On the Reduction of the Optimal Non-Destructive System Exploitation Problem to the Mathematical Programming Problem

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Abstract

This article is devoted to the use of a renewable resource based on its withdrawal from the certain system, not leading to its destruction. Such problems arise, for example, for resources that are managed by harvesting cohort-structured biological populations. Our aim is to present a unified approach to modeling and managing such resource systems. The purpose of optimization is to obtain the maximum admissible total effect of system exploitation. It is shown that this problem can be reduced to some problem of mathematical programming.

1 Introduction

This work is devoted to problems of rational exploitation of ecological populations. Much of the research in this area includes the use of matrix models (see the review, for example, in the book [Caswell, 2001]) to search for the maximum operating regime for specific commercial populations (eg, in fisheries, forestry, etc.); a relatively small group of papers is devoted to the theoretical study of resource management problems.

The problem of searching of the balanced (steady) level of operation of population, on the one hand, maximal in these conditions, and, on the other hand, keeping its stable existence, was put and studied by many authors both in theoretical researches, and in works of practical [Getz & Haight, 1989]. The concept of steady exploitation variously was defined at different authors. G. Dunkel [Dunkel, 1970] introduced the concept of Sustainable Yield and formulated the corresponding optimization problem. He proved the solvability of the optimization problem in linear formulation. Doubleday [Doubleday, 1975] formalized some of the natural concepts and formulations for the exploitation of ecosystems and proved the corresponding statements rigorously. He first noticed and clearly formulated the connection between the issues of optimal exploitation of populations and the solution of linear programming problems. The approach he proposed was later used by many authors in both theoretical studies and practical work.

The characteristic property of optimal solutions of the proposed optimization problem was the number of age classes to be exploited: it turned out that there necessarily exist optimal solutions having a maximum

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of two such classes. Early evidence of this property had a private, non-systemic character; later used linear programming methods. Apparently, these results were first obtained by R. Beddington, D. B. Taylor [Beddington & Taylor, 1973] , and C. Rorres, W. Fair [Rorres & Fair, 1975]. Their authors established the existence of bimodal optimal control, i.e. management, allowing the exploitation (withdrawal, partial or complete) of not more than two age classes — a partial removing of one age class and a full removing the other (the older). Such "two-age" strategies were obtained later by many authors not only for linear models, but also for their generalizations with nonlinearities [Getz & Haight, 1989]. Approach to optimal exploitation, based on the stationary structure of the population represented by an eigenvector corresponding to the dominant eigenvalue, has long been a prevailing.

The first successful attempts of generalizations of these results to the case of density dependence associated with the work of W. Reed [Reed, 1980], which dealt with the density-dependence only for the first age class. This allowed them to decompose the problem into two parts, which respectively have been used methods of linear or nonlinear optimization. It was found that the quality of this already non-linear optimization problem is not fundamentally different from the corresponding linear, in the sense that the obtained optimal solutions have "two-age" character too.

The task of optimizing the operation in this formulation — the task of finding the maximum balanced (permissible, stable) level of operation was called Maximum Sustainable Yield Problem (MSY-Problem) [Getz & Haight, 1989].

Attempts to investigate optimal strategies for general classes of discrete dynamical systems also continued. So, in work [Smirnov, 1980], the asymptotic properties of system with constant additive control for the class of concave separable mappings were characterized.

In this work, we proposed a general formulation of the stationary control of the system for a class of concave mappings and reduce its solution to a solution of a certain mathematical programming problem.

2 Some Preliminary Results

We will use the following notation: $\mathbb{R}^q_+$ denotes the non-negative orthant of the space $\mathbb{R}^q$; $x \leqq y$ means $y - x \in \mathbb{R}^q_+$; $x < y$ means $y - x \in \operatorname{int} \mathbb{R}^q_+$; $\operatorname{int} M$ — the interior of the set M; $x \leq y$ means $x \leqq y$ and $x \neq y$; $\overline{m,n} = \{i \in \mathbb{Z} : m \leq i \leq n\}$; $\mathbb{Z}$ — the set of integers. Vectors $x = (x_1, x_2, \ldots, x_q)$, $F(x) = (f_1(x),\ f_2(x), \ldots, f_q(x))$ will be briefly written in the form $x = (x_i)$, $F(x) = (f_i(x))$ respectively. The iterations of the map F are denoted by $F^t(x)$ $(t = 1, 2, \ldots)$, $F^0(x) \equiv x$. All mappings considered further are monotonous (order-preserving), i.e. $x \leqq y \Rightarrow F(x) \leqq F(y)$ $(\forall x, y \in \mathbb{R}^q_+)$.

