(l)$ is unique. Further, we mention one important property of dual functional [Vikhtenko & Namm, 2016].

Theorem 3. The dual functional $\underline{M}(l)$ is Gateaux differentiable in $L_2(\gamma)$ and it's derivative $\nabla\underline{M}(l)$ satisfies the Lipschitz condition with the constant r^{-1}; that is, for all $l_1, l_2 \in L_2(\gamma)$, it holds that

$$\|\nabla\underline{M}(l_1) - \nabla\underline{M}(l_2)\|_{L_2(\gamma)} \le r^{-1}\|l_1 - l_2\|_{L_2(\gamma)}.$$

In the proof of this theorem it shows that the functional subdifferential consists of a single element $\partial(\underline{M}(l)) = \{m(l)\}$. Therefore $\nabla\underline{M}(l) = m(l)$.

Using equations (4) and (5) for the dual functional, we get

$$\nabla\underline{M}(l) = m(l) = \max\left\{-\frac{l}{r}, -[v]\right\}.$$

Let us consider the dual problem

$$\left\{ \begin{array}{l} \underline{M}(l) - \sup, \\ l \in L_2(\gamma). \end{array} \right. \tag{7}$$

Since the gradient of the functional $\underline{M}(l)$ satisfies the Lipschitz condition, the dual problem (7) can be solved by using the gradient method for maximizing a functional

$$l_{k+1} = l_k + r\,\nabla\underline{M}(l_k) = l_k + r\,m(l_k), \quad k = 0, 1, 2, \ldots \tag{8}$$

Here $l_0 \in L_2(\gamma)$ is an arbitrary initial value.

Theorem 4. The sequence $\{l_k\}$ constructed by the gradient method (8) satisfies the limit equality

$$\lim_{k\to\infty} \|m(l_k)\|_{L_2(\gamma)} = 0.$$

Using gradient method (8), we can construct the following algorithm Uzawa method for solving the problem (1)

$$\begin{array}{ll} (i) & u_{k+1} = \arg \min\limits_{v\in H^1(\Omega_\gamma)} M(v, l_k); \\ (ii) & l_{k+1} = l_k + r \max\left\{-\dfrac{l_k}{r}, -[u_{k+1}]\right\}. \end{array}$$

Justifying the convergence presented method is complicated by the fact that the problem (7) can be unsolvable. The question of solvability of problem (7) is closely connected with the regularity of a solution u of problem (1). It can be proved if $u \in H^2(\Omega_\gamma)$, then $-[u_{x_2}] \in H^{1/2}(\gamma)$ is a solution of problem (7) [Khludnev, 2010]. This assumption looks unnatural in crack problem. Despite this problem, the duality ratio can be proved for initial and dual problem [Vikhtenko & Namm, 2016].

Theorem 5. There is a duality ratio

$$\sup_{l\in L_2(\gamma)} \underline{M}(l) = \inf_{v\in K} J(v).$$

Note that if the dual problem (7) has a solution, then $\{l_k\}$ is a bounded sequence in $L_2(\gamma)$ [4]. Together with theorem 4 it means that method (i), (ii) converges according the initial functional $J(u)$, that is

$$\lim_{k\to\infty} J(u_k) = \inf_{v\in K} J(u).$$

4 Numerical Implementation of Method

$\Omega = \{(x, y) \in R^2\colon\ 0 \le x \le 1;\ 0 \le y \le 1\}$ and $\gamma = \{(x, y) \in \Omega\colon\ a < x < b,\ y = c\}$. Approximation of the problem made by means of finite element method. To perform a triangulation of the region by step h, as shown in figure 1.

Introduce the notation:

I_γ is the set of indices of internal cracks nodes;

I_Ω is the set of indices of all other nodes of the triangulation.

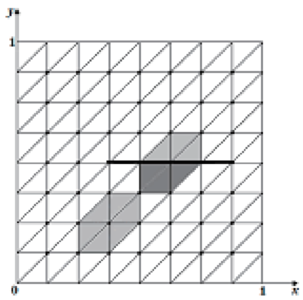

Figure 1: Triangulation of the computational domain

For each node of triangulation we define a piecewise affine basis function $\varphi_k(x,y)$. Also define functions $\varphi_k^+(x,y)$ and $\varphi_k^-(x,y)$ for each top and bottom node on the crack faces.

