Constructions of Input and Output Isoquants for Nonconvex Multidimensional Models

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Abstract

In this paper, we develop methods for frontier visualization of nonconvex Free Disposal Hull (FDH) models. Our approach is based on constructions of input and output isoquants for multidimensional nonconvex frontier. Our theoretical results are confirmed by computational experiments using real-life data sets from different areas.

1 Introduction

Free Disposal Hull (FDH) models were proposed by Deprins, Simar and Tulkens [Deprins et al., 1984]. In these models the production possibility set is a nonconvex one. For this reason, it is required to develop special methods in order to calculate different characteristics of production unit's behavior. These methods are divided into two groups. In the first group, methods are based on mathematical programming (MP) approach. The second group of methods involved enumeration algorithms.

Kerstens and Vanden Eeckaut [Kerstens & Vanden, 1999] proposed a method for the estimation returns to scale (RTS) of decision making units in FDH models. In their method, one has to solve mixed integer nonlinear programming problems and to compare related efficiency scores. In paper [Podinovski, 2004] a method was proposed where one has to solve mixed integer linear programming problems instead of nonlinear ones. However, the size of LP-models is increased significantly. In the paper [Soleimani-damaneh et al., 2006] enumeration algorithms were proposed for estimating the RTS in FDH models. In the paper [Leleu, 2006], it was proposed to use linear programming framework in estimating RTS in FDH models.

However, Cesaroni, Kerstens and Van de Woestyne [Cesaroni et al., 2017] noted the absence of papers devoted to methods for frontier visualization in FDH models.

In this paper, we develop methods for frontier reconstruction in FDH models. Our approach is based on methods that were proposed for convex DEA models [Krivonozhko et al., 2004]. We propose methods for constructions of input and output isoquants for multidimensional frontier using both optimization and enumeration methods. Our theoretical results are confirmed by computational experiments using real-life data sets from different areas.

2 Background

Consider a set of n observations of actual production units (X_j, Y_j), $j = 1, \ldots, n$, where the vector of outputs $Y_j = (y_{1j}, \ldots, y_{rj}) > 0$ is produced from the vector of inputs $X_j = (x_{1j}, \ldots, x_{mj}) > 0$. The traditional Free

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Disposal Hull technology proposed by Deprins, Simar and Tulkens [Deprins et al., 1984] is formulated as follows:

$$T^{FDH} = \Big\{(X,Y) \;\Big|\; \sum_{j=1}^{n} X_j, \lambda_j \leq X,\, \sum_{j=1}^{n} Y_j, \lambda_j \geq Y,\, \sum_{j=1}^{n} \lambda_j = 1,\, \lambda_j \in \{0,1\},\, j = 1,\dots,n\Big\}, \tag{1}$$

where λ_j, $j = 1,\dots,n$ are integer variables, taking on values 0 or 1.

The FDH input-oriented model for evaluating unit (X_o, Y_o) under variable RTS assumption is written as follows

$$\begin{aligned} \theta_o^{FDH} = \; & \min \theta \\ & \textstyle\sum_{j=1}^{n} X_j \lambda_j \leq \theta X_o, \\ & \textstyle\sum_{j=1}^{n} Y_j \lambda_j \geq Y_o, \\ & \textstyle\sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0,1\},\; j = 1,\dots,n, \end{aligned} \tag{2}$$

where λ_j, $j = 1,\dots,n$ are integer variables, taking on values 0 or 1.

Define the two-dimensional plane in space E^{m+r} as

$$\mathrm{Pl}(X_o, Y_o, d_1, d_2) = (X_o, Y_o) + \alpha d_1 + \beta d_2, \tag{3}$$

where $(X_o, Y_o) \in T$, α and β are any real numbers, directions $d_1, d_2 \in E^{m+r}$, and d_1 is not parallel to d_2. The plane (3) goes through point (X_o, Y_o) in E^{m+r} and is spanned by vectors d_1 and d_2.

