Optimal Stabilization of Multiply Connected Dynamical Systems

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Abstract

The conditions of optimal stabilization of controlled dynamical systems described by nonlinear multiply connected systems of ordinary differential equations are considered. The properties of stabilizing control and form of integrand in criterion of quality transient are used taking into account that subsystems are asymptotically stable. The results are obtained for the case of a part of phase variables and for the case when right parts consist of homogeneous vector-functions. Conditions of optimal stabilization with respect to a part of variables are suggested. The algorithms of optimal stabilization of controlled multiply connected dynamical systems are designed.

1 Introduction

The search of conditions of optimal stabilization and synthesis of corresponding stabilization algorithms is a significant problem in research of behavior of nonlinear controlled systems [Rumyantsev, 1970], [Krasovsky, 1966], [Rumyantsev, 1987], [Andreev, 1997]. Fundamental approach to optimal stabilization for systems of ordinary differential equations was developed by V.V. Rumyantsev with using of condition of minimization for a functional characterizing the quality of control. Solving of the problems of optimal stabilization for different types of multiply connected dynamic systems is based on the fundamental results [Rumyantsev, 1970] about optimal stabilization of nonlinear system of differential equations of perturbed motion with the additional forces. The functional is given in the form of a definite integral with the upper infinite limit. Integrand function of the functional is defined in the proof of the theorem, in this case the known Lyapunov function for the system of differential equations of perturbed motion without the control becomes the optimal Lyapunov function for the specified system under the action of additional forces.

Methods of optimal stabilization are considered in [Krasovsky, 1966], [Rumyantsev, 1987], [Andreev, 1997]. Critical cases are allocated and ways of a finding of stabilizing control in critical cases are developed [Galperin, 1963], [Hitrov, 1979]. The convenient way and the basis of general scheme of stabilization for multiply

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connected systems is two-level stabilization [Shil'yak, 1994]. Some methods for solving of the problem of optimal stabilization to respect to all variables and to a part of phase variables for multiply connected nonlinear controlled dynamic systems are given in [Shchennikova, 2006], [Druzhinina et al., 2011].

In this work we suggest the conditions and algorithms of optimal stabilization for controlled dynamic systems described by nonlinear multiply connected systems of ordinary differential equations. We consider the common case and the case when right-hand sides consists of homogeneous vector functions. The properties of stabilizing control and form of integrand in criteria of quality of transient are used taking into account that subsystems are asymptotically stable. Optimal control is synthesized at the level of the initial system. The results can be used in problems of control of motion of complicated spatial mechanisms, and also in problems of stabilization of motion of multiply connected systems of different types.

2 The Optimal Stabilization with Using of Homogeneous Vector Functions

It is known that for nonlinear systems of differential equations of general form the conditions of theorems on asymptotic stability on the first nonlinear approach are hardly verified. However, in some cases is possible to search for fairly easily verifiable conditions under which we prove the asymptotic stability of the equilibrium of the first nonlinear approach. Nonlinear system with right-hand sides are homogeneous (generalized homogeneous) vector functions have been studied in [Kosov, 1997], [Alexandrov, 2004], which shows the theorems about asymptotic stability on the first nonlinear approach with the conditions which are easily verified. These results can be used in the solving of problems of optimal stabilization of nonlinear controlled multiply connected systems.

We consider multiply connected nonlinear controlled dynamic system

$$\frac{dx_s}{dt} = X_s^{(\mu_s)}(x_s) + \sum_{j=1}^{q} R_{sj}(t,x) + B_s(x_s)u_s \equiv \Phi_s(t,x,u). \tag{1}$$

Here $x_s \in R^{n_s}$, $x = \left(x_1^T, ..., x_q^T\right)$, $X_s^{(\mu_s)}(x_s)$ are homogeneous of order $\mu_s > 1$ continuously differentiable vector functions, $\mu_s = p_s/q_s$, p_s and q_s are odd numbers, $u_s \in R^{r_s}$,$R^{n_1} \times ... \times R^{n_q} = R^n$, $R^{r_1} \times ... \times R^{r_q} = R^r$, $s = \overline{1,q}$. Continious functions $R_{sj}(t,x)$ are defined in domain

$$\Omega = \{t, x : t \geq t_0, ||x|| < h, 0 < h = \text{const}\}. \tag{2}$$

It should be noted that we use the Euclidean norm of the vector in formula (2). It is accepted that the conditions

$$||R_{sj}(t,x)|| \leq c_{sj}||x_1||^{\alpha_{sj}^{(1)}} \dots ||x_q||^{\alpha_{sj}^{(q)}},$$

