On a Solution of Fractional Programs via D.C. Optimization Theory

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Abstract

This paper addresses the problems of fractional programming which data is given by d.c. functions. Such problems are, in general, nonconvex (with numerous local extrema). We develop an efficient global search method for fractional program based on two different approaches. The first approach develops the Dinkelbach's idea and uses a solution of an equation with the optimal value of an auxiliary d.c. optimization problem with a vector parameter. The second one deals with another auxiliary problem with nonconvex inequality constraints. Both auxiliary problems are d.c. optimization problems, which allows us to apply the Global Optimization Theory and develop two corresponding global search algorithms. The algorithms have been substantiated and tested on an extended set of sum-of-ratios problems with up to 200 variables and 200 terms in the sum. Also we propose an algorithm for solving problems of fractional programming, which combines the two approaches.

1 Introduction

During the recent decades the problems with the functions representable as a difference of two convex functions (i.e. d.c. functions) can be considered as one of most attractive in nonconvex optimization [Hiriart-Urruty, 1985, Horst&Pardalos, 1995, Horst&Tuy, 1996, Evtushenko& Posypkin, 2013, Khamisov, 1999, Strekalovsky, 2003, Strongin&Sergeyev, 2000, Tuy, 1998]. Actually, any continuous optimization problem can be approximated by a d.c. problem with any prescribed accuracy [Hiriart-Urruty, 1985, Horst&Tuy, 1996, Tuy, 1998]. In addition, the space of d.c. functions is a linear space, any polynomial and moreover any twice differentiable function belong to the space of d.c. functions [Hiriart-Urruty, 1985, Horst&Pardalos, 1995, Horst&Tuy, 1996, Strekalovsky, 2003, Strongin&Sergeyev, 2000, Tuy, 1998]. Besides, this space is closed with respect to all standard operations used in Analysis and Optimization. On account of this situation the general

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problem of mathematical optimization

$$(\mathcal{P}_{dc}) \qquad \begin{cases} f_0(x) := g_0(x) - h_0(x) \downarrow \min\limits_x, \quad x \in S, \\ f_j(x) := g_j(x) - h_j(x) \leq 0, \quad j = 1, \dots, m, \end{cases}$$

where the functions $g_j(\cdot)$, $h_j(\cdot)$, $j = 1, \dots, m,$, are convex on $I\!R^n$, S is a convex set, $S \subset I\!R^n$, can be considered as a rather attractive object from theoretical and practical points of view.

Recently, we have succeeded in developing the Global Optimality Conditions for Problem $(\mathcal{P}_{dc})$, which perfectly fits optimization theory and proved to be rather efficient in terms of computations [Strekalovsky, 2003, Strekalovsky, 2014]. Now we apply this theory to solving the following problem of the fractional optimization [Bugarin et al., 2016, Schaible&Shi, 2003]

$$(\mathcal{P}_f) \qquad f(x) := \sum_{i=1}^{m} \frac{\psi_i(x)}{\varphi_i(x)} \downarrow \min_x, \quad x \in S,$$

where $S \subset I\!R^n$ is a convex set and $\psi_i : I\!R^n \to I\!R$, $\varphi_i : I\!R^n \to I\!R$,

$$(\mathcal{H}_0) \qquad \psi_i(x) > 0, \ \varphi_i(x) > 0 \ \ \forall x \in S, \quad i = 1, \dots, m.$$

It is well-known that the fractional programming problem is NP-complete [Freund&Jarre, 2001]. Surveys on methods for solving this problem can be found in [Bugarin et al., 2016, Schaible&Shi, 2003], but the development of new efficient methods for a fractional program still remains an important field of research in mathematical optimization, because the sum-of-ratios programs arise in various economic applications and real-life problems [Schaible&Shi, 2003].

