Investigation of Optimal Control Problem without Initial Conditions

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Abstract

In this paper, we propose an approach to the optimal control of processes, described by differential equations of hyperbolic type with a set of initial conditions. The influence of initial conditions on the current state of the process weakens over time, taking into account the participation of dissipative terms in real processes, which characterize the resistance of internal or external nature. There are also possible cases, when it is impossible to measure the initial state of the process accurately for some reason. In this case, we have an information about the set of possible initial states depending on the parameters from the given parametric set and it is necessary to control the state of the process without knowing the exact information about the initial state of the process.

1 Introduction

First A.N. Tikhonov studied the boundary value problems without initial conditions for differential equations of parabolic and hyperbolic types [Tikhonov & Samarskii, 1977], and offered the method of study the boundary-value problems without initial conditions, gave their first strict solutions. In [Tikhonov, 1935] he proved the uniqueness of the solution to the problem without initial conditions for heat equation.

In the paper, we study optimal control of processes, described by differential equations of hyperbolic type without accurate information about the initial state of the process. Such problems arise at controlling of real long time functioning evolutionary processes, the value of initial state of the processes which doesn't impact to their current state. A practical example of such problem is the optimal control of pipeline transportation of hydrocarbon raw material [Charniy, 1981], [Wichowski, 2006], [Aidazade & Rahimov, 2012], [Aidazade & Ashrafova, 2015]. There are also possible cases, when it is impossible to measure the initial state of the process accurately for some reason. In this case, we have an information about the set of possible initial states depending on the parameters from the given parametric set and it is necessary to control the state of the process without knowing the exact information about the initial values of the state of process [Aidazade & Rahimov, 2013], [Aidazade & Rahimov, 2012], [Allwright, 1985],

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[Rahimov, 2013], [Ashrafova, 2015], [Bokalo & Lorenzi,2009], [Bokalo, 1986], [Kirilich & Myshkis, 1992], [Lavrenyuk & Ptashnik, 2000], [Moiseev & Vafodorova, 2010], [Vafodorova, 2003]. In this paper, we propose an approach to the optimal control of processes, described by differential equations of hyperbolic type with a set of initial conditions.

2 Problem statement

Let's consider the wave process, described by the following system of differential equations of hyperbolic type [Charniy, 1981], [Wichowski, 2006]:

$$\begin{cases} -\frac{\partial P(x,t)}{\partial x} = \frac{\rho}{S}\frac{\partial Q(x,t)}{\partial t} + a\frac{\rho}{S}Q(x,t), \ t \in (0,T), \ x \in (0,l), \\ -\frac{\partial P(x,t)}{\partial t} = c^2\frac{\rho}{S}\frac{\partial Q(x,t)}{\partial x}, \end{cases} \tag{1}$$

$$P(0,t) = \upsilon_0(t), \ P(l,t) = \upsilon_l(t), \quad t \in (0,T), \tag{2}$$

where $P(x,t)$, $Q(x,t)$– are the phase state of the process (for instance, the pressure and the rate of transported raw materials in the point x of the pipeline at the moment of time t), determined from the solutions to the system (1)-(2) for corresponding admissible value of optimizing control vector-function $\upsilon = (\upsilon_0(t), \upsilon_l(t))$, $t \in [0,T]$; a – is the friction coefficient. If the wave process (1), (2) is observed for a long time, i.e. T is large enough, the impact of the initial conditions have substantially no effect on the behavior of the system starting from a certain point of time $t_0 > 0$, and $t_0 << T$ [Ashrafova, 2015]. We assume an unknown the initial state of the process at the time its origin $t = 0$, i.e. the values

$$Q(x,0) = Q_0(x), \ P(x,0) = P_0(x), \ x \in [0,l], \tag{3}$$

determining initial state of the process are no given accurately, but on the basis of a priori information on the investigating process it is known that the functions determining the possible initial state of the process, belong to some admissible set

$$Q_0(x) = Q_0(x,\gamma^q) \in L_2[0,l], \ P_0(x) = P_0(x,\gamma^p) \in L_2[0,l], \ x \in [0,l]\},$$

which depend on the parameters $\gamma^p, \gamma^q \in D \subset R^r$:

$$D = \{\gamma = (\gamma^p, \gamma^q) \in R^r : \underline{\gamma_i^p} \le \gamma_i^p \le \bar{\gamma}_i^p, \underline{\gamma_i^q} \le \gamma_i^q \le \bar{\gamma}_i^q, i = 1,...,r\} \tag{4}$$

and the density function $\rho_D(\gamma)$ is given. It is possible the existence of a set D_N, instead of the set D, which is determined with a certain finite number of functions