$$c_{sj} \geq 0,\ \alpha_{sj}^{(i)} \geq 0, \quad \sum_{i=1}^{q} \alpha_{sj}^{(i)} > 1,$$

are hold. We assume that $\Phi_s(t,0,0) \equiv 0$, $s = \overline{1,q}$, and that equilibrium states of systems

$$\frac{dx_s}{dt} = X_s^{(\mu_s)}(x_s), \quad s = \overline{1,q}, \tag{3}$$

are asymptotically stable. As a Lyapunov function for system (1) in this case we consider the function

$$v(x) = \sum_{s=1}^{q} v_s(x_s),$$

where $v_s(x_s)$ are Lyapunov functions for systems (3) satisfying the conditions:
(i) $v_s(x_s)$ and $w_s(x_s)$ are positive definite functions;
(ii) $v_s(x_s)$ and $w_s(x_s)$ are gomogeneously positive functions of order $m_s + 1 - \mu_s$ and m_s, where m_s are enough large rational numbers with odd denominator and even numerator;
(iii) functions $v_s(x_s)$ are continuously differentiable and

$$(\nabla v_s)^T X_s^{(\mu_s)}(x_s) = -w_s(x_s).$$

The problem of optimal stabilization for the system (1) has a unique solution in closed form. Krasovsky function $B[v;t,x,u]$ has a form

$$B[v;t,x,u] = \sum_{s=1}^{q}\Big[-w_s(x_s) - ((\nabla v_s(x_s))^T(\sum_j R_{sj}(t,x) + B_s(x_s)u_s)) +$$

$$+\frac{1}{2}u_s^T\beta_s(x_s)u_s)\Big] + \Psi_1(t,x).$$

According to Rumyantsev and Krasovsky theorems optimal control

$$u^0 = \left(u_1^{T^0}, ..., u_s^{T^0}\right)^T,$$

and optimal Lyapunov function we obtain from a system

$$\frac{\partial B}{\partial u_s} = B_s(x_s) + \beta_s(x_s)u_s^0 = 0,\ s = \overline{1,q}. \tag{4}$$

From (4) we have

$$u_s^0 = -\beta_s^{-1}(x_s)(\nabla v_s)^T B_s(x_s) = -\beta_s^{-1}(x_s)B_s^T(x_s)\nabla v_s. \tag{5}$$

Substituting u_s^0 in (5) to function $B[v;t,x,u]$ we have the algebraic equation $B[v^0;t,x,u^0] = 0$ with respect to function $\Psi_1(t,x)$.

The function

$$\Psi_1 = -\sum_{s=1}^{q} w_s(x_s) - ((\nabla v_s(x_s))^T(\sum_{j=1}^{q} R_{sj}(t,x))) + (u_s^T)^0\beta_s(x_s)u_s^0, \tag{6}$$

will be positive definite and functional of control finally becomes

$$J(u_0) = \int_{t_0}^{\infty}\sum_{s=1}^{q}\Big[-w_s(x_s) - (\nabla v_s(x_s))^T(\sum_{j=1}^{q} R_{sj}(t,x)) + \\ + (u_s^0)^T\beta_s(x_s)u_s^0 + u_s^T\beta_s(x_s)u_s\Big]dt. \tag{7}$$

We applied to system (1) and to functional (7) the results about optimal stabilization of common nonhomogeneous multiply connected systems [Shchennikova, 2006]. In the case under consideration we use general scheme of stabilization without assumption of positive definition of (6) because this function has required property. The algorithms of optimal stabilization of system (1) with respect to all and to a part of variables are developed.

3 The Optimal Stabilization with Respect to a Part of Variables

We consider multiply connected nonlinear controlled dynamic system

$$\frac{dx_s}{dt} = f_s(t,x_s,u_s^{\text{loc}}) + F_s(t,x,u_s^{\text{glob}}) \equiv \Phi_s(t,x,u_s^{\text{loc}},u_s^{\text{glob}}),\quad s = \overline{1,q}, \tag{8}$$

where $x = \left(x_1^T, ..., x_q^T\right)^T$, $x_s \in R^{n_s}$, $R^{n_1} \oplus ... \oplus R^{n_q} = R^n$, $u_s^{\text{loc}}(t,0) = 0$, $u_s^{\text{glob}}(t,0) = 0$, $\Phi_s(t,0,0,0) \equiv 0$.

