On Limit of Value Functions for Various Densities

Dmitry V. Khlopin
Krasovskii Institute of Mathematics and Mechanics
16, S.Kovalevskaja St.,
620990 Yekaterinburg, Russia.
Ural Federal University
4, Turgeneva St.,
620083, Yekaterinburg, Russia
khlopin@imm.uran.ru

Abstract

The paper is concerned with zero-sum differential games and the asymptotic properties of their value functions. The games with a common dynamics, a running cost, and capabilities of players are considered. Each payoff represents an average of the running cost with respect to the given discount functions (densities); these games differ in densities only. We prove a Tauberian-type Theorem, that is, the fact that the existence of a uniform limit of the value functions for uniform density or for exponential density implies that the value functions uniformly converge to the same limit for arbitrary piecewise continuous density as the time scale parameter tends to zero.

1 Introduction

In dynamic optimization, it is not uncommon to normalize the payoff by taking the average over time with respect to a certain probability distribution—for example, when the terminal time is large yet not exactly specified. In this case, for a realization of the process (a function $t \mapsto z(t)$), in addition to the running cost (a function $t \mapsto g(z(t))$), one also considers the payoff in the form of a certain average of the running cost,

$$\int_0^\infty \varrho(t)g(z(t))\,dt,$$

with respect to a certain discount function, a probability density function ϱ. Most often, when the problem is considered on infinite horizon, the potential infinity of the interval is emulated by considering the problems where the payoff is taken over increasingly large intervals $[0,T]$ or in view of increasingly small discounts λ; then, the limits of these problems are studied if such exist. Thus, effectively, for the payoffs

$$\int_0^\infty \lambda\varrho(\lambda t)g(z(\lambda t))\,dt, \tag{1}$$

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one considers the asymptotic behavior of the corresponding value functions as the scale parameter λ tends to zero. Usually, the densities of the uniform $\varrho(t) = 1_{[0,1]}(t)$ (Cezaro mean) and exponential $\varrho(t) = e^{-t}$ (Abel mean) distributions are applied.

The existence of such a limit of value functions in view of some density means that the value function's response to a change in the scale parameter λ is very weak when this parameter is sufficiently small. In particular, in the stochastic statement, this value (the asymptotic value) is customarily considered the game value when the planning horizon is infinite [Bewley et al., 1976]. In these statements, one could often obtain, in addition, an asymptotically optimal strategy such that a payoff is close to the optimal one (uniform value)—for sufficiently small values of the scale parameter [Mertens et al., 1981]; however, in this paper, we only consider the value function asymptotics. The existence of uniform limits of value functions for payoffs (1) when averaged with respect to the uniform and/or exponential densities was proved for a broad class of stochastic games [Ziliotto, 2016b], for optimal control problems [Gaitsgori, 1985, Lions et al., 1986, Grüne, 1998, Artstein et al., 2000, Li et al., 2016, Gaitsgori et al., 2013], and for certain classes of differential games [Buckdahn et al., 2011, Cannarsa et al., 2015] in the so-called nonexpansive-like case.

It turns out that, for stochastic games with a finite number of states and actions [Mertens et al., 1981], for discrete-time control problems [Lehrer et al., 1992], for general control problems [Oliu-Barton et al., 2013], for differential games [Khlopin, 2016], and a broad class of stochastic games [Ziliotto, 2016a, Ziliotto, 2016b], there holds the following Tauberian theorem: the uniform convergence of value functions under payoffs (1) with one of these densities (uniform or exponential) guarantees that the value functions in view of the other payoff also converge uniformly—to the same limit. The general approach, which deduces these Tauberian theorems from the Dynamic Programming Principle, is considered in [Khlopin, in print].

Such Tauberian theorems guarantee that if there is a uniform asymptotics for one of these densities (uniform or exponential) then, in addition to the value function's insensitivity to the choice of the discount parameter λ for payoff (1), this asymptotics is also insensitive to the choice between these two densities. Often, it is also possible to prove insensitivity to the choice of the density ϱ from quite a broad class.