Next, define two intersections of the frontier with two-dimensional planes

$$\mathrm{Sec}_1(X_o, Y_o, e_p, e_s) = \big\{(X,Y) \;\big|\; (X,Y) \in \mathrm{Pl}(X_o, Y_o, d_1, d_2) \cap \mathrm{WEff}_P\, T, d_1 = (e_p, 0), d_2 = (e_s, 0)\big\}, \tag{4}$$

$$\mathrm{Sec}_2(X_o, Y_o, g_1, g_2) = \big\{(X,Y) \;\big|\; (X,Y) \in \mathrm{Pl}(X_o, Y_o, g_1, g_2) \cap \mathrm{WEff}_P\, T, g_1 = (0, \bar{e}_q), g_2 = (0, \bar{e}_t)\big\}, \tag{5}$$

where e_p and e_s are m-identity vectors with a one in positions p and s, respectively, $\bar{e}_q$ and $\bar{e}_t$ are r-identity vectors with a one in positions q and t, respectively. Here $\mathrm{WEff}_P\, T$ is a set of weakly efficient points of set T^{FDH}. Using the same approach as in [Krivonozhko et al., 2004], we can prove that set $\mathrm{WEff}_P\, T$ coincides with set $\mathrm{Bound}\, T$, where $\mathrm{Bound}\, T$ denotes the set of boundary points of T^{FDH}.

By taking different directions d_1 and d_2, g_1 and g_2, we can obtain various sections going through unit (X_o, Y_o) and cutting the frontier. Moreover, we can construct the curves generalizing the well-known functions in macro- and microeconomics. Section (4) is a generalized input isoquant, and section (5) is a generalized output isoquant.

3 Main Results

Now, consider an optimization algorithm for construction of the generalized input isoquant (4) for unit (X_o, Y_o). This isoquant is determined by directions $e_p \in E^m$ and $e_s \in E^m$, where e_p and e_s are unity vectors.

Algorithm 1.

Step 1. Find a leftmost point on the input isoquant going through point (X_o, Y_o) and determined by directions $e_p \in E^m$ and $e_s \in E^m$, where e_p and e_s are unity vectors.

$$\begin{aligned} & \min \theta \\ & \textstyle\sum_{j=1}^{n} x_{pj} \lambda_j \leq \theta x_{po}, \\ & \textstyle\sum_{j=1}^{n} x_{sj} \lambda_j \leq \tau x_{so}, \\ & \textstyle\sum_{j=1}^{n} x_{kj} \lambda_j \leq x_{ko}, k = 1,\dots,m, k \neq p, k \neq s, \\ & \textstyle\sum_{j=1}^{n} Y_j \lambda_j \geq Y_o, \\ & \textstyle\sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0,1\},\, j = 1,\dots,n, \\ & \tau \text{ is a free variable.} \end{aligned} \tag{6}$$

Set $\alpha_1 = \theta^*$, $l = 1$.

Step 2. Find two adjacent angular points on the input isoquant. Solve the following two optimization problems.

a) Problem A:

$$\begin{array}{l}\min \eta \\ \sum_{j=1}^{n} x_{pj}\lambda_j \le \alpha_l x_{po}, \\ \sum_{j=1}^{n} x_{sj}\lambda_j \le \eta x_{so}, \\ \sum_{j=1}^{n} x_{kj}\lambda_j \le x_{ko}, k = 1,\dots,m, k \ne p, k \ne s, \\ \sum_{j=1}^{n} Y_j\lambda_j \ge Y_o, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \in \{0,1\},\ j = 1,\dots,n. \end{array} \tag{7}$$

Set $\beta_l = \eta^*$.

b) Problem B:

$$\begin{array}{l}\min \theta \\ \sum_{j=1}^{n} x_{pj}\lambda_j \le \theta x_{po}, \\ \sum_{j=1}^{n} x_{sj}\lambda_j \le \tau x_{so}, \\ \sum_{j=1}^{n} x_{kj}\lambda_j \le x_{ko}, k = 1,\dots,m, k \ne p, k \ne s, \\ \sum_{j=1}^{n} Y_j\lambda_j \ge Y_o, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \tau \le \beta_l(1-\varepsilon), \\ \lambda_j \in \{0,1\},\ j = 1,\dots,n, \\ \tau \text{ is a free variable.} \end{array} \tag{8}$$

Here ε is a small parameter.