Here we develop efficient global search method for fractional program, which is based on two following approaches. Generalizing the Dinkelbach's idea, we propose to reduce the fractional program with d.c. functions to solving an equation with the vector parameter that satisfies the nonnegativity assumption. In this case, we need to solve the auxiliary d.c. minimization problems, i.e. Problem $(\mathcal{P}_{dc})$ with $f_i(x) \equiv 0, \quad i \in I$. We also propose reduction of the sum-of-ratios problem to a minimization of a linear function on the nonconvex feasible set given by d.c. inequality constraints, i.e. Problem $(\mathcal{P}_{dc})$ where $f_0(x)$ is a linear function. Thus, based on the solution of these two particular cases of general d.c. optimization problem $(\mathcal{P}_{dc})$ we develop two-method technology for solving a general fractional program, based on the Global Search Theory for d.c. optimization [Strekalovsky, 2003, Strekalovsky, 2014].

This paper is an extension of our research [Gruzdeva&Strekalovsky, 2017], where the theoretical framework and some computational simulations were presented. The outline of the paper is as follows. In Sect. 2, we describe two approaches to solving Problem $(\mathcal{P}_f)$ using auxiliary d.c. optimization problems. In Sect. 3, we show some comparative computational testing of two approaches on fractional program instances with up to 200 variables and 200 terms in the sum generated randomly and elaborate an algorithm for solving sum-of-ratios problem combining the two proposed methods.

2 D.C. Programming Approach to Sum-of-Ratios Problems

Instead of considering a fractional program directly, we develop efficient global search methods, which are based on two approaches. The first method adopts a reduction of the fractional minimization problem to the solution of an equation with an optimal value of the d.c. optimization problem with a vector parameter. The second method is based on the reduction of the sum-of-ratios problem to the optimization problem with nonconvex constraints.

2.1 Reduction to the D.C. Minimization Problem

To solve fractional program using d.c. minimization we consider the following auxiliary optimization problem

$$(\mathcal{P}_\alpha) \qquad \Phi(x, \alpha) := \sum_{i=1}^{m} [\psi_i(x) - \alpha_i \varphi_i(x)] \downarrow \min_x, \quad x \in S,$$

where $\alpha = (\alpha_1, \dots, \alpha_m)^\top \in I\!R^m$ is the vector parameter.

In this subsection we recall some results [Gruzdeva&Strekalovskiy, 2016] concerning the relations between Problems $(\mathcal{P}_f)$ and $(\mathcal{P}_\alpha)$. Note, that the data of Problem $(\mathcal{P}_f)$ must satisfy "the nonnegativity condition", i.e. the following inequalities hold

$$(\mathcal{H}(\alpha)) \qquad \psi_i(x) - \alpha_i\varphi_i(x) \geq 0 \ \ \forall x \in S, \ \ i = 1, \ldots, m.$$

Let us introduce the function $\mathcal{V}(\alpha)$ of the optimal value to Problem $(\mathcal{P}_\alpha)$:

$$\mathcal{V}(\alpha) := \inf_x \{\Phi(x, \alpha) \ \mid \ x \in S\} = \inf_x \left\{ \sum_{i=1}^{m} [\psi_i(x) - \alpha_i\varphi_i(x)] \ : \ x \in S \right\}.$$

In addition, suppose that the following assumptions are fulfilled:

$$(\mathcal{H}_1) \quad \begin{array}{l} (a) \ \ \mathcal{V}(\alpha) > -\infty \ \ \forall \alpha \in \mathcal{K}, \text{where } \mathcal{K} \text{ is a convex set from } I\!R^m; \\ (b) \ \ \forall \alpha \in \mathcal{K} \subset I\!R^m \ \text{ there exists a solution } z = z(\alpha) \text{ to Problem } (\mathcal{P}_\alpha) \end{array}$$

Further, suppose that in Problem $(\mathcal{P}_{\alpha_0})$ the following equality takes place:

$$\mathcal{V}(\alpha_0) : = \inf_x \left\{ \sum_{i=1}^{m} [\psi_i(x) - \alpha_{0i}\varphi_i(x)] \ : \ x \in S \right\} = 0. \tag{1}$$

In [Gruzdeva&Strekalovskiy, 2016] we proved that any solution $z = z(\alpha_0)$ to Problem $(\mathcal{P}_{\alpha_0})$ is a solution to Problem $(\mathcal{P}_f)$, so that $z \in Sol(\mathcal{P}_{\alpha_0}) \subset Sol(\mathcal{P}_f)$.