Thus, we can find a sufficient asymptotic condition on the densities ϱ_λ, $\lambda > 0$, under which the uniform convergence of value functions as $\lambda \to 0$ for the payoffs with uniform or/and exponential densities (for Cesaro and/or Abel means) implies the uniform convergence to the same limit (as $\lambda \to 0$) of the value functions for the payoffs

$$\int_0^\infty \varrho_\lambda(t) g(z(t))\, dt. \tag{2}$$

For example, to this end, for discrete-time control processes, in paper [Monderer et al., 1993], the following sufficient asymptotic condition on a family of ϱ_λ, $\lambda > 0$ was proposed: all densities ϱ_λ are nonincreasing and

$$\lim_{\lambda\to 0} \int_0^T \varrho_\lambda(t)\, dt = 0 \qquad \forall T > 0. \tag{3}$$

For Markov decision processes, the sufficiency of the asymptotic condition

$$\lim_{\lambda\to 0} V_0^\infty[\varrho_\lambda] = 0 \tag{4}$$

was refined in [Ziliotto, 2016c]; here, $V_0^\infty[\mu]$ is the total variation of a real-valued function μ on $\mathbb{R}_+ \stackrel{\triangle}{=} [0,\infty)$.

For zero-sum differential games with the Isaacs condition, from the uniform convergence of the value functions for Cesaro means, it follows (see [Khlopin, 2015]) that the value functions with payoffs (2) converge to the same limit as $\lambda \to 0$ for a family of densities $\varrho_\lambda, \lambda > 0$ if this family enjoys (3) and

$$\limsup_{\lambda\to 0} V_0^{q[\varrho_\lambda](r)}[\ln \varrho_\lambda] < +\infty \qquad \forall r \in (0,1); \tag{5}$$

here, for each $r \in (0,1)$, the quantile $q[\varrho_\lambda](r)$ is the minimal solution of the equation $\int_0^{q[\varrho_\lambda](r)} \varrho_\lambda(t)dt = r$.

The main aim of this paper to prove that in zero-sum differential games with the Isaacs condition the existence of a uniform limit of the value functions for uniform density or for exponential density implies the uniform convergence (to the same limit) for the values in view of payoffs (1) for every piecewise continuous density ϱ.

To this end, first, we will improve condition (5) (see (12)), and, then, apply this improved sufficient asymptotic condition for payoffs (2). The cornerstone of this proof is the sufficiency of (3)&(5), proved in [Khlopin, 2015] for zero-sum differential games with Isaacs condition.

2 Differential Game

Consider a system in $\mathbb{R}^m$ controlled by two players,

$$\dot{x} = f(x,p,q),\ x(0) \in \mathbb{R}^m,\ t \geq 0,\ p(t) \in \mathbb{P},\ q(t) \in \mathbb{Q}; \tag{6}$$

here, $\mathbb{P}$ and $\mathbb{Q}$ are non-empty compact subsets of finite-dimensional Euclidean spaces.

Assume that the functions $f : \mathbb{R}^m \times \mathbb{P} \times \mathbb{Q} \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{P} \times \mathbb{Q} \to [0,1]$ are continuous, and let these functions be Lipschitz continuous in the state variable; namely, there exists a constant $L > 0$ such that, for all $x, y \in \mathbb{R}^m, p \in \mathbb{P}$, and $q \in \mathbb{Q}$,

$$\big|\big|f(x,p,q) - f(y,p,q)\big|\big| + \big|g(x,p,q) - g(y,p,q)\big| \leq L\big|\big|x - y\big|\big|.$$

Denote by $\mathcal{P}$ and $\mathcal{Q}$ the sets of all Borel measurable functions $\mathbb{R}_+ \ni t \mapsto p(t) \in \mathbb{P}$ and $\mathbb{R}_+ \ni t \mapsto q(t) \in \mathbb{Q}$, respectively. So, for each pair $(p,q) \in \mathcal{P} \times \mathcal{Q}$, for every initial condition $x(0) = x_* \in \mathbb{R}^m$, system (6) generates the unique solution $x(\cdot) = y(\cdot; x_*, p, q)$ defined for the whole $\mathbb{R}_+$.