It is accepted that right part of system (8) is defined in domain

$$\Omega_1 = \{t,x,u_s^{\text{loc}},u_s^{\text{glob}} : t \ge t_0 \ge 0,\ ||x|| < H, \\ ||u_s^{\text{loc}}|| < \infty,\ ||u_s^{\text{glob}}|| < \infty,\ 0 < H = \text{const},\ s = \overline{1,q}\}, \tag{9}$$

and conditions of existence and uniqueness of solution are satisfied. Let us assume than system (8) can be represented as

$$\frac{dy_s}{dt} = Y_s(t,y_s,z_s,u_s^{\text{loc}}) + \sum_{j=1}^{q} Y_{1sj}(t,y,z)u_s^{\text{glob}}, \\ \frac{dz_s}{dt} = Z_s(t,y_s,z_s,u_s^{\text{loc}}) + \sum_{j=1}^{q} Z_{1sj}(t,y,z)u_s^{\text{glob}}, \tag{10}$$

where $x_s = (y_s^T, z_s^T)^T$, $x = (y^T, z^T)^T$, where $y_s \in R^{k_s}$, $z_s \in R^{m_s}$, $k_s + m_s = n_s$, $s = \overline{1,q}$. For system (10) domain (9) takes the form

$$\Omega_2 = \{t, x, u_s^{\text{loc}}, u_s^{\text{glob}} : t \geq t_0 \geq 0, ||y_s|| < H_s, ||z_s|| \leq \infty, \\ ||u_s^{\text{loc}}|| < \infty, ||u_s^{\text{glob}}|| < \infty, 0 < H = \text{const}, s = \overline{1,q}\},$$

and each solution is z-extendible.

We consider the subsystems of the form

$$\frac{dy_s}{dt} = Y_s(t, y_s, z_s, u_s^{\text{loc}}) \\ \frac{dz_s}{dt} = Z_s(t, y_s, z_s, u_s^{\text{loc}}),\ s = \overline{1,q}. \tag{11}$$

Further, we will solve the problem of optimal y_s-stabilization of multiply connected dynamical systems of the form (10), $s = \overline{1,q}$, $y = (y_1^T, ..., y_q^T)^T$, using the method of Lyapunov vector-functions. In this case the strategy of solving the problem of stabilization is that each subsystem must be y_s-stabilized with the help of local controls u_s^{loc}, $s = \overline{1,q}$, i.e. it must be y_s-stabilized on the level of subsystems, and then the asymptotic y_s-stability of interconnected subsystems must be checked. The general scheme of a two-level stabilization scheme [Shil'yak, 1994] is that the global control u_s^{glob}, $s = \overline{1,q}$, is added to the decentralized control in order to weaken the effect of interrelated subsystems. In this work the problem of optimal stabilization of multiply connected system is sold also using a two-level stabilization scheme with respect to a part of variables.

We consider the case when right parts of (11) can be written in the form

$$Y_s(t, y_s, z_s, u_s^{\text{loc}}) \equiv \overline{Y}_s(t, y_s, z_s) + b_{1s}(t, y_s, z_s) u_s^{\text{loc1}}, \\ Z_s(t, y_s, z_s, u_s^{\text{loc}}) \equiv \overline{Z}_s(t, y_s, z_s) + b_{2s}(t, y_s, z_s) u_s^{\text{loc2}},\ s = \overline{1,q}, \tag{12}$$

where $b_{1s}(t, y_s, z_s)$ and $b_{2s}(t, y_s, z_s)$ are matrixes of appropriate dimensions, and controls u_s^{loc1} and u_s^{loc2} are built considering the choice of Lyapunov vector functions.

It was shown that equilibrium state of system (11) taking into account (12) is uniformly asymptotic y_s-stable. In this case system (10) can be represented in the form

$$\frac{dy_s}{dt} = \varphi_s(t, y_s, x_s) + Y_{1s}(t, \widetilde{y}, \widetilde{z}) u_s^{\text{glob}}, \\ \frac{dz_s}{dt} = \psi_s(t, y_s, z_s) + Z_{1s}(t, \widetilde{y}, \widetilde{z}) u_s^{\text{glob}}, \tag{13}$$

where

$$\varphi_s(t, y_s, z_s) = \overline{Y}(t, y_s, z_s) + b_{1s}(t, y_s, z_s) u_s^{\text{loc}}(t, y_s, z_s), \\ \psi_s(t, y_s, z_s) = \overline{Z}_s(t, y_s, z_s) + b_{2s}(t, y_s, z_s) u_s^{\text{loc}}(t, y_s, z_s).$$

We consider the problem of optimal stabilization for system (13). Criterion of quality control we write in the integral form

$$J = \int_0^\infty w(t, y[t], z[t], u_s^{\text{glob}}[t]) dt, \tag{14}$$

in this case we define the function $w(t, x, u)$ in the process of solving.