Therefore in order to verify the equality (1), we should be able to find a global solution to Problem $(\mathcal{P}_\alpha)$ for every $\alpha \in I\!R^m_+$. Since $\psi_i(\cdot)$, $\varphi_i(\cdot)$, $i = 1, \ldots, m$, are simply convex or generally d.c. functions it can be readily seen that Problem $(\mathcal{P}_\alpha)$ belongs to the class of d.c. minimization. As a consequence, in order to solve Problem $(\mathcal{P}_\alpha)$, we can apply the Global Search Theory [Strekalovsky, 2003, Strekalovsky, 2014].

Due to the theoretical foundation developed in [Gruzdeva&Strekalovskiy, 2016], we are able to avoid the consideration of Problem $(\mathcal{P}_f)$, and address the parametrized problem $(\mathcal{P}_\alpha)$ with $\alpha \in I\!R^m_+$. Hence, instead of solving Problem $(\mathcal{P}_f)$, we propose to combine solving of Problem $(\mathcal{P}_\alpha)$ with a search with respect to the parameter $\alpha \in I\!R^m_+$ in order to find the vector $\alpha_0 \in I\!R^m_+$ such that $\mathcal{V}(\alpha_0) = 0$.

Denote $\Phi_i(x) := \psi_i(x) - \alpha_i^k\varphi_i(x)$, $i = 1, \ldots, m$. Let $[v^k, u^k]$ be a k-segment for varying α.

Let a solution $z(\alpha^k)$ to Problem $(\mathcal{P}_{\alpha^k})$ be given and we computed $\mathcal{V}_k := \mathcal{V}(\alpha^k)$.

α_k-bisection algorithm

Step 1. If $\mathcal{V}_k > 0$, then set $v^{k+1} = \alpha^k$, $\alpha^{k+1} = \frac{1}{2}(u^k + \alpha^k)$.

Step 2. If $\mathcal{V}_k < 0$, then set $u^{k+1} = \alpha^k$, $\alpha^{k+1} = \frac{1}{2}(v^k + \alpha^k)$.

Step 3. If $\mathcal{V}_k = 0$ and $\min\limits_i \Phi_i(z(\alpha^k)) < 0$, then set $\alpha_i^{k+1} = \dfrac{\psi_i(z(\alpha^k))}{\varphi_i(z(\alpha^k))}$ $\forall i : \Phi_i(z(\alpha^k)) < 0$,
$\alpha_i^{k+1} = \alpha_i^k$ $\forall i : \Phi_i(z(\alpha^k)) \geq 0$.
In addition, set $v^{k+1} = 0$, $u^{k+1} = t_{k+1}\alpha^{k+1}$, where $t_{k+1} = \dfrac{\alpha^+}{\max\limits_i \alpha_i}$.

Stop: we computed the values α^{k+1}, v^{k+1} and u^{k+1}.

Further, let $[0, \alpha^+]$ be an initial segment for varying α. To choose α^+ we should take into account that due to $(\mathcal{H}(\alpha))$ and $(\mathcal{H}_0)$, we have

$$\forall i = 1, \ldots, m : \alpha_i \leq f_i(x) \triangleq \frac{\psi_i(x)}{\phi_i(x)} \leq \sum_{i=1}^{m} \frac{\psi_i(x)}{\phi_i(x)} = f(x) \ \ \forall x \in S,$$

so, for example, α^+ can be chosen as $\alpha_i^+ = f_i(x^0)$, $i = 1, \ldots, m$.

Algorithm for solving fractional problem

Step 0. (Initialization) $k = 0,\ v^k = 0,\ \ u^k = \alpha^+,\ \ \alpha^k = \frac{\alpha^+}{2} \in [v^k, u^k]$.