We will essentially refer to the results proved in [Khlopin, 2015]. As a consequence, we need to admit all assumptions on differential games from [Khlopin, 2015]. To make it happen, we also impose the Isaacs condition ("the saddle point condition in a small game") [Krasovskii et al., 1988]

$$\max_{p\in\mathbb{P}}\min_{q\in\mathbb{Q}}\Big[\langle s, f(x,p,q)\rangle + g(x,p,q)\Big] = \min_{q\in\mathbb{Q}}\max_{p\in\mathbb{P}}\Big[\langle s, f(x,p,q)\rangle + g(x,p,q)\Big] \quad \forall x, s \in \mathbb{R}^m.$$

It is easy to see that, for each nonnegative function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$, it implies that, for all $t \in \mathbb{R}_+, x, s \in \mathbb{R}^m$,

$$\max_{p\in\mathbb{P}}\min_{q\in\mathbb{Q}}\Big[\langle s, f(x,p,q)\rangle + \varrho(t)g(x,p,q)\Big] = \min_{q\in\mathbb{Q}}\max_{p\in\mathbb{P}}\Big[\langle s, f(x,p,q)\rangle + \varrho(t)g(x,p,q)\Big]. \tag{7}$$

Let $\mathfrak{D}$ be the set of all probability density functions having their support in $\mathbb{R}_+$. For a density $\varrho \in \mathfrak{D}$ and a number $r \in (0,1)$, let the quantile $q[\varrho](r)$ be the minimum number such that

$$\int_0^{q[\varrho](r)} \varrho(t)\, dt = r.$$

For a given density $\varrho \in \mathfrak{D}$ and an initial position $x_* \in \mathbb{R}^m$, let the goal of the first player be to maximize the payoff function

$$c[\varrho](x_*, p, q) \triangleq \int_0^\infty \varrho(t) g(y(t; x_*, p, q), p(t), q(t))\, dt, \tag{8}$$

and let the task of the second one be to minimize it.

There are many ways to define a game and the sets of strategies for each player; for a very well made review encompassing a large number of formalizations, refer to [Subbotin, 1995, Subsect.14,15]. We will consider the nonanticipating strategies (see [Eliott et al., 1972]).

Definition 1 *A map $\alpha\colon \mathcal{Q} \to \mathcal{P}$ is called a nonanticipating strategy of the first player if, for all $t > 0$ and $q, q' \in \mathcal{Q}$, from $q|_{[0,t]} = q'|_{[0,t]}$ it follows that $\alpha[q]|_{[0,t]} = \alpha[q']|_{[0,t]}$.*

A map $\beta\colon \mathcal{P} \to \mathcal{Q}$ is called a nonanticipating strategy of the second player if, for all $t > 0$ and $p, p' \in \mathcal{P}$, from $p|_{[0,t]} = p'|_{[0,t]}$ it follows that $\beta[p]|_{[0,t]} = \beta[p']|_{[0,t]}$.

We denote by $\mathcal{A}$ and $\mathcal{B}$ the sets of all nonanticipating strategies of the first player and of the second player, respectively.

For each density $\varrho \in \mathfrak{D}$, define the corresponding value function by the following rule:

$$\mathbb{V}[\varrho](x_*) \triangleq \sup_{\alpha\in\mathcal{A}}\inf_{q\in\mathcal{Q}} \int_0^\infty \varrho(t) g\big(y\big(t; x_*, \alpha[q], q\big), \alpha[q](t), q(t)\big)\, dt \quad \forall x_* \in \mathbb{R}^m; \tag{9}$$

also, define

$$\mathbb{V}_+[\varrho](x_*) \triangleq \inf_{\beta\in\mathcal{B}}\sup_{p\in\mathcal{P}} \int_0^\infty \varrho(t) g\big(y\big(t; x_*, p, \beta[p]\big), p(t), \beta[p](t)\big)\, dt \quad \forall x_* \in \mathbb{R}^m.$$