Set $l = l + 1$.

If the solution of problem (8) is infeasible, then $\alpha_l = M$, where M is a large number, go to Step 3.

Else $\alpha_l = \theta^*$, go to the beginning of Step 2.

Step 3. Stop.

Points (α_l, β_l), $l = 1,\dots,L$, are angular points of the stepwise input isoquant of FDH model for unit (X_o, Y_o) with directions e_p and e_s, where L is a number of angular points of input isoquant.

Next, proceed to description of an algorithm for construction of stepwise output isoquant (5) determined by directions $\bar{e}_q \in E^r$ and $\bar{e}_t \in E^r$, where $\bar{e}_q$ and $\bar{e}_t$ are unity vectors with a one in position q and t, respectively.

Algorithm 2.

Step 1. Find a leftmost point on the output isoquant going through point (X_o, Y_o). For this purpose solve the following optimization problem

$$\begin{array}{l}\max \eta \\ \sum_{j=1}^{n} X_j\lambda_j \le X_o, \\ \sum_{j=1}^{n} y_{ij}\lambda_j \ge y_{io}, i = 1,\dots,r, i \ne q, i \ne t, \\ \sum_{j=1}^{n} y_{tj}\lambda_j \ge \eta y_{to}, \\ \sum_{j=1}^{n} y_{qj}\lambda_j \ge \tau y_{qo}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \in \{0,1\},\ j = 1,\dots,n, \\ \tau \text{ is a free variable.} \end{array} \tag{9}$$

Set $\alpha_1 = 0$, $\beta_1 = \eta^*$, $l = 1$.

Step 2. Find two adjacent points on the output isoquant. Solve the following two optimization problems.

a) Problem A:

$$\begin{array}{l}\max \theta \\ \sum_{j=1}^{n} X_j\lambda_j \le X_o, \\ \sum_{j=1}^{n} y_{ij}\lambda_j \ge y_{io}, i = 1,\dots,r, i \ne q, i \ne t, \\ \sum_{j=1}^{n} y_{tj}\lambda_j \ge \beta_l y_{to}, \\ \sum_{j=1}^{n} y_{qj}\lambda_j \ge \theta y_{qo}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \in \{0,1\},\ j = 1,\dots,n. \end{array} \tag{10}$$

Set $l = l + 1$, $\alpha_l = \theta^*$.

b) Problem B:

$$\begin{array}{l} \max \eta \\ \sum_{j=1}^{n} X_j \lambda_j \leq X_o, \\ \sum_{j=1}^{n} y_{ij} \lambda_j \geq y_{io}, i = 1, \dots, r, i \neq q, i \neq t, \\ \sum_{j=1}^{n} y_{tj} \lambda_j \geq \eta y_{to}, \\ \sum_{j=1}^{n} y_{qj} \lambda_j \geq \tau y_{qo}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \tau \geq \alpha_l (1+\varepsilon), \\ \lambda_j \in \{0,1\},\ j = 1, \dots, n, \\ \tau \text{ is a free variable.} \end{array} \tag{11}$$

Here ε is a small parameter.

If the solution of Problem B (11) is infeasible, then $\beta_l = 0$, go to Step 3.

Else $\beta_l = \eta^*$, go to the beginning of Step 2.

Step 3. Stop.

Points (α_l, β_l), $l = 1, \dots, L$, are angular points of the output stepwise isoquant of FDH model for unit (X_o, Y_o) with directions $\bar{e}_q$ and $\bar{e}_t$, where L is a number of angular points of output isoquant.

Main steps of Algorithm 2 are explained on the Figure 1.