Step 1. Find a solution $z(\alpha^k)$ to Problem $(\mathcal{P}_{\alpha^k})$ using the global search strategy for d.c. minimization [Strekalovsky, 2003, Strekalovsky, 2014].

Step 2. (Stopping criterion) If $\mathcal{V}_k := \mathcal{V}(\alpha^k) = 0$ and $\min_i \Phi_i(z(\alpha^k)) \geq 0$, then STOP: $z(\alpha^k) \in Sol(\mathcal{P}_f)$.

Step 3. Implement α_k-bisection algorithm to find new parameters $\alpha^{k+1},\ v^{k+1}$ and u^{k+1}; $k = k+1$ and go to Step 1.

Let us emphasize the fact that the algorithm for solving Problem $(\mathcal{P}_f)$ of fractional optimization consists of 3 basic stages: the (a) local and (b) global searches in Problem $(\mathcal{P}_\alpha)$ with a fixed vector parameter α (Step 1) and (c) the method for finding the vector parameter α (Step 3) at which the optimal value of Problem $(\mathcal{P}_\alpha)$ is zero, i.e. $\mathcal{V}(\alpha) = 0$.

2.2 Reduction to the Problem with D.C. Constraints

In this subsection we reduce the fractional program to the following optimization problem with a nonconvex feasible set

$$(\mathcal{P}) \qquad \left\{ \begin{array}{c} f_0 := \sum\limits_{i=1}^{m} \alpha_i \downarrow \min\limits_{(x,\alpha)},\ \ x \in S, \\ f_i := \psi_i(x) - \alpha_i \varphi_i(x) \leq 0,\ \ i = 1, \ldots, m. \end{array} \right.$$

In [Gruzdeva&Strekalovsky, 2016] it was proved that for any solution $(x_*, \alpha_*) \in I\!R^n \times I\!R^m$ to the problem $(\mathcal{P})$, the point x_* will be a solution to Problem $(\mathcal{P}_f)$.

Further we solved problem $(\mathcal{P})$ using the exact penalization approach for d.c. optimization developed in [Strekalovsky, 2016, Strekalovsky, 2017]. Therefore, we introduce the penalized problem as follows

$$(\mathcal{P}_\sigma) \qquad \theta_\sigma(x) = f_0(x) + \sigma \max\{0, f_i(x), i \in I\} \downarrow \min, \quad x \in S.$$

It can be readily seen that the penalized function $\theta_\sigma(\cdot)$ is a d.c. function, because the functions $f_i(\cdot) = g_i(\cdot) - h_i(\cdot),\ i \in I \cup \{0\}$, are the same, and we can apply the global search theory to solving this class of nonconvex problems [Strekalovsky, 2003, Strekalovsky, 2014].

Actually, since $\sigma > 0$, $\theta_\sigma(x) = G_\sigma(x) - H_\sigma(x),\ H_\sigma(x) := h_0(x) + \sigma \sum\limits_{i \in I} h_i(x)$,

$$G_\sigma(x) := \theta_\sigma(x) + H_\sigma(x) = g_0(x) + \sigma \max\left\{ \sum_{i=1}^{m} h_i(x); \max_{i \in I}[g_i(x) + \sum_{j \in I,\ j \neq i} h_j(x)] \right\},$$

it is clear that $G_\sigma(\cdot)$ and $H_\sigma(\cdot)$ are convex functions.

Let the Lagrange multipliers, associated with the constraints and corresponding to the point $z^k,\ \ k \in \{1, 2, ...\}$, be denoted by $\lambda := (\lambda_1, \ldots, \lambda_m) \in I\!R^m$.

Global search scheme

Step 1. Using the local search method from [Strekalovsky&Gruzdeva, 2007, Strekalovsky, 2015], find a critical point z^k in $(\mathcal{P})$.