Construct an approximate solution as linear combination of basis functions $\varphi_k(x,y)$.

$$v_h(x,y)=\sum_{k\in I_\Omega} v_k\,\varphi_k(x,y)+\sum_{k\in I_\gamma} v_k^+\,\varphi_k^+(x,y)+\sum_{k\in I_\gamma} v_k^-\,\varphi_k^-(x,y)\,.$$

Let us substitute function $v_h(x,y)$ in a modified Lagrangian functional $M(v,l)$ taking change of variables

$$v_i^+=v_i^-+t_i,\quad t_i=[v_i]\ge 0,\ i\in I_\gamma.$$

Also we perform trapezoid approximation of the integral

$$\int_\gamma\left(\left((l-r[v])^+\right)^2-l^2\right)d\Gamma\approx\frac{h}{2}\sum_{i\in I_\gamma}\left\{\left((l_i-r\,t_i)^+\right)^2-l_i^2\right\}.$$

As result we obtain finite-dimensional functional of the following form

$$M\left(v_h,v_h^-,t_h,l_h\right)=\frac{1}{2}\left\{\sum_{i\in I_\Omega}\sum_{j\in I_\Omega}a_{ij}v_iv_j+\sum_{i\in I_\gamma}\sum_{j\in I_\gamma}a_{ij}^+\left(v_i^-+t_i\right)\left(v_j^-+t_j\right)+\sum_{i\in I_\gamma}\sum_{j\in I_\gamma}a_{ij}^-v_i^-v_j^-+\right.$$

$$\left.+2\sum_{i\in I_\Omega}\sum_{j\in I_\gamma}b_{ij}^+v_i\left(v_j^-+t_j\right)+2\sum_{i\in I_\Omega}\sum_{j\in I_\gamma}b_{ij}^-v_iv_j^-\right\}-$$

$$-\sum_{i\in I_\Omega}f_iv_i-\sum_{i\in I_\gamma}f_i^+\left(v_i^-+t_i\right)-\sum_{i\in I_\gamma}f_i^-v_i^-+\frac{h}{2r}\sum_{i\in I_\gamma}\left\{\left((l_i-r\,t_i)^+\right)^2-l_i^2\right\},$$

where

$$a_{ij}=\int_{\Omega_\gamma}\nabla\varphi_i\nabla\varphi_j d\Omega,\quad i,j\in I_\Omega,$$

$$a_{ij}^+=\int_{\Omega_\gamma}\nabla\varphi_i^+\nabla\varphi_j^+d\Omega,\quad a_{ij}^-=\int_{\Omega_\gamma}\nabla\varphi_i^-\nabla\varphi_j^-d\Omega,\quad i,j\in I_\gamma,$$

$$b_{ij}^+=\int_{\Omega_\gamma}\nabla\varphi_i\nabla\varphi_j^+d\Omega,\quad b_{ij}^-=\int_{\Omega_\gamma}\nabla\varphi_i\nabla\varphi_j^-d\Omega,\quad i\in I_\Omega,\quad j\in I_\gamma,$$

$$f_i=\int_{\Omega_\gamma}f\,\nabla\varphi_i d\Omega,\quad f_i^+=\int_{\Omega_\gamma}f\,\nabla\varphi_i^+d\Omega,\quad f_i^-=\int_{\Omega_\gamma}f\,\nabla\varphi_i^-d\Omega,\quad i\in I_\gamma.$$

According to the Uzawa method on the iteration number k it is necessary to solve auxiliary problem

$$\begin{cases} M\left(v_h,v_h^-,t_h,l_h^k\right)-\min,\\ -t_i\le 0,\quad i\in I_\gamma.\end{cases}$$

To solve this problem we apply the pointwise relaxation method.

Computational experiments were conducted with the following fixed parameters, which are presented in the table 1.

Consider three examples differing in way of defining function f in the computational domain Ω. The values of the function and the corresponding solution of the problem (1) is shown in figures 2–4.

In example 1 we attempt to raise the portion positioned below the crack. For this purpose, we define a positive value of the function f on the lower face of the crack, and negative on the top face. As seen from the corresponding graph solution in figure 2, the condition $[v]=v^+-v^-\ge 0$ is satisfied on γ. Disconnection of crack faces does not occur; we obtain the solution of the Dirichlet problem.

In example 2, the value of the function f is reversed. In this case, we observe opening faces along the entire length of the crack. Corresponding graph solution is presented in figure 3.

Table 1: Fixed parameters of computational experiments

Parameters	Parameter value
Computational domain	$\Omega = \{(x, y) \in R^2 : 0 \le x \le 1,\ 0 \le y \le 1\}$
Positioning of crack in domain	$\gamma = \{(x, y) :\ 0.2 < x < 0.8,\ y = 0.4\}$
Step triangulation	$h = 2^{-6}$
Computational accuracy	$\varepsilon = 10^{-8}$

