Quasi- and Pseudo-pyramidal Tours
for Generalized Traveling Salesman Problem

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Abstract

In this paper, we introduce notions of l-quasi-pyramidal and l-pseudo-pyramidal tours extending the classic notion of pyramidal tour to the case of Generalized Traveling Salesman Problem (GTSP). We show that, for the instance of GTSP on n cities and k clusters with arbitrary weights, optimal l-quasi-pyramidal and l-pseudo-pyramidal tours can be found in time $O(4^l n^3)$ and $O(2^l k^{l+4} n^3)$, respectively. Consequently, we show, that, in the most general setting, GTSP belongs to FPT for parametrizations induced by these special kinds of tours. Also, we describe a non-trivial polynomially solvable subclass of GTSP, for which the existence of optimal l-quasi-pyramidal tour (for some fixed value of l) is proved.

1 Introduction

The Traveling Salesman Problem (TSP) is the famous combinatorial optimization problem having many valuable applications in operations research and attracting interest of scientists for decades (see, e.g. [Gutin & Punnen, 2007, Pardalos et al., 2013]).

It is known that TSP is strongly NP-hard and hardly approximable in its general setting [Sahni & Gonzales, 1976]. At the same time, the problem remains intractable in metric and Euclidean settings. TSP can be approximated well in these cases admitting fixed-ratio algorithms for an arbitrary metric [Christofides, 1975] and Polynomial Time Approximation Schemes (PTAS) for Euclidean spaces of any fixed dimension [Arora, 1998]. Many generalizations of TSP, e.g. Cycle Cover Problem [Gimadi & Rykov, 2016, Khachay & Neznakhina, 2016a, Khachai & Neznakhina, 2015], Peripatetic Salesman Problem [Baburin et al., 2009, Gimadi et al., 2014], have the similar approximation behaviour.

Algorithmic issues of finding optimal restricted tours, for several kinds of restrictions, e.g. precedence constraints, are also actively investigated (see.,e.g. [Balas, 1999, Balas & Simonetti, 2001, Chentsov et al., 2016]). Among others, the restriction of TSP to considering so called *pyramidal tours* (see, e.g. [Burkard et al., 1998]) seems to be especially popular. The pyramidal tour respects the initial order $\{1, \ldots, n\}$ defined on the nodeset of a given graph and has the form

$$1 = v_{i_1}, v_{i_2}, \ldots, v_{i_r} = n, v_{i_{r+1}}, \ldots, v_{i_n},$$

where $v_{i_j} < v_{i_{j+1}}$ for any $j \in \{1, \ldots, r-1\}$ and $v_{i_j} > v_{i_{j+1}}$ for any $j \in \{r+1, \ldots, n-1\}$. It is widely known [Klyaus, 1976] that an optimal pyramidal tour can be found in time of $O(n^2)$ for any weighting function. Recently it was shown [Berg et al., 2016] that, for the Euclidean setting, optimal pyramidal tour can be found in $O(n \log^2 n)$ time. In papers [Enomoto et al., 1998, Oda & Ota, 2001], several generalizations of pyramidal tours, for which an optimal tour also can be found efficiently, were introduced. Despite their fame, pyramidal tours have one shortcoming. Known settings of TSP and its generalizations, for which existence of optimal pyramidal tours is proven [Burkard & Deineko, 1998, Baki & Kabadi, 1999, Baki, 2006], remain very rare so far. Actually, they are mostly exhausted with settings satisfying the well known sufficient conditions by Demidenko and van der Veen (see, e.g. [Gutin & Punnen, 2007]).

Contribution of this paper is two-fold. At first, we introduce (in Section 2) notions of l-quasi-pyramidal and l-pseudo-pyramidal tours extending the classic notion of pyramidal tour and results of [Oda & Ota, 2001] to the case of Generalized Traveling Salesman Problem (GTSP). We show that optimal l-quasi-pyramidal and l-pseudo-pyramidal tours can be found in time $O(4^l n^3)$ and $O(2^l k^{l+4} n^3)$, respectively. At second, we describe (in Section 3) a non-trivial polynomially solvable subclass of GTSP, for which the existence of optimal l-quasi-pyramidal tour (for some fixed l) is proved.