For each density $\varrho \in \mathfrak{D}$ with bounded supp ϱ, condition (7) guarantees ([Krasovskii et al., 1988], [Subbotin, 1995], [Cardaliaguet et al., 2000]) the equality

$$\mathbb{V}_+[\varrho] \equiv \mathbb{V}[\varrho]. \tag{10}$$

In the general case, for each density $\varrho \in \mathfrak{D}$, define the sequence of densities $\varrho_n \stackrel{\triangle}{=} \frac{n+1}{n}\varrho \cdot 1_{[0,q[\varrho](\frac{n}{n+1})]} \in \mathfrak{D}$. Since supp ϱ_n is compact, passing to the limit as $n \to \infty$ in (8) and (10), we see that the payoffs $c[\varrho_n]$ converge to $c[\varrho]$ and the value functions $\mathbb{V}[\varrho_n] = \mathbb{V}_+[\varrho_n]$ converge to $\mathbb{V}[\varrho] = \mathbb{V}_+[\varrho]$ as $n \to \infty$. Now, we have proved (10) for all $\varrho \in \mathfrak{D}$.

3 The Main Result

For each density $\varrho \in \mathfrak{D}$ and an arbitrary $\lambda > 0$, it is also possible to introduce the density $\varrho^\lambda_{\text{scale}}$ by the rule

$$\varrho^\lambda_{\text{scale}}(t) = \lambda\varrho(\lambda t) \qquad \forall t \geq 0.$$

Set

$$\varpi_\lambda(t) = \lambda \cdot 1_{[0,1/\lambda]}, \quad \pi_\lambda(t) = \lambda \cdot e^{-\lambda t}, \qquad \forall \lambda > 0, t \geq 0.$$

Thus, we define the uniform and exponential density families.

For an interval $[a, b) \subset \mathbb{R}$ and a function $y : [a, b) \to \mathbb{R} \cup \{\infty\}$, denote by $V_a^b[y]$ the total variation of the function y in $[a, b)$.

Theorem 1 *Let a non-empty subset $\Omega \subset \mathbb{R}^m$ be strongly invariant with respect to system (6).*

For a given map $U_* : \Omega \to [0, 1]$, the following conditions are equivalent:

1) *for each piecewise continuous on $(0, \infty)$ density $\mu \in \mathfrak{D}$, there holds*

$$\lim_{\lambda\to 0} \sup_{x_*\in\Omega} \Big|\mathbb{V}[\mu^\lambda_{scale}](x_*) - U_*(x_*)\Big| = 0;$$

2) *the value functions $\mathbb{V}[\varpi_\lambda]$ converge to U_* uniformly in Ω as $\lambda \to 0$, i.e.,*

$$\lim_{\lambda\to 0} \sup_{x_*\in\Omega} \Big|\mathbb{V}[\varpi_\lambda](x_*) - U_*(x_*)\Big| = 0; \tag{11}$$

3) *the value functions $\mathbb{V}[\pi_\lambda]$ converge to U_* uniformly in Ω as $\lambda \to 0$, i.e.,*

$$\lim_{\lambda\to 0} \sup_{x_*\in\Omega} \Big|\mathbb{V}[\pi_\lambda](x_*) - U_*(x_*)\Big| = 0;$$

4) *for every family of densities $\mu_\lambda \in \mathfrak{D}, \lambda > 0$, it follows from (3) and*

$$\limsup_{\lambda\to 0} V_0^{q[\mu_\lambda](r)}[\mu] \cdot q[\mu_\lambda](r) < +\infty \qquad \forall r \in (0, 1) \tag{12}$$

that the value functions $\mathbb{V}[\mu_\lambda]$ converge to U_* uniformly in Ω as $\lambda \to 0$, i.e.,

$$\lim_{\lambda\to 0} \sup_{x_*\in\Omega} \Big|\mathbb{V}[\mu_\lambda](x_*) - U_*(x_*)\Big| = 0. \tag{13}$$

4 The Proof of Theorem 1

The implications $2) \Rightarrow 3), 3) \Rightarrow 2)$ were proved in [Khlopin, 2016].