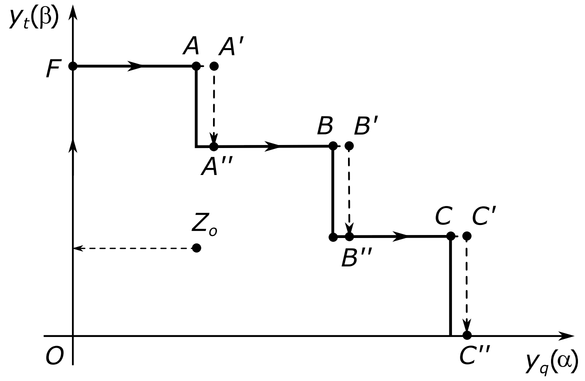

Figure 1: Construction of stepwise output isoquant for the FDH model

At Step 1 of the algorithm some leftmost point F on the isoquant is found, where $F = (0, \eta^*)$. At Step 2a an angular point $A = (\alpha_1, \beta_1)$ is calculated using the solution of Problem A (10). Thus, an angular point A on the curve is determined. Then, algorithm moves slightly out the production possibility set and takes point A'. After this the algorithm moves the current point parallel to horizontal line until this point reaches the feasible point A''. The algorithm stops if it discovers an infinite horizontal line or, in other words, the solution of Problem B at some iteration will be infeasible.

Now, we dwell on algorithms for construction of stepwise input and output isoquant for the nonconvex FDH model using enumeration approach.

Again, let input isoquant for unit (X_k, Y_k) be determined by directions $e_p \in E^m$ and $e_s \in E^m$, where e_p and e_s are unity vectors with a one in positions p and s, respectively.

Algorithm 3.

Step 1. Determine set

$$D_{ps}(k) = \Big\{ j \,\Big|\, x_{ij} \leq x_{ik},\ i = 1, \dots, m,\ i \neq p,\ i \neq s,\ y_{ij} \geq y_{ik},\ i = 1, \dots, r,\ j = 1, \dots, n \Big\}. \tag{12}$$

Let

$$\alpha_1^p = \min_{j \in D_{ps}(k)} \frac{x_{pj}}{x_{pk}}, \quad \beta_1^s = \min_{\substack{j \in D_{ps}(k) \\ x_{pj} = \alpha_1^p}} \frac{x_{sj}}{x_{sk}}. \tag{13}$$

Determine a vertical ray of isoquant from point (α_1^p, β_1^s)

$$S = \big\{ (\alpha_1^p, \beta_1^s) + \gamma(0,1),\ \gamma \geq 0 \big\}. \tag{14}$$

Let $l = 1$.

Step 2. While $D_{(l+1)} = \left\{ j \;\middle|\; \dfrac{x_{pj}}{x_{pk}} > \alpha_l^p,\, \dfrac{x_{sj}}{x_{sk}} < \beta_l^s,\, j \in D_{ps}(k) \right\} \neq \emptyset$

do

$$\alpha_{(l+1)}^p = \min_{j \in D_{(l+1)}} \frac{x_{pj}}{x_{pk}} \tag{15}$$

Add a horizontal and vertical segments to the isoquant

$$S = S \cup \left[(\alpha_l^p, \beta_l^s), (\alpha_{(l+1)}^p, \beta_l^s) \right], \tag{16}$$

$$\beta_{(l+1)}^s = \min_{\substack{j \in D_{(l+1)} \\ x_{pj} = \alpha_{(l+1)}^p}} \frac{x_{sj}}{x_{sk}}, \tag{17}$$

$$S = S \cup \left[(\alpha_{(l+1)}^p, \beta_l^s), (\alpha_{(l+1)}^p, \beta_{(l+1)}^s) \right], \tag{18}$$

$$l = l + 1. \tag{19}$$

Step 3. Add a horizontal ray to the isoquant

$$S = S \cup \left\{ (\alpha_l^p, \beta_l^s) + \gamma(1, 0),\, \gamma \geq 0 \right\}. \tag{20}$$

Step 4. $Sec(X_k, Y_k, e_p, e_s) = S$. Construction of the isoquant is completed.

Points (α_l^p, β_l^s), $l = 1, \ldots, L$ are angular points of the input isoquant, where L is a number of these points computed by the algorithm.

Next, we proceed to construction of output stepwise isoquant using enumeration methods. Let this output isoquant be determined by production unit (X_k, Y_k) and directions $\bar{e}_q \in E^r$ and $\bar{e}_t \in E^r$, respectively.

Algorithm 4.