2 Quasi- and Pseudo-pyramidal Tours

We proceed with the common setting of the Generalized Traveling Salesman Problem (GTSP). Instance of the GTSP is defined by complete edge-weighted graph $G = (V, E, w)$ with weighting function $w \colon E \to \mathbb{R}_+$, and by a given partition $V_1 \cup \ldots \cup V_k = V$ of the nodeset $V = V(G)$ of graph G. Feasible solutions are cyclic tours $v_{i_1}, \ldots, v_{i_k}$ visiting each cluster V_i once. Hereinafter, we call such routes *Clustered Hamiltonian tours* or *CH-tours*. The problem is to find a CH-tour of the minimum weight[1].

In this section, we extend the well-known notion of a pyramidal tour to the case of partial orders defined implicitly by the orderings of clusters. Indeed, linear ordered finite set $(V_1, \ldots, V_k)$ of clusters induces a partial order on the nodeset V of the graph G as follows, for any $u \in V_i$ and $v \in V_j$, $u \prec v$ if $i < j$.

Definition 1 *Let τ be a CH-tour $v_1, v_{i_1}, \ldots, v_{i_r}, v_k, v_{j_{k-r-2}}, \ldots, v_{j_1}$ such that $v_t \in V_t$ for any t. We call τ an l-quasi-pyramidal tour, if $i_p - i_q \leq l$ and $j_{p'} - j_{q'} \leq l$ for any $1 \leq p < q \leq r$ and $1 \leq p' < q' \leq k - r - 2$.*

The following theorems extend the results proposed in [Oda & Ota, 2001] for the classic TSP.

Theorem 1 *For any weighting function $w \colon E \to \mathbb{R}_+$, a minimum cost l-quasi-pyramidal CH-tour can be found in time of $O(4^l n^3)$.*

Remark 1 *Evidently, result of Theorem 1 can be considered in the context of the parameterized complexity. Actually, Theorem 1 claims that, in the most general setting, GTSP is fixed-parameter tractable with respect to parametrization induced by quasi-pyramidal tours.*

In Section 3, we describe a subclass of geometric GTSP, each whose instance has l-quasi-pyramidal optimal tours for some fixed l. Nevertheless, this class seems to be very specific, and the scheme proposed can hardly be extended to more general settings. To overcome this gap, we propose a more common notion of pyramidal-like tours. We call them *pseudo-pyramidal*.

Definition 2 *Let τ be a CH-tour $v_1, v_{i_1}, \ldots, v_{i_r}, v_k, v_{j_{k-r-2}}, \ldots, v_{j_1}$ such that $v_t \in V_t$ for any t. We call τ an l-pseudo-pyramidal tour, if $i_p - i_{p+1} \leq l$ and $j_q - j_{q+1} \leq l$ for any $1 \leq p \leq r - 1$ and $1 \leq q \leq k - r - 1$.*

It easy to verify that any l-quasi-pyramidal tour is an l-pseudo-pyramidal as well.

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[1] In this paper, we restrict ourselves to the case of undirected graphs. Although, the similar argument can be provided to the case of digraphs and asymmetric weighting functions w.

Theorem 2 *For any weighting function $w\colon E \to \mathbb{R}_+$, a minimum cost l-pseudo-pyramidal CH-tour can be found in time of $O(2^l k^{l+4} n^3)$.*

Remark 2 *As for Theorem 1, Theorem 2 states that, for any weighting function, GTSP belongs to FPT with respect to parameters k and l. Also, since $O(2^l (\log n)^{l+4} n^3) = 2^{O(l^2)} \cdot O(n^4)$, the problem has FPT algorithms with respect to parameter l only any time when $k = O(\log n)$.*

For the brevity, we skip proofs of Theorems 1 and 2, which will be come in forthcoming paper.