From $(\varpi_1)^\lambda_{\text{scale}} = \varpi_\lambda$ for all positive λ, it follows that $1) \Rightarrow 2)$.

It remains to verify $2) \Rightarrow 4) \Rightarrow 1)$. To do this, we need the following proposition proved in [Khlopin, 2015]:

Proposition 1 *Assume that the value functions $\mathbb{V}[\varpi_\lambda]$, $\lambda > 0$ converge to a function U_* uniformly in Ω as $\lambda \to 0$, i.e., (11) holds.*

Let a family of $\mu_\lambda \in \mathfrak{D}, \lambda > 0$, satisfy (3) and (5).

Then, for all positive $\delta < 1$, there exists a positive λ_δ such that, for all positive $\lambda < \lambda_\delta$,

$$\mathbb{V}[\mu_\lambda](x_*) > U_*(x_*) - 8\delta \ln \frac{1}{\delta} \qquad \forall x_* \in \Omega.$$

4.1 The Proof of 2) $\Rightarrow$ 4).

The proof is by reductio ad absurdum. Assume the converse. Then, for a positive $\varepsilon < 1/20$, there exists a family of densities $\hat\mu_\lambda \in \mathfrak{D}, \lambda > 0$ such that (3), (12), and

$$\limsup_{\lambda\to 0} \sup_{x_*\in\Omega} |\mathbb{V}[\hat\mu_\lambda](x_*) - U_*(x_*)| \geq 3\varepsilon \tag{14}$$

hold. Choose positive $\delta < 1$ and M such that

$$8\delta \ln\frac{1}{\delta} < \varepsilon,\ \limsup_{\lambda\to 0} V_0^{q[\mu_\lambda](1-\varepsilon)}[\mu]\cdot q[\mu_\lambda](1-\varepsilon) < M.$$

Now, for all positive λ, define the mapping $\mu_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ by the following rule:

$$\mu_\lambda(t) = \hat\mu_\lambda(t) + \frac{\varepsilon}{q[\hat\mu_\lambda](1-\varepsilon)} \qquad \forall t \in [0, q[\hat\mu_\lambda](1-\varepsilon)]$$

and $\mu_\lambda(t) = 0$ otherwise. Then,

$$\int_0^\infty \mu_\lambda(t)dt = \int_0^{q[\hat\mu_\lambda](1-\varepsilon)} \mu_\lambda(t)dt = \int_0^{q[\hat\mu_\lambda](1-\varepsilon)} \hat\mu_\lambda(t)dt + \varepsilon = 1, \tag{15}$$

that is, $\mu_\lambda \in \mathfrak{D}$.

Now, since $\hat\mu_\lambda$ enjoys (3), we see that $q[\hat\mu_\lambda](1-\varepsilon)$ tends to ∞ as $\lambda \to 0$, and

$$\limsup_{\lambda\to 0} \int_0^T \mu_\lambda(t)\,dt \leq \lim_{\lambda\to 0}\int_0^T \hat\mu_\lambda(t)\,dt + \lim_{\lambda\to 0}\int_0^{\min\{T, q[\hat\mu_\lambda](1-\varepsilon)\}} \frac{\varepsilon}{q[\hat\mu_\lambda](1-\varepsilon)}\,dt = 0$$

holds for all positive T; thus, the family of μ_λ enjoys (3).

