Hölder Continuity of Quasiconformal Mappings

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Abstract

We prove that every K-quasiconformal mapping w of the unit ball $\mathbb{B}^n$ onto a C^2-Jordan domain Ω is Hölder continuous with constant $\alpha = 2 - \frac{n}{p}$, provided that its weak Laplacean Δw is in $L^p(\mathbb{B}^n)$ for some $n/2 < p < n$. In particular it is Hölder continuous for every $0 < \alpha < 1$ provided that $\Delta w \in L^n(\mathbb{B}^n)$.

1 Introduction

B^n denotes the unit ball in $\mathbb{R}^n$, $n \geq 2$ and S^{n-1} denotes the unit sphere. Also we will assume that $n > 2$ (the case $n = 2$ has been already treated in [Kalaj & Pavlović, 2011]). We will consider the vector norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ and the matrix norms $|A| = \sup\{|Ax| : |x| = 1\}$.

A homeomorphism $u : \Omega \to \Omega'$ between two open subsets Ω and Ω' of Euclid space R^n will be called a K ($K \geq 1$) *quasi-conformal* or shortly a q.c mapping if

(i) u is absolutely continuous function in almost every segment parallel to some of the coordinate axes and there exist the partial derivatives which are locally L^n integrable functions on Ω. We will write $u \in ACL^n$ and

(ii) u satisfies the condition

$$|\nabla u(x)|^n/K \leq J_u(x) \leq K l(\nabla u(x))^n,$$

at almost everywhere x in Ω where

$$l(\nabla u(x)) := \inf\{|\nabla u(x)\zeta| : |\zeta| = 1\}$$

and $J_u(x)$ is the Jacobian determinant of u (see [Reshetnyak, 1968]).

Notice that, for a continuous mapping u the condition (i) is equivalent to the condition that u belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega)$.

Let P be Poisson kernel i.e. the function

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n},$$

and let G be the Green function i.e. the function

$$G(x, y) = c_n \begin{cases} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{(|\,x|y| - y/|y|\,|)^{n-2}}\right), & \text{if } n \geq 3; \\ \log \frac{|x-y|}{|1-x\bar{y}|}, & \text{if } n = 2 \text{ and } x, y \in \mathbb{C} \cong \mathbb{R}^2. \end{cases} \tag{1}$$

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where $c_n = \frac{1}{(n-2)\Omega_{n-1}}$, and Ω_{n-1} is the measure of S^{n-1}. Both P and G are harmonic for $|x| < 1$, $x \neq y$.

Let $f : S^{n-1} \to \mathbb{R}^n$ be a L^p, $p > 1$ integrable function on the unit sphere S^{n-1} and let $g : \mathbb{B}^n \mapsto \mathbb{R}^n$ be continuous. The weak solution of the equation (in the sense of distributions) $\Delta u = g$ in the unit ball satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by

$$u(x) = P[f](x) - G[g](x) := \int_{S^{n-1}} P(x,\eta)f(\eta)d\sigma(\eta) - \int_{B^n} G(x,y)g(y)dy, \tag{2}$$

$|x| < 1$. Here $d\sigma$ is Lebesgue $n-1$ dimensional measure of Euclid sphere satisfying the condition: $P[1](x) \equiv 1$. It is well known that if f and g are continuous in S^{n-1} and in $\overline{B^n}$ respectively, then the mapping $u = P[f] - G[g]$ has a continuous extension $\tilde{u}$ to the boundary and $\tilde{u} = f$ on S^{n-1}. If $g \in L^\infty$ then $G[g] \in C^{1,\alpha}(\overline{B^n})$. See [Gilbarg & Trudinger, 1983, Theorem 8.33] for this argument.

We will consider those solutions of the PDE $\Delta u = g$ that are quasiconformal as well and investigate their Lipschitz character.

A mapping f of a set A in Euclidean n-space $\mathbf{R}^n$ into $\mathbf{R}^n$, $n \geq 2$, is said to belong to the Hölder class $\mathrm{Lip}_\alpha(A)$, $\alpha > 0$, if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x-y|^\alpha \tag{3}$$

for all x and y in A. If D is a bounded domain in $\mathbf{R}^n$ and if f is quasiconformal in D with $f(D) \subset \mathbf{R}^n$, then f is in $\mathrm{Lip}_\alpha(A)$ for each compact $A \subset D$, where $\alpha = K_I(f)^{1/(1-n)}$ and $K_I(f)$ is the inner dilatation of f. Simple examples show that f need not be in $\mathrm{Lip}_\alpha(D)$ even when f is continuous in $\overline{D}$.

However O. Martio and R. Näkki in [Martio & Näkki, 1991] showed that if f induces a boundary mapping which belongs to $\mathrm{Lip}_\alpha(\partial D)$, then f is in $\mathrm{Lip}_\beta(D)$, where

$$\beta = \min(\alpha, K_I(f)^{1/(1-n)});$$

the exponent β is sharp.

In a recent paper of the first author and Saksman [Kalaj & Saksman, 2014] it is proved the following result, if f is quasiconformal mapping of the unit disk onto a Jordan domain with C^2 boundary such that its weak Laplacean $\Delta f \in L^p(\mathbb{B}^2)$, for $p > 2$, then f is Lipschitz continuous. The condition $p > 2$ is necessary also. Further in the same paper they proved that if $p = 1$, then f is absolutely continuous on the boundary of $\partial\mathbb{B}^2$.