3   Polynomial Time Solvable Subclass of GTSP on Grid Clusters

In this section, we describe polynomially solvable subclass of Generalized Traveling Salesman Problem on Grid Clusters, GTSP-GC for short. In this special case of the GTSP, an undirected edge-weighted graph $G = (V, E, w)$ is given where the set of vertices V correspond to a set of points in the planar rectangular grid. Every nonempty 1×1 cell of the grid forms a cluster. The weighting function is induced by distances between the appropriate points with respect to some metric. To put it simple, we consider Euclidean distances, but the similar results can be easily obtained for some other metrics, e.g. for l_1. In Fig. 1, we present an instance of the Euclidean GTSP-GC with 6 clusters.
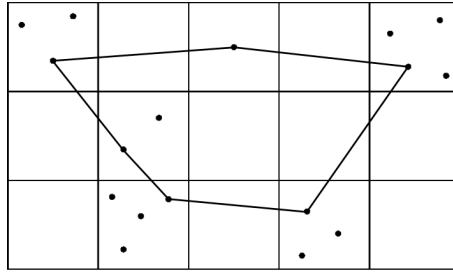

Figure 1: An instance of the Euclidean GTSP-GC and its optimal solution

It is known that for two special cases of the problem, when the number k of clusters is $O(logn)$ or $n - O(logn)$, polynomial time approximation schemes (PTAS) were proposed [Khachay & Neznakhina, 2016b]. Meanwhile, the question of systematic description of polynomial time solvable subclasses of GTSP-GC, which is closely related to complexity analysis of Hamiltonian cycle problem on grid graphs, is still far from its complete answer.

Let H and W be *height* and *width* (number of rows and columns) of the given grid, respectively. We consider a special case of the GTSP-GC, for which one of these parameters, say H does not exceed 2 (while another one is unbounded). We call this case GTSP-GC(H2). We show that any instance of GTSP-GC(H2) has an optimal l-quasi-pyramidal CH-tour for some l independent on n. Therefore, this subclass of GTSP-GC is polynomially solvable due to Theorem 1.

Our argument is based on the introduced *Tour straightening transformation* (Algorithm 1), which is closely related to the well-known class of *local search heuristics*. To describe the transformation, assign to columns of the grid defining the given instance of GTSP-GC(H2), integer numbers $1, 2, \ldots, W$ (from the left to the right). Consider an arbitrary CH-tour τ. Assigning to each node v_i of τ the number c_i of the column it belongs, obtain a sequence σ of column numbers presented in the order induced by the tour τ. Without loss of generality, assume that σ has the form

$$1 = c_1, c_2, \ldots, c_r = W, c_{r+1}, \ldots, c_s = 1 \tag{1}$$

for some appropriate numbers r and s.

Suppose, for some integer number t, whose value will be specified later, there exist indexes

$$1 \le p < q \le r, \quad \text{such that} \quad c_p - c_q \ge t - 1, \text{ or} \tag{2}$$

$$r + 1 \le p' < q' \le s, \quad \text{such that} \quad c_{q'} - c_{p'} \ge t - 1. \tag{3}$$

In this case, we say that the tour τ has *t-zigzag* (Fig. 2). Obviously, any l-quasi-pyramidal tour contains no t-zigzags, for any $t \ge l$. Algorithm 1 replaces all segments of the tour τ having t-zigzags with subtours of the special kind (see. Fig. 3).

Algorithm 1 Tour straightening transformation

Outer Parameter: t.
Input: an instance of GTSP-GC(H2) and a CH-tour τ.
Output: a CH-tour τ' without t-zigzags.