Step 2. Set $\sigma_k := \sum\limits_{i=1}^{m} \lambda_i$. Choose a number $\beta :\ \inf(G_\sigma, S) \leq \beta \leq \sup(G_\sigma, S)$.
Choose an initial $\beta_0 = G_\sigma(z^k)$, $\zeta_k = \theta_\sigma(z^k)$.

Step 3. Construct a finite approximation $R_k(\beta) = \{v^1, \ldots, v^{N_k} \mid H_\sigma(v^i) = \beta + \zeta_k,\ i = 1, \ldots, N_k,\ N_k = N_k(\beta)\}$ of the level surface $\{H_\sigma(x) = \beta + \zeta_k\}$ of the function $H_\sigma(\cdot)$.

Step 4. Find a δ_k-solution $\bar{u}^i$ of the following Linearized Problem:

$$(P_\sigma L_i) \qquad G_\sigma(x) - \langle \nabla H_\sigma(v^i), x \rangle \downarrow \min_x,\ x \in S.$$

so that $G_\sigma(\bar{u}^i) - \langle \nabla H_\sigma(v^i), \bar{u}^i \rangle - \delta_k \leq \inf\limits_x \{G_\sigma(x) - \langle \nabla H_\sigma(v^i), x \rangle\}$.

Step 5. Starting from the point $\bar{u}^i$, find a KKT-point u^i by the local search method from [Strekalovsky&Gruzdeva, 2007, Strekalovsky, 2015].

Step 6. Choose the point u^j : $f(u^j) \leq \min\{f_0(u^i), \ i = 1, ..., N\}$.

Step 7. If $f_0(u^j) < f_0(z^k)$, then set $z_{k+1} = u^j$, $k = k + 1$ and go to Step 2.

Step 8. Otherwise, choose a new value of β and go to Step 3.

The point z^* resulting from the global search strategy will be a solution to the original fraction program [Gruzdeva&Strekalovsky, 2016]. It should be noted that, in contrast to the approach from Subsect. 2.1, α_i will be found simultaneously with the solution vector x, because they are variables, although auxiliary ones, of the optimization problem.

3 Computational Simulations

Two approaches from above for solving the fractional programs $(\mathcal{P}_f)$ via d.c. optimization problems were verified. The algorithm based on the method for solving fractional problem from Subsect. 2.1 (F1-algorithm) and the global search scheme from Subsect. 2.2 (F2-algorithm) were coded in C++ language and applied to an extended set of test examples generated randomly with up to 200 variables and 200 terms in the sum. All computational experiments were performed on the Intel Core i7-4790K CPU 4.0 GHz. All convex auxiliary problems (linearized problems) on the steps of F1-, F2-algorithms were solved by the software package IBM ILOG CPLEX 12.6.2.

Some of the computational experiments of testing the proposed algorithms on fractional problems with quadratic functions in the numerators of ratios is presented here.

We generated the problems of the type proposed by [Jong, 2013] as follows:

$$f(x) := \sum_{i=1}^{m} \frac{\langle x, A_i x \rangle + \langle b^i, x \rangle}{\langle c^i, x \rangle} \downarrow \min_x, \quad Qx \leq q, \quad x \in [1, 5]^n, \tag{2}$$

where $A_i = U_i D_{0i} U_i^T$, $U_i = V_1 V_2 V_3$, $i = 1, \ldots, n$; $V_j = I - 2\frac{w_j w_j^T}{\|w_j\|^2}$, $j = 1, 2, 3$; $w_1 = -i + rand(n, 1)$, $w_2 = -2i + rand(n, 1)$, $w_3 = -3i + rand(n, 1)$; $c_i = i - i \cdot rand(n, 1)$, $b_i = i + i \cdot rand(n, 1)$, $Q = -1 + 2 \cdot rand(5, n)$, $q = 2 + 3 \cdot rand(5, 1)$[Jong, 2013]. (We denote by $rand(k_1, k_2)$ the random matrix with k_1 rows, k_2 columns and the elements generated randomly on $[0, 1]$.)