$$\varphi(x;\gamma_i) = (Q_0(x;\gamma_i^q), P_0(x;\gamma_i^p)), \quad x \in [0,l], \ i = 1,2,..,N. \tag{5}$$

which depend on the parameters from $D_N = \{\gamma_i : \underline{\gamma_i^p} \le \gamma_i^p \le \bar{\gamma}_i^p, \underline{\gamma_i^q} \le \gamma_i^q \le \bar{\gamma}_i^q, i = 1,...,N\}$ with density functions of distribution $\rho_{D_N}(\gamma_i)$, $i = 1,...,N$. We assume the restrictions on the optimized values of the boundary conditions on the basis of technological and technical conditions. The control functions belong to the set

$$V = \{\upsilon(t) = (\upsilon_0(t), \upsilon_l(t)) \in L_2[0,T] : \underline{\upsilon_0} \le \upsilon_0(t) \le \bar{\upsilon}_0, \underline{\upsilon_l} \le \upsilon_l(t) \le \bar{\upsilon}_l \ a.e. \ on \ [0,T]\}, \tag{6}$$

where $\underline{\upsilon_0}, \underline{\upsilon_l}, \bar{\upsilon}_0, \bar{\upsilon}_l$–are the given upper and lower admissible values of the pressure and the rate of transported raw materials respectively, by which the pumping station can operate.

It is required, the wave process characterized by pressure and rate of raw material to lead the regime where it differs as little as possible from a given state, to the moment of time $T > 0$, by controlling boundary conditions. The problem consists of finding such values of boundary controls $\upsilon_0(t), \upsilon_l(t), t \in (0,T]$, in which the functional gets it's minimum value defined particularly in the following form:

$$J(\upsilon) = \int_D I(\upsilon,\varphi)\rho_D(\gamma)d\gamma \to \min_{\upsilon \in V}, \tag{7}$$

$$I(\upsilon,\varphi) = \int_0^l \left[(Q(x,T;\upsilon,\varphi) - q_T(x))^2 + (P(x,T;\upsilon,\varphi) - p_T(x))^2\right]dx + \alpha \left\|\upsilon(t)\right\|^2_{L_2^2[0,T]},$$

Here $\|\cdot\|_{L_2^2[0,T]}$ – is the norm of vector-function; $\alpha > 0$ –is a weight coefficient. The functional (7) determines the assessment of the mean value of deviation of the state of the process at $t = T$ from the desired state of $(q_T(x), p_T(x))$ for all possible initial conditions $(Q_0(x,\gamma), P_0(x,\gamma)), \gamma \in D$; $\rho_D(\gamma)$ is the given density function of the initial values distributions on the set of D.

In the study of the boundary-value and optimal control problems the interval $[t_0, T]$ plays an important role, when the state of the process is almost does not depend on the values of the initial conditions at $t = 0$.

3 Formulas for the Numerical Solution

We suggest using the iterative methods of optimization of the first order for the numerical solution of optimal control problems, based on the use of analytical formulas derived below for control functions of the target functional gradient. One can use the methods of gradient projection

$$\upsilon^{k+1} = Pr_V(\upsilon^k - \lambda_k gradJ(\upsilon^k)), k = 0, 1, ...$$

or projection of conjugate gradient [Vasilev,2002]. Here $\upsilon^0 = [\upsilon_0^0(t), \upsilon_l^0(t)]$ – is any given initial value of control; $gradJ(\upsilon)$– is the gradient of target functional with respect to optimizable controls; λ_k – is the value of the step for one-dimensional search in the direction of anti gradient of the functional ; $\mathrm{Pr}_V(\cdot)$ – is the operator of projection of function on the set of admissible controls V (for positional restrictions of the form (6) this operator has a simple form [Vasilev,2002]).

Next we'll obtain the formulas for the gradient $gradJ(\upsilon; \varphi)$ of the functional for any one arbitrary selected admissible initial condition φ of the considered problem

Let $\upsilon = \upsilon(t)$ and $\upsilon + \Delta\upsilon = \upsilon(t) + \Delta\upsilon(t)$ are two admissible controls; $P(x,t;\upsilon,\varphi), P(x,t;\upsilon + \Delta\upsilon)$, $Q(x,t;\upsilon,\varphi), Q(x,t;\upsilon+\Delta\upsilon)$– are the solutions to boundary-value problem (1)-(3) corresponding to these controls for any arbitrary selected admissible initial condition and

$$\Delta P(x,t) = P(x,t;\upsilon + \Delta\upsilon,\varphi) - P(x,t;\upsilon,\varphi), \quad \Delta Q(x,t) = Q(x,t;\upsilon + \Delta\upsilon,\varphi) - Q(x,t;\upsilon,\varphi),$$