1: set $\tau' := \tau$
2: **while** τ' has t-zigzag **do**
3:     assume that equation (2) is valid (without loss of generality, we assume that $c_p - c_q = t - 1$), the case of (3) can be treated similarly;
4:     let C be the set of columns with numbers $c_q, \ldots, c_p$ (see Fig. 2);
5:     let $Y = (y_1, \ldots, y_{2t+4})$ be the ordinate sequence of nodes visited by τ in C augmented by ordinates of left and right crossing points;
6:     find an optimal 2-medians clustering for Y with medians m_1 and m_2;
7:     replace segments of the tour τ' belonging to C by horizontal lines at height m_1 and m_2 connected to all points mentioned in Step 5 by line segments (Fig. 3)
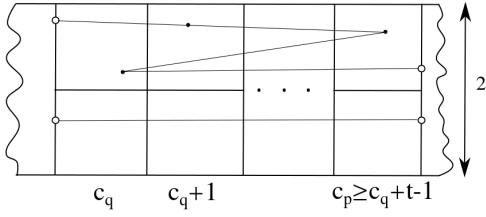8: **end while**
9: output the CH-tour τ'.
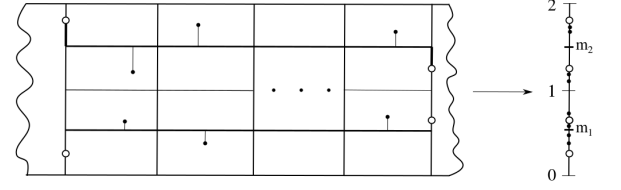---

Figure 2: Segment of τ with t-zigzag

Figure 3: Replacing t-zigzag with tour segments of special kind

Figure 4: Cluster ordering

To specify the value of t, notice that the weight of eliminated segments of τ has an evident lower bound

$$t + 2(t-1) + t - 2 = 4t - 4.$$

Meanwhile, the weight of their replacement in Step 7 at any iteration of Algorithm 1 is at most $2t + 2F(Y, [0, 2])$, where $F(Y, S)$ is an optimum value of 2-medians clustering objective function for a sample Y taken from a line segment S.

To estimate an upper bound for $F(Y, S)$ we need the following technical lemma, which can be easily proved by recurrent variable elimination method of linear programming.

Lemma 1 *For any sample $\xi = (p_1, \ldots, p_n)$, $p_i \in [0, 1]$ there exist numbers m_1 and $m_2 \in [0, 1]$ such that*

$$F(\xi, [0, 1]) = \sum_{i=1}^{n} \min\{|p_i - m_1|, |p_i - m_2|\} \le n/6. \tag{4}$$

Getting back to discussion of Algorithm 1, we obtain from Lemma 1 that $F(Y, [0, 2]) \le 2 \cdot 1/6(2t + 4)$. Therefore, at any iteration of Algorithm 1, the tour τ' becomes cheaper if $2t + 4t/3 + 8/3 \le 4t - 4$, i.e. $t \ge 10$.

Further, let cells of the grid be ordered as in Fig. 4 (i.e., top-down and left-right). For $t = 10$, any CH-tour of the given GTSP-GC(H2) instance can be transformed to l-quasi-pyramidal CH-tour for $l = 20$ without increasing its weight. Hence, we are proved the following theorem.

Theorem 3 *Any instance of GTSP-GC(H2) has an optimal 20-quasi-pyramidal CH-tour.*

As a consequence of Theorem 1 and Theorem 3, we obtain that GTSP-GC(H2) can be solved to optimality in time $O(n^3)$.

4   Conclusion

In this paper, a new notion of l-quasi-pyramidal and l-pseudo-pyramidal tours extending the classic notion of pyramidal tour is introduced. We show that, similar to the case of pyramidal tours and TSP, an optimal l-quasi-pyramidal tour for the Generalized Traveling Salesman Problem can be found efficiently (for an arbitrary weighting function). Also, we describe a non-trivial polynomially solvable geometric special case of GTSP. Each instance of the problem in question has an l-quasi-pyramidal tour as an optimal solution. Actually, an instance of this problem is defined by unit 2-row rectangular grid on the Euclidean plane. Although, the the trick with 2-medians can not be applied straightforward even to the case $h = 3$, we believe that we can prove soon the existence of optimal l-pseudo-pyramidal tours for the case of GTSP-GC(Hh) defined by a grid of an arbitrary fixed height h.

Acknowledgements

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