Step 1. Determine set

$$D_{qt}(k) = \left\{ j \;\middle|\; x_{ij} \leq x_{ik},\, i = 1, \ldots, m,\, y_{ij} \geq y_{ik},\, i = 1, \ldots, r,\, i \neq q,\, i \neq t,\, j = 1, \ldots, n \right\}. \tag{21}$$

Let

$$\beta_1^t = \max_{j \in D_{qt}(k)} \frac{y_{tj}}{y_{tk}}, \quad \alpha_1^q = \max_{\substack{j \in D_{qt}(k) \\ y_{tj} = \beta_1^t}} \frac{y_{qj}}{y_{qk}}. \tag{22}$$

Determine the first segment of the output isoquant

$$S = \left[(0, \beta_1^t), (\alpha_1^q, \beta_1^t) \right]. \tag{23}$$

Let $l = 1$.

Step 2. While $D_{(l+1)} = \left\{ j \;\middle|\; \dfrac{y_{qj}}{y_{qk}} > \alpha_l^q,\, \dfrac{y_{tj}}{y_{tk}} < \beta_l^t,\, j \in D_{qt}(k) \right\} \neq \emptyset$

do

$$\beta_{(l+1)}^t = \max_{j \in D_{(l+1)}} \frac{y_{tj}}{y_{tk}} \tag{24}$$

Add a horizontal and vertical segments to the output isoquant

$$S = S \cup \left[(\alpha_l^q, \beta_l^t), (\alpha_l^q, \beta_{(l+1)}^t) \right], \tag{25}$$

$$\alpha_{(l+1)}^q = \max_{\substack{j \in D_{(l+1)} \\ y_{tj} = \beta_{(l+1)}^t}} \frac{y_{qj}}{y_{qk}}, \tag{26}$$

$$S = S \cup \left[(\alpha_l^q, \beta_{(l+1)}^t), (\alpha_{(l+1)}^q, \beta_{(l+1)}^t) \right], \tag{27}$$

$$l = l + 1. \tag{28}$$

Step 3. Add a vertical segment to the isoquant

$$S = S \cup \left[(\alpha_l^q, \beta_l^t), (\alpha_l^q, 0) \right]. \tag{29}$$

Step 4. $Sec(X_k, Y_k, e_q, e_t) = S$. Construction of the output stepwise isoquant is completed.

4 Conclusions

Computational experiments were accomplished to check our algorithms using real-life data sets taken from different areas: Russian banks, Swedish electricity utilities and Norwegian municipalities. Original data sets are described in detail in papers [Krivonozhko et al., 2012, Forsund et al., 2007, Erlandsen & Førsund, 2002]. Computational experiments were conducted on the basis of personal computer with Intel Core i3 CPU 3.33 GHz and lp_solve solver, version 5.5.2.0. Computational results are presented in Table 1 and Table 2.

Table 1: Computations of efficiency scores

Real-life data sets	Number of input and output indicators	Number of production units	Optimization methods		Enumeration methods
			Time, s	Number of iterations	Time, s
Russian banks, 2008	6	200	1.0	37244	0.5
Swedish electricity utilities,1987	8	163	0.9	36096	0.1
Norwegian municipalities, 1997	13	469	20.1	564591	0.3

Table 2: Constructions of input and output isoquants

Real-life data sets	Number of input and output indicators	Number of production units	Optimization methods all input isoquants		Enumeration methods all output isoquants	
			Time, s	Avg. time per isoquant, ms	Time, s	Avg. time per isoquant, ms
Russian banks, 2008	6	200	15.2	30	0.14	0.235
Swedish electricity utilities,1987	8	163	10.9	20	0.187	0.234
Norwegian municipalities, 1997	13	469	254.7	180	17.48	0.828

It is well known in scientific literature that enumeration methods have a great computational advantage over optimization methods [Kerstens & Vanden, 1999]. Table 2 confirms that computational time for constructions of isoquants is much less for enumeration method than for the method based on optimization algorithms. However, there are a lot of standard optimization programs in scientific literature. For this reason it may be easier to interface a new model with standard optimization solver than to develop new enumeration methods.

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