It follows from (1)-(3) that, the pair of functions $\Delta P(x,t), \Delta Q(x,t)$ are the solutions to the next boundary-value problem:

$$\begin{cases} -\frac{\partial \Delta P(x,t)}{\partial x} = \frac{\rho}{S}\frac{\partial \Delta Q(x,t)}{\partial t} + 2a\frac{\rho}{S}\Delta Q(x,t), \\ -\frac{\partial \Delta P(x,t)}{\partial t} = c^2\frac{\rho}{S}\frac{\partial \Delta Q(x,t)}{\partial x}, \quad (x,\, t) \in \Omega = (0,\, l) \times (0,\, T]. \end{cases}$$

$$\Delta P(0,t) = \Delta\upsilon_0(t),\ \Delta P(l,t) = \Delta\upsilon_l(t),\ t \in (0,T),$$

$$\Delta\varphi(x) = (\Delta P(x,0), \Delta Q(x,0)) = 0,\ x \in [0,l].$$

Then the increment of the functional (7) can be written in the following form:

$$\begin{aligned} \Delta I(\upsilon,\varphi) = 2\int_0^l [(Q(x,T;\upsilon,\varphi) - q_T(x))\,\Delta Q(x,T) + \ (P(x,T;\upsilon,\varphi) - p_T(x))\,\Delta P(x,T)] + \\ +2\alpha\int_0^T (\upsilon(t), \Delta\upsilon(t))dt + o(\Delta P^2(x,T)) + o(\Delta Q^2(x,T)) + o(\|\Delta\upsilon(t)\|^2). \end{aligned} \tag{8}$$

Let $\psi(x,t) = (\psi_1(x,t), \psi_2(x,t))$ is the solution to the auxiliary adjoint boundary-value problem

$$\begin{cases} -\frac{\partial\psi_1(x,t)}{\partial x} = \frac{\partial\psi_2(x,t)}{\partial t},\ x \in (0,l),\ t \in (0,T), \\ -\frac{\partial\psi_1(x,t)}{\partial t} = c^2\frac{\partial\psi_2(x,t)}{\partial x} - 2a\psi_1(x,t), \end{cases} \tag{9}$$

$$\psi_1(x,T) = -\frac{2S[Q(x,T) - q_T(x)]}{\rho c^2},$$

$$\psi_2(x,T) = -2[P(x,T) - p_T(x)], x \in [0,l], \tag{10}$$

$$\psi_1(0,t) = 0,\ \psi_1(l,t) = 0,\ t \in (0,T). \tag{11}$$

We obtain the following form for the increment of the functional (7):

$$\Delta I(\upsilon,\varphi) = \int_0^T (\psi_2(l,t)\Delta\upsilon_l(t) - \psi_2(0,t)\Delta\upsilon_0(t))dt + 2\alpha \int_0^T (\upsilon(t),\Delta\upsilon(t))dt + o(\|\Delta\upsilon(t)\|^2),$$

by using (8)-(11) and the estimation analogically to obtained one in [Vasilev,2002], [Ladijenskaya, 1973] for a more general case of controls from the class of measurable functions:

$$\int_0^T \int_0^l (|\Delta P(x,t)|^2 + |\Delta Q(x,t)|^2) dxdt \le C \int_0^T |\Delta\upsilon(t)|^2 dt,$$

where $C > 0$ –is a constant which doesn't depend on the choice of $\Delta\upsilon$. So, the formulas for the components of the gradient of target functional for control functions $\upsilon_0(t), \upsilon_l(t)$ are determined in the following form for the considered problem:

$$grad_{\upsilon_0(t)} J(\upsilon) = \int_D (-\psi_2(0,t) + 2\alpha\upsilon_0(t))\rho_D(\gamma)d\gamma,\ t \in [0,T],$$

$$grad_{\upsilon_l(t)} J(\upsilon) = \int_D (\psi_2(l,t) + 2\alpha\upsilon_l(t))\rho_D(\gamma)d\gamma,\ t \in [0,T],$$

4 The Results of Numerical Experiments

To carry out computer experiments on numerical investigation of optimal boundary control problems with inaccurately given initial conditions, we consider the following test problem for the example of controlling the process of fluid flow in a linear section of the main pipeline, described by a system of differential equations of hyperbolic type (1).
Suppose the transportation of oil with kinematic viscosity $\nu = 1.5 \cdot 10^{-4}, m^2/\sec$ and with the density $\rho = 920\ (kg/m^3)$ in the pipeline with the length $l = 100\ km$ and with the diameter $d = 0.53\ m$. In this case $2a = \frac{32\nu}{d^2} = 0.017$. The speed of sound in the medium is assumed equal to the $c = 1200\ (m/\sec)$.