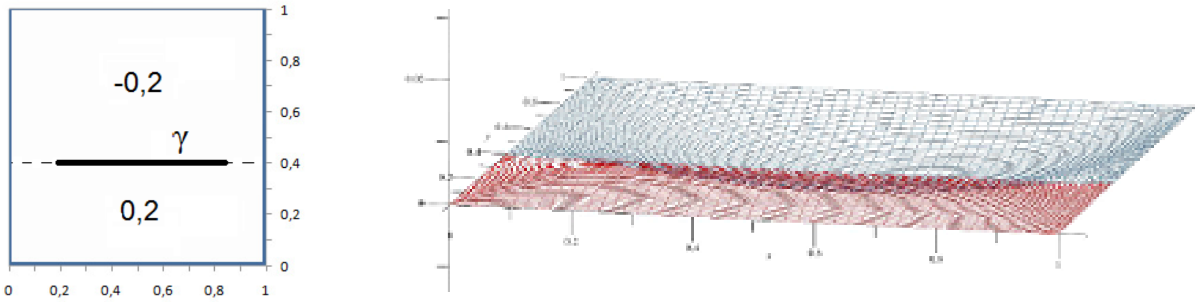

Figure 2: The calculation results. Example 1

Finally, in example 3, the function is defined so that to the upper face of crack the portions are adjacent with a positive and negative value of the function f. As seen from the figure 4 there is a partial disconnection of the crack faces only, in that part of the domain Ω where the value of function f is negative at the upper face. Condition $[v] = v^+ - v^- \ge 0$ is still satisfied on γ.

Let's consider the question of the relationship between the number of iterations by Uzawa method with parameter r which is the shift step value by the dual variable. We distinguish two types of iteration:

1. The external iterations is the number of step (ii) of the Uzawa method to achieve the required accuracy.
2. The internal iterations is the number of step (i) of the Uzawa method to achieve the required accuracy.

The table 2 shows the calculation results for example 3 for different values of parameter r. Here null vectors chosen as an initial approximation.

Table 2: Number of iterations by Uzawa method for example 3

Value r	Number of internal iterations	Number of external iterations
10	6029	51
10^2	3208	11
10^3	2519	5
10^4	2321	3

Conducted computing experiments have shown that the convergence of the method to the saddle point of the Lagrangian functional is good enough. At that, increasing the value of the parameter r leads to growth convergence rate. For considered example on the grid in steps of $h = 2^{-6}$, with an accuracy of $\varepsilon = 10^{-8}$, it is necessary less than sixty external iterations.

5 Conclusion

These results allow us to conclude about the effectiveness of the chosen approach for solving model crack problem, one of the formulations of this is a minimization problem (1). The main advantage of using a modified Lagrangian functional is the shift step value by the dual variable is large positive value in distinct from the classical scheme of duality. Therefore duality method with a modified Lagrangian functional superior to the standard scheme uses classical functional for computing speed.

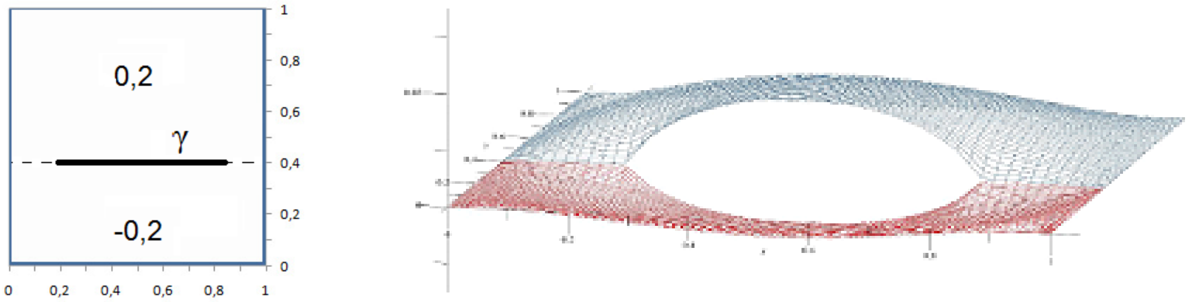

Figure 3: The calculation results. Example 2

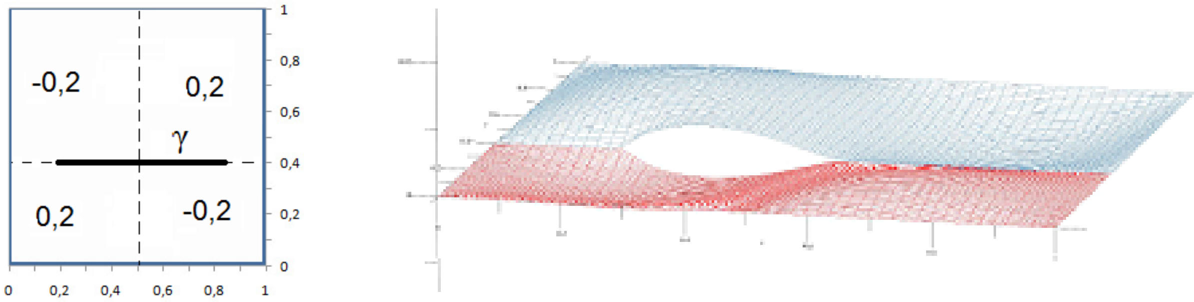

Figure 4: The calculation results. Example 3

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