Next, note that, for all $x, y > 0$, there holds

$$|\ln x - \ln y| = \ln\frac{\max\{x,y\}}{\min\{x,y\}} \leq \frac{\max\{x,y\}}{\min\{x,y\}} - 1 = \frac{|x-y|}{\min\{x,y\}};$$

moreover, by (15), $q[\mu_\lambda](r) < q[\hat\mu_\lambda](1-\varepsilon)$ for all $\lambda > 0$ and $r \in (0,1)$. Then, we obtain

$$V_0^{q[\mu_\lambda](r)}[\ln\mu_\lambda] \leq V_0^{q[\hat\mu_\lambda](1-\varepsilon)}[\ln\mu_\lambda] \leq \frac{V_0^{q[\hat\mu_\lambda](1-\varepsilon)}[\mu_\lambda]}{\inf_{t\in[0,q[\hat\mu_\lambda](1-\varepsilon))}\mu_\lambda(t)} \leq \frac{V_0^{q[\hat\mu_\lambda](1-\varepsilon)}[\hat\mu_\lambda]\cdot q[\hat\mu_\lambda](1-\varepsilon)}{\varepsilon} < \frac{M}{\varepsilon}$$

for all $r \in (0,1)$, $\lambda > 0$. Thus, the family of μ_λ enjoys (5).

Since the family of μ_λ enjoys all assumptions of Proposition 1, we can find a positive λ_δ such that

$$\mathbb{V}[\mu_\lambda](x_*) > U_*(x_*) - 8\delta\ln\frac{1}{\delta} \qquad \forall x_* \in \Omega, \lambda \in (0, \lambda_\delta). \tag{16}$$

Consider a new differential game. Define the sets $\mathbb{P}^- \triangleq \mathbb{Q}, \mathbb{Q}^- \triangleq \mathbb{P}$ and the maps $f^-(x,p^-,q^-) \triangleq f(x,q^-,p^-)$, $g^-(x,p^-,q^-) \triangleq 1 - g(x,q^-,p^-)$ for all $x \in \mathbb{R}^m$, $p^- \in \mathbb{P}^-$, $q^- \in \mathbb{Q}^-$. By (7) with $s = -s^-$, we have

$$\max_{p^-\in\mathbb{P}^-}\min_{q^-\in\mathbb{Q}^-}\Big[\langle s^-, f^-(x,p^-,q^-)\rangle + \varrho(t)g^-(x,p^-,q^-)\Big] = \min_{q^-\in\mathbb{Q}^-}\max_{p^-\in\mathbb{P}^-}\Big[\langle s^-, f^-(x,p^-,q^-)\rangle + \varrho(t)g^-(x,p^-,q^-)\Big]$$

for all $s^-, x \in \mathbb{R}^m, t \geq 0, \varrho \in \mathfrak{D}$. Thus, the Isaacs condition also holds.

In addition, by $\mathcal{P}^- = \mathcal{Q}$ and $\mathcal{Q}^- = \mathcal{P}$, we obtain $y^-(x_*,p^-,q^-) \triangleq y(x_*,q^-,p^-)$ for all $x_* \in \mathbb{R}^m$, $p^- \in \mathcal{P}^-$, $q^- \in \mathcal{Q}^-$. Then, Ω is a strongly invariant set for this dynamics.

Moreover, thanks to $\mathcal{Q}^- = \mathcal{P}$ and $\mathcal{A}^- = \mathcal{B}$, for each density $\varrho \in \mathfrak{D}$, the map

$$\begin{aligned}
\Omega \ni x_* \mapsto 1 - \mathbb{V}[\varrho](x_*) &= 1 - \mathbb{V}_+[\varrho](x_*) \\
&= \sup_{\beta\in\mathcal{B}}\inf_{p\in\mathcal{P}}\int_0^\infty \varrho(t)\Big(1 - g\big(y(t;x_*,p,\beta[p]), p(t), \beta[p](t)\big)\Big)\,dt \\
&= \sup_{\alpha^-\in\mathcal{A}^-}\inf_{q^-\in\mathcal{Q}^-}\int_0^\infty \varrho(t)\Big(1 - g\big(y^-\big(t;x_*,\alpha^-[q^-],q^-\big), q^-(t), \alpha^-[q^-](t)\big)\Big)\,dt \\
&= \sup_{\alpha^-\in\mathcal{A}^-}\inf_{q^-\in\mathcal{Q}^-}\int_0^\infty \varrho(t)g^-\Big(y^-\big(t;x_*,\alpha^-[q^-],q^-\big), \alpha^-[q^-](t), q^-(t)\Big)\,dt
\end{aligned}$$

is the value function (9) of the new game. In particular, (11) holds for this game with asymptotics $U_*^- \equiv 1 - U_*$.