We assume that the process has functioned for a long time at $t < 0$ and the mode is not exactly specified at the time of the beginning of the control by the boundary conditions at $t = 0$, but it is known that the flow and pressure can take values in the following ranges (the set D):

$$200 \le \hat{Q}_0(x) \le 210\ (m^3/hour), \tag{12}$$

$$1500000 - 2a\hat{Q}_0(x)x \le \hat{P}_0(x) \le 1700000 - 2a\hat{Q}_0(x)x\ (Pa),\ x \in [0, 100000]. \tag{13}$$

We use a set of initial conditions with parameters in D satisfying the constraints (12), (13) instead of exact initial conditions. Particularly, let us assume the possible initial conditions for flow rate and pressure are defined on a finite set D_N, obtained from D by a uniform partition of with N points, where the step $h_N = 10/N$. So, we have a the set of initial conditions as follows:

$$Q_0(x;\gamma_i) = 200 + \gamma_i h_N, \gamma_i \in D_N$$

$$P_0(x;\gamma_i) = 1500000 + 20000\gamma_i h_N - 2a\hat{Q}_0(x)x, \gamma_i \in D_N\ , x \in [0, 100000]. \tag{14}$$

We use the sweep method for the numerical solution of initial-boundary value problems of the form (1)-(3) with the initial conditions $Q_0(x;\gamma_i)$, $P_0(x;\gamma_i)$, $i = 1, 2, ..., N$ i.e. to calculate the values of the flow mode in the pipeline, the step for the time variable is $h_t = 0.83$ (sec), and the step for the spatial variable is $h_x = 100m$ (in dimensionless quantities $h_t = 0.009$, $h_x = 0.001$, these values were determined by the results of specially conducted experiments to determine the effective values of these parameters).

Let it is required to obtain a new desired steady-state regime at the moment of time $t = T = 90$ sec in the whole pipeline, for which the flow rate and pressure along the sections should have the following values:

$$q_T(x) = 130\ (m^3/hour),\ p_T(x) = 2400000 - 2.2x\ (Pa),\ x \in [0, 100000]. \tag{15}$$

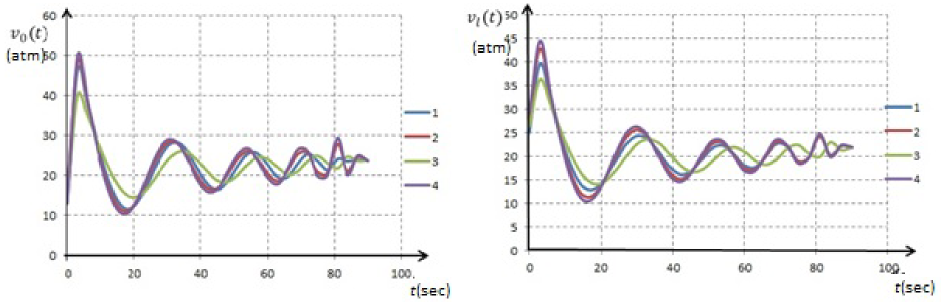

Figure 1: Plots of optimal controls for $N = 3$.

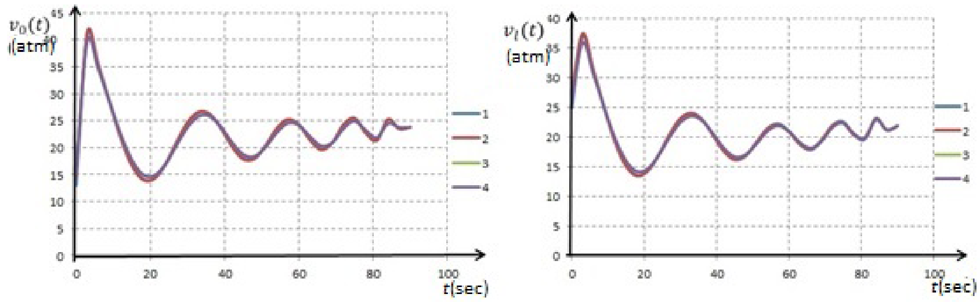

Figure 2: Plots of optimal controls for $N = 5$.