As optimal Lyapunov function for system (13) we choose the function

$$V(t, y, z) = \sum_{s=1}^{q} \alpha_s V_s(t, y_s, z_s),$$

where α_s are positive real constants, $V_s(t, x_s)$ are Lyapunov functions which guarantee uniform asymptotic y_s-stability of systems (11), $s = \overline{1,q}$.

We introduce Krasovsky–Bellman function $B(t, x, u, v, u^{\text{glob}})$ with special component $\Psi(t, y, z)$ allowing to consider the function w in integral (14) in the form

$$w(t, y, z, u^{\text{glob}}) = \Psi(t, y, z, u^{\text{glob}}) + \frac{1}{2} \sum_{s=1}^{q} (u^{\text{glob}})^T \theta_s u_s^{\text{glob}}.$$

According to Rumyantsev theorem function $B(t, x, u, v, u^{\text{glob}})$ is positive-definite with respect to y. Along optimal control $(u_s^0)^{\text{glob}}$ we have that

$$\frac{\partial B}{\partial u_s^{\text{glob}}}\Big|_{(u_s^0)^{\text{glob}}} = 0,\ s = \overline{1, q}. \tag{15}$$

In result we obtain positive-definite function with respect to y-component of phase vector of system (10). In this case we can write the criterion of quality control in the form

$$J = \int_{t_0}^{\infty} (\sum_{s=1}^{q} \alpha_s W_s(t, y_s, z_s) + \sum_{s,j=1}^{q} \theta_{sj} (u_s^0)^{\text{glob}} (u_j^0)^{\text{glob}} + \sum_{s,j=1}^{q} \theta_{sj} u_s^{\text{glob}} u_j^{\text{glob}}) dt. \tag{16}$$

Thus if (i) for systems (11) y_s-stabilizing (to uniform asymptotic stability) controls $u_s^{\text{loc}} = u_s^{\text{loc}}(t, x_s)$ exist, $s = \overline{1, q}$, (ii) function $\Psi(t, y, z)$ is positive-definite with respect to vector y of system (10), then controls $(u_s^0)^{\text{glob}}$ defined from system (15) are the functions solving the problem of optimal y-stabilization of system (10) with respect to functional (16).

By the aid of this two-level scheme of stabilization with applications of results of [Shchennikov, 2001], [Shestakov, 2010], [Druzhinina, 2002], [Shestakov, 2009] the algorithms of optimal stabilization for multiply connected nonlinear controlled systems are developed.

We note that the problem of optimal stabilization with respect to a part of the variables under permanently acting perturbations is of theoretical and practical interest. In particular, we consider a system

$$\frac{dx}{dt} = Ax^{(\mu)} + F(t, x, u), \tag{17}$$

where A is a constant $(n \times n)$-matrix, $x = (x_1, ..., x_n)^T,\ x = (y, z),\ x^{(\mu)} = (x_1^{\mu}, ..., x_n^{\mu})^T,\ \mu = 2p - 1,\ p = 2, 3, ...$, and the function $F(t, x, u)$ is given in the domain in which the uniqueness condition for the solution of the Cauchy problem and the condition of the z-extendibility of solutions are satisfied. We use a single-level stabilization scheme for system (17). However, by generalizing systems (17) to a multiply connected case, we can use a two-level stabilization scheme analogous to the stabilization scheme for system (8).

4 Conclusions

The conditions of optimal stabilization for multiply connected systems are obtained by the aid of two-level stabilization scheme and the generalization of Rumyantsev method on indicated class class of dynamical systems. The basic algorithm of the solving of optimal stabilization problem concerning all phase variables consists of the following stages:

1) to carry out stabilization at the level of the interconnected subsystems, i.e. to find local controls;

2) to present optimal Lyapunov function in the form of a linear combination of functions of Lyapunov for subsystems;

3) to find Krasovsky-Bellman function;

4) to make algebraic system taking into account that Krasovsky-Bellman function vanishes on the optimal solution;

5) to find optimal control for initial multiply connected system from the algebraic system;

6) to define functional in relation to which control is optimal.

By the aid of basic algorithm the optimal stabilization algorithms for different types of multiply connected dynamic systems are offered. The method of Lyapunov functions [Matrosov, 2001] is essentially used for research. Modifications of dynamical models of manipulation robots [Miroshnik et al., 2000], [Vukobratovic et al., 1985] are considered and for them the corresponding algorithms of stabilization with application of multilevel stabilization are developed.

This algorithm is modified to the case of a part of phase variables and to the case of permanently acting perturbations. The results can be used in problems of motion control of complicated spartial mechanisms and in research of robotics systems dynamics.

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