We are interested in the condition under which the quasiconformal mapping is in $\mathrm{Lip}_\alpha(B^n)$, for every $\alpha < 1$. It follows form our results that the condition that u is quasiconformal and $|\Delta u| \in L^p$, such that $p > n/2$ guaranty that the selfmapping of the unit ball is in $\mathrm{Lip}_\alpha(B^n)$, where $\alpha = 2 - \frac{p}{n}$. In particular if $p = n$, then $f \in \mathrm{Lip}_\alpha(B^n)$ for $\alpha < 1$.

It should be noted that the topic is very active area of research in geometric function theory, and the following people have obtained some substantial results in this area: Pavlović and Kalaj, Mateljević, Partyka, Sakan ([Kalaj & Pavlović, 2011, Kalaj & Pavlović, 2005, Pavlović, 2002, Kalaj, 2011, Kalaj, 2012, Kalaj, 2013, Kalaj & Mateljevic, 2012, Kalaj & Mateljevic, 2011a, Kalaj & Mateljevic, 2011b, Kalaj & Mateljevic, 2006, Zhu & Kalaj, 2017, Kalaj, 2015, Partyka & Sakan, 2007]). The pioneering work on this topic have been done by Martio [Martio, 1968].

Our new result in several-dimensional case is the following:

Theorem 1 *Let $n \geq 2$ and let $p > n/2$ and assume that $g \in L^p(\mathbb{B}^n)$. Assume that w is a K-quasiconformal solution of $\Delta w = g$, that maps the unit ball onto a bounded Jordan domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary.*

- *If $p < n$, then w is Hölder continuous with the Hölder constant $\alpha = 2 - \frac{n}{p}$.*
- *If $p = n$, then w is Hölder continuous for every $\alpha \in (0,1)$.*

The sketch of the proof is given in the next section.

2 Sketch of the Proofs

In what follows, we say that a bounded Jordan domain $\Omega \subset \mathbb{R}^n$ has C^2-boundary if it is the image of the unit ball $\mathbb{B}^n$ under a C^2-diffeomorphism of the whole Euclidean space. For Jordan domains $\Omega \subset \mathbb{R}^n$ this is well-known to be equivalent to the more standard definition, that requires the boundary to be locally isometric to the graph of a C^2-function on $\mathbb{R}^{n-1}$. In what follows, Δ refers to the distributional Laplacian. We shall make use of the following well-known facts.

Proposition 2.1 (Morrey's inequality) *Assume that $n < p \leq \infty$ and assume that U is a domain in $\mathbf{R}^n$ with C^1 boundary. Then there exists a constant C depending only on n, p and U so that*

$$\|u\|_{C^{0,\alpha}(U)} \leq C\|u\|_{W^{1,p}(U)} \tag{4}$$

for every $u \in C^1(U) \cap L^p(U)$, where

$$\alpha = 1 - \frac{n}{p}.$$

Lemma 1 *See e.g.[Astala & Manojlovic, 2015]. Suppose that $w \in W^{2,1}_{loc}(\mathbb{B}^n) \cap C(\overline{\mathbb{B}^n})$, that $h \in L^p(\mathbb{B}^n)$ for some $1 < p < \infty$ and that*

$$\Delta w = h \ \text{ in } \mathbb{B}^n, \ \text{ with } w\big|_{\mathbb{S}^{n-1}} = 0,$$

a) If $1 < p < n$, then

$$\|\nabla w\|_{L^q(\mathbb{B}^n)} \leq c(p,n)\|h\|_{L^p(\mathbb{B}^n)}, \qquad q = \frac{pn}{n-p}.$$

b) If $p = n$ and $1 < q < \infty$ then

$$\|\nabla w\|_{L^q(\mathbb{B}^n)} \leq c(q,n)\|h\|_{L^n(\mathbb{B}^n)}.$$

c) if $p > n$, then

$$\|\nabla w\|_{L^\infty(\mathbb{B}^n)} \leq c(p,n)\|h\|_{L^n(\mathbb{B}^n)}.$$

Now we formulate the following fundamental result of Gehring

Proposition 2.2 *[Gehring, 1973] Let f be a quasiconformal mapping of the unit ball $\mathbb{B}^n$ onto a Jordan domain Ω with C^2 boundary. Then there is a constant $p = p(K,n) > n$ so that*

$$\int_{\mathbb{B}^n} |Df|^p < C(n, K, f(0), \Omega).$$

Now we formulate two lemmas, whose proofs are easily, but the details will be printed elsewhere

Lemma 2 *If $\Delta u = g \in L^p$ and $r < 1$, then $Du \in L^q(r\mathbb{B})$ for $q \leq \frac{np}{n-p}$.*

To prove Theorem 1, we need as well this lemma.

Lemma 3 *If $H : \mathbf{R}^n \to \mathbf{R}$ and $w = (w_1, \ldots, w_n) : A \to B$ (where A, B are open subsets in $\mathbf{R}^n$) are functions from C^2 class, then:*

$$\Delta(H \circ w) = \sum_{i=1}^{n} \frac{\partial^2 H}{\partial w_i^2} |\nabla w_i|^2 + 2 \sum_{1 \leq i < j \leq n} \frac{\partial^2 H}{\partial w_i \partial w_j} \langle \nabla w_i, \nabla w_j \rangle + \sum_{i=1}^{n} \frac{\partial H}{\partial w_i} \Delta w_i$$

Sketch of the proof of Theorem 1.

In addition to the previous propositions, the proof depends on an approach discovered in [Kalaj & Saksman, 2014]. Details will be published elsewhere.

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