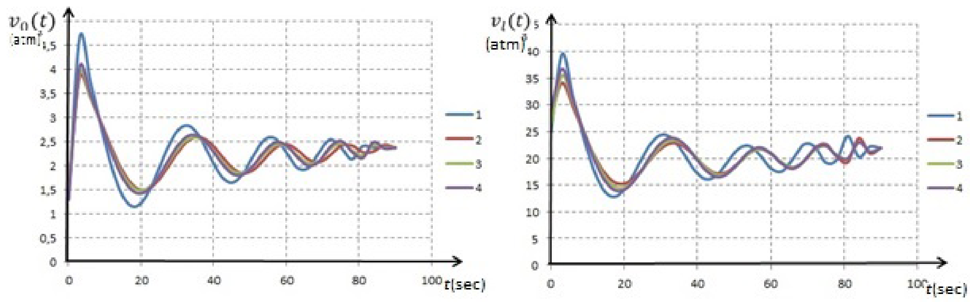

Figure 3: Plots of optimal controls for $N = 6$.

Table 1 shows the results of minimization of the functional (7) for different quantities of initial conditions of the form (14), i.e., for different values of N.

Figures 1-3 show the plots of optimal boundary controls, and in Figure 4 shows the optimal trajectories for each problem.

It can be seen from the results the practically independence of optimal values of controls from the exact values of the initial condition, i.e. the problem of optimal control for inaccurately given initial conditions can be reduced to the optimal control problem with the set of admissible initial states.

The obtained results can find application in the studies related with the controlling of many long-time functioning processes with distributed parameters, described by the systems of partial differential equations.

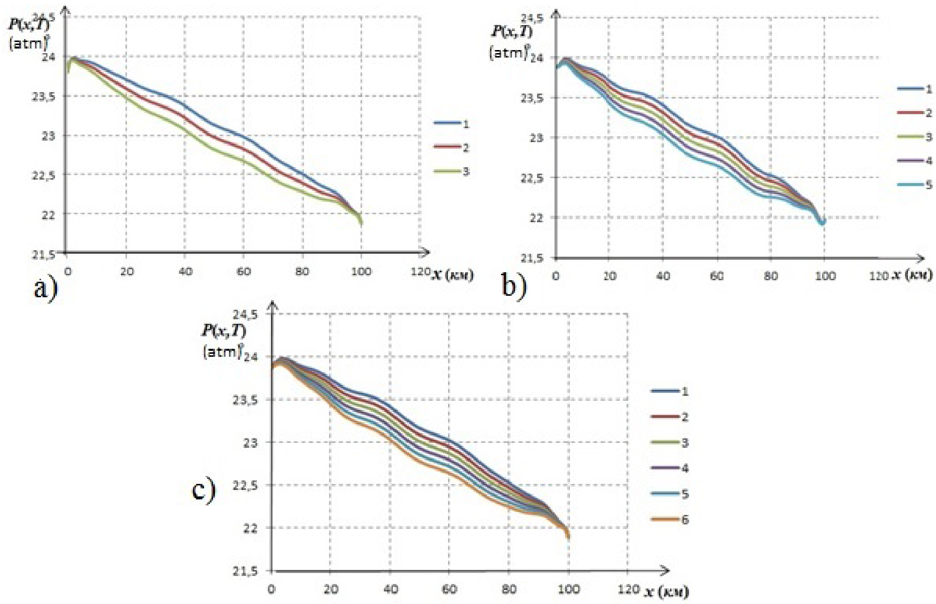

Figure 4: a),b),c) Plots of the pressures, calculated for optimal controls for different N.

Table 1: The results of minimization of the functional (7) for different N

	$\upsilon_0^0(t)\,(Pa)$	$\upsilon_l^0(t)\,(Pa)$	$J_0(\upsilon)$ $N=3$	$J_0(\upsilon)$ $N=5$	$J_0(\upsilon)$ $N=6$	$J_{\min}(\upsilon)$
1	1300000-100000t	2500000-300000t	0,80091 0,80924 0,81786	0,80090 0,80587 0,81094 0,81611 0,82140	0,80091 0,80503 0,80924 0,81351 0,81786 0,82229	0,00017
2	1500000-100000t	2800000-500000t	0,60363 0,60619 0,60904	0,60363 0,60513 0,60674 0,60845 0,61026	0,60363 0,60488 0,60619 0,60758 0,60904 0,61057	0,00017
3	15600000	26400000	0,61091 0,60922 0,60783	0,61091 0,60986 0,60892 0,60808 0,60735	0,61091 0,61003 0,60922 0,60849 0,60783 0,60724	0,00017
4	1300000-100000t	2900000-200000t	1,12287 1,12355 1,12452	0,61091 0,60986 0,60892 0,60808 0,60735	1,12287 1,12317 1,12355 1,12400 1,12452 1,12512	0,00017

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