Table 1: Randomly generated problems (2) with quadratic functions

n	m	$f(x_0)$	$f(z)$	Time F1	Time F2
10	10	36.738565	36.429457	1.73	2.22
10	50	161.613719	157.926620	10.98	18.49
10	100	303.448935	298.051190	20.90	122.03
10	200	609.211089	606.136790	38.68	501.31
50	10	33.151416	31.676804	4.05	124.25
50	50	156.152869	152.686192	292.17	918.91
50	100	304.067383	298.776803	179.12	2634.33
50	200	602.000116	588.454982	340.89	18132.53
100	10	31.809077	30.521716	104.81	1241.75
100	50	153.990301	149.812955	410.82	4986.00
100	100	302.938895	296.580627	492.11	13206.50
100	200	603.905018	591.954144	3678.78	55462.61
200	10	31.878970	30.021172	262.79	12878.94
200	50	152.632702	146.398920	1546.64	81156.93
200	100	306.462441	295.557105	2034.73	108858.27
200	200	603.988798	587.862890	9552.40	318381.50

Note that Problems $(\mathcal{P}_\alpha)$ corresponding to the problems (2) happened to be convex, because the constructed functions $\psi_i = \langle x, A_i x \rangle + \langle b^i, x \rangle$, $i = 1, \ldots, m$, are convex. That is why the F1-algorithm's run-time turned out to be much less than the run-time of F2-algorithm, in which it is necessary to solve nonconvex problems with m d.c. constraints.

The shortcoming of the F2-algorithm is the run-time which is spent on solving problems with d.c. constraints, but F1-algorithm wastes a lot of run-time for finding the vector parameter α at which the optimal value of Problem $(\mathcal{P}_\alpha)$ is zero.

The results of computational experiments suggest that the solving of the fraction programs need a combination of the two approaches. For example, we can use the solution to Problem $(\mathcal{P})$ to search for the parameter α that reduces the optimal value function of Problem $(\mathcal{P}_\alpha)$ to zero. This idea could be implemented by the following algorithm.

Two-component algorithm

Step 0. (Initialization) $k = 0,\ v^k = 0,\ \ u^k = \alpha^+$.

Step 1. Starting from feasible point x^k, implement the local search method from [Strekalovsky&Gruzdeva, 2007, Strekalovsky, 2015] to find a critical point $z^k = (\hat{z}(\alpha^k), \alpha_k)$ in d.c. constraints problem $(\mathcal{P})$.

Step 2. (Stopping criterion) If $\mathcal{V}_k := \mathcal{V}(\alpha^k) = 0$ and $\min\limits_i \Phi_i(\hat{z}(\alpha^k)) \geq 0$, then STOP: $\hat{z}(\alpha^k) \in Sol(\mathcal{P}_f)$.

Step 3. Find a solution $z(\alpha^k)$ to Problem $(\mathcal{P}_{\alpha^k})$ using the global search strategy for d.c. minimization [Strekalovsky, 2003, Strekalovsky, 2014].

Step 4. (Stopping criterion) If $\mathcal{V}_k := \mathcal{V}(\alpha^k) = 0$ and $\min\limits_i \Phi_i(z(\alpha^k)) \geq 0$, then STOP: $z(\alpha^k) \in Sol(\mathcal{P}_f)$.

Step 5. Implement α_k-bisection algorithm to find new parameters α^{k+1}, v^{k+1} and u^{k+1}.
Set $x^{k+1} = (z(\alpha^k), \alpha_{k+1})$, $k = k + 1$ and go to Step 1.

The two-component algorithm was tested on a lots of test examples and its run-time turned out to be less than the run-time of F1-algorithm.

4 Conclusions

In this paper, we showed how fractional programs can be solved by applying the Global Search Theory of d.c. optimization. The methods developed were substantiated and tested on an extended set of test examples generated randomly. In addition, we proposed a two-component technology based on both the global search algorithm for d.c. minimization problem and algorithm for solving d.c. constraints problem.

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