Applying Proposition 1 for this game, we can choose a positive λ_δ^- such that

$$1 - \mathbb{V}[\mu_\lambda](x_*) > 1 - U_*(x_*) - 8\delta \ln \frac{1}{\delta} \qquad \forall x_* \in \Omega$$

holds for all positive $\lambda < \lambda_\delta^-$. Together with (16), it implies

$$\limsup_{\lambda \to 0} \sup_{x_* \in \Omega} \Big| \mathbb{V}[\mu_\lambda](x_*) - U_*(x_*) \Big| \le 8\delta \ln \frac{1}{\delta} < \varepsilon.$$

Then, thanks to (14), we obtain

$$\limsup_{\lambda \to 0} \sup_{x_* \in \Omega} \Big| \mathbb{V}[\hat{\mu}_\lambda](x_*) - \mathbb{V}[\mu_\lambda](x_*) \Big| > 2\varepsilon. \tag{17}$$

However,

$$\int_0^\infty |\mu_\lambda(t) - \hat{\mu}_\lambda(t)| dt = \varepsilon + \int_{q[\hat{\mu}_\lambda](1-\varepsilon)}^\infty \hat{\mu}_\lambda(t) dt = 2\varepsilon,$$

therefore, by $0 \le g \le 1$, for all $x_* \in \Omega$, $\alpha \in \mathcal{A}, q \in \mathcal{Q}, \lambda > 0$, we have

$$\int_0^\infty |\mu_\lambda(t) - \hat{\mu}_\lambda(t)| g(y(t; x_*, \alpha[q](t), q(t)), \alpha[q](t), q(t)) dt \le 2\varepsilon,$$

$$\Big| \inf_{q \in \mathcal{Q}} \int_0^\infty \mu_\lambda(t) g(y(t; x_*, \alpha[q](t), q(t)), \alpha[q](t), q(t)) dt -$$

$$- \inf_{q \in \mathcal{Q}} \int_0^\infty \hat{\mu}_\lambda(t) g(y(t; x_*, \alpha[q](t), q(t)), \alpha[q](t), q(t)) dt \Big| \le 2\varepsilon$$

$$\Big| \mathbb{V}[\mu_\lambda](x_*) - \mathbb{V}[\hat{\mu}_\lambda](x_*) \Big| \le 2\varepsilon.$$

We obtain a contradiction with (17). This contradiction proves the implication 2) $\Rightarrow$ 4).

4.2 The Proof of 4) $\Rightarrow$ 1)

Consider a piecewise continuous in $(0, \infty)$ density ϱ. Fix a number $\varepsilon > 0$; now, there exists a sufficiently large natural $n > 3$ such that $\frac{5}{n} < \varepsilon$. For all $n \in \mathbb{N}, n > 3$, set

$$r_n \triangleq q[\varrho](1/n), \quad s_n \triangleq q[\varrho](1 - 1/n).$$

Since the piecewise continuous function ϱ is Riemann integrable on $[r_n, s_n]$, there exists a staircase function $\mu_n : \mathbb{R}_+ \to \mathbb{R}$, $\operatorname{supp} \mu_n \subset [r_n, s_n]$ such that $\int_{r_n}^{s_n} \mu_n(t)\, dt = \int_{r_n}^{s_n} \varrho(t)\, dt = \frac{n-2}{n}$, $\int_{r_n}^{s_n} |\mu_n(t) - \varrho(t)|\, dt < 1/n$ hold. In particular, its total variation in $[0, \infty)$ is finite. Since $\int_0^\infty \mu_n(t) dt = \frac{n-2}{n}$, put

$$\bar{\mu}_n \triangleq \frac{n}{n-2} \mu_n \in \mathfrak{D}, \qquad M \triangleq s_n V_0^\infty[\bar{\mu}_n] = \frac{n s_n}{n-2} V_0^\infty[\mu_n] \in \mathbb{R}.$$

Now, for all $\lambda > 0$, we have

$$\begin{aligned} V_0^\infty\big[(\bar{\mu}_n)^\lambda_{\text{scale}}\big] \cdot q\big[(\bar{\mu}_n)^\lambda_{\text{scale}}\big](1-\varepsilon) &= \lambda V_0^\infty[\bar{\mu}_n] \frac{q[\bar{\mu}_n](1-\varepsilon)}{\lambda} = V_0^\infty[\bar{\mu}_n] q[\bar{\mu}_n](1-\varepsilon) \le s_n V_0^\infty[\bar{\mu}_n] = M, \quad (18) \\ \int_0^\infty \Big| \varrho^\lambda_{\text{scale}}(t) - (\bar{\mu}_n)^\lambda_{\text{scale}}(t) \Big| dt &= \int_0^\infty \Big| \varrho(t) - \bar{\mu}_n(t) \Big| dt < \frac{3}{n} + \int_{r_n}^{s_n} \Big(\frac{n}{n-2} - 1 \Big) \varrho(t)\, dt = \frac{5}{n} < \varepsilon. \end{aligned}$$

Thus, we have

$$\sup_{x_* \in \Omega} \big| \mathbb{V}[\varrho^\lambda_{\text{scale}}](x_*) - \mathbb{V}[(\bar{\mu}_n)^\lambda_{\text{scale}}](x_*) \big| \le \varepsilon \qquad \forall \lambda > 0. \tag{19}$$

Consider some positive T. For all positive $\lambda < r_n/T$, we have $\bar{\mu}_n|_{[0,\lambda T]} \equiv 0$ and

$$\int_0^T (\bar{\mu}_n)^\lambda_{\mathrm{scale}}(t)\, dt = \lambda \int_0^T \bar{\mu}_n(\lambda t)\, dt = 0.$$

Thus, (3) holds for densities $(\bar{\mu}_n)^\lambda_{\mathrm{scale}}, \lambda > 0$.

Thanks to (18), the densities $(\bar{\mu}_n)^\lambda_{\mathrm{scale}}, \lambda > 0$ also satisfy (12). Applying condition 4) for this family, we give

$$\lim_{\lambda\to 0} \sup_{x_*\in\Omega} |\mathbb{V}[(\bar{\mu}_n)^\lambda_{\mathrm{scale}}](x_*) - U_*(x_*)| = 0.$$

Accounting for (19), we obtain

$$\limsup_{\lambda\to 0} \sup_{x_*\in\Omega} |\mathbb{V}[\varrho^\lambda_{\mathrm{scale}}](x_*) - U_*(x_*)| \le \varepsilon.$$

Since the choice of a positive number ε was arbitrary, the implication 4) ⇒ 1) is proved. □

5 Conclusion

Based on the results from [Khlopin, 2015], for differential games with the Isaacs condition, we managed to prove that their value functions' insensitivity to the choice of the scale parameter for the exponential or uniform distribution family implies the same result with respect to all densities of a relatively general form. Apparently, it can be proved for all dynamic games by the game value map method (similar [Khlopin, in print]), however, this should be investigated further.

In addition, we propose a new condition (3)&(12), which is sufficient for (13). For control systems on a compact invariant set under the non-expansive dynamics assumptions, a more weak (than (3)&(5) or (4)) sufficient asymptotic condition was proposed in paper [Li et al., 2016]. Their tightness under the non-expansive dynamics assumption for control problems and for differential games remains to be tested. Whether asymptotic condition (4) is sufficient in the deterministic framework is likewise unknown.

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