It is supposed that there is an object (a system) whose state is described by vector $x \in \mathbb{R}^q_+$. We assume that the successive transitions of system states can be simulated by the iterative process:

$$x_{t+1} = F(x_t), \qquad t = 0, 1, 2, \ldots \tag{1}$$

The detailed characteristics of the mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ are not available; however, we know its aggregated properties. It is supposed that the mapping F is concave on $\mathbb{R}^q_+$, hence, it is monotone. We assume that the mapping F has the trivial fixed point: $F(0) = 0$.

We shall need the concept of irreducibility (indecomposability) of a mapping. Along with the classical definition of the irreducibility [Nikaido, 1968], there exists its local aspect [Smirnov, 2016c]. The mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is called *reducible at the point* $y \in \mathbb{R}^q_+$, if

$$\exists x \in \mathbb{R}^q_+ : \quad x \geq y, \quad I^0(x, y) \neq \varnothing, \quad I^0(x, y) \subseteq I^0(F(x), F(y)).$$

The mapping, reducible at every point of the set M is called *reducible on the set* M. Accordingly monotone mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is called *irreducible at point* of y, if

$$\forall x \in \mathbb{R}^q_+ : \quad x \geq y,\ I^0(x, y) \neq \varnothing \Rightarrow I^0(x, y) \cap I^+(F(x), F(y)) \neq \varnothing.$$

The mapping, irreducible at every point of the set M is called *irreducible on the set* M.

We consider separately the case of irreducibility of the mapping at the point $y = 0$. The mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is called *reducible at point* $y = 0$ (*reducible at zero*), if

$$\exists x \in \mathbb{R}^q_+ : \quad x \geq 0, \quad I^0(x) \neq \varnothing, \quad I^0(x) \subseteq I^0(F(x)). \tag{2}$$

Accordingly monotone mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is called *irreducible at point* $y = 0$ (*irreducible at zero*), if

$$\forall x \in \mathbb{R}^q_+ : \quad x \geq 0,\ I^0(x) \neq \varnothing \Rightarrow I^0(x) \cap I^+(F(x)) \neq \varnothing. \tag{3}$$

An irreducible mapping in the sense [Nikaido, 1968] will be called *globally irreducible.* It is clear that the global irreducibility of a mapping means irreducibility at any point $\mathbb{R}^q_+$ and, in particular, irreducibility at zero.

The mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is called *primitive in point* $y \in \mathbb{R}^q_+$ [Nikaido, 1968] if for every $x \geq y$ holds the inequality $F^k(x) > F^k(y)$ for some integer non-negative k. Here F^k denotes the iteration of the mapping F; $F^0(x) \equiv x$.

Irreducibility at zero is a weaker property in comparison with the primitivity at zero [Smirnov, 2016b, Lemma 2.1.12]. Nevertheless, irreducibility at zero also guarantees the positiveness of non-zero fixed points. This important property holds for any mappings that are irreducible at zero.

The conditions for local irreducibility of subhomogeneous mappings and their properties are considered in the paper [Mazurov & Smirnov, 2016a], like some conditions for the coincidence of the concept of irreducibility at zero with the concept of global irreducibility.

It will be recalled that the mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is called *subhomogeneous* [Lemmens & Nussbaum, 2012] if

$$F(\alpha x) \geqq \alpha F(x) \quad (\forall x \in \mathbb{R}^q_+,\ \alpha \in [0,1]). \tag{4}$$

Note that concave mapping $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is monotone subhomogeneous mapping.

A monotone subhomogeneous mapping is nonexpansive in the Birkhoff-Thompson metric [Lemmens & Nussbaum, 2012] (and, consequently, continuously) on the interior of a positive cone in a Banach space, in particular, on $\operatorname{int} \mathbb{R}^q_+$. Furthermore, a subhomogeneous monotone mapping always has a continuous extension, in the case of the space $\mathbb{R}^q$, to the whole cone $\mathbb{R}^q_+$ [Lemmens & Nussbaum, 2012, Lemma 5.1.5]; so we can consider a monotone monotone mapping as continuous on the whole $\mathbb{R}^q_+$.

It is easy to show that subhomogeneity equivalent to the property

$$F(\beta x) \leqq \beta F(x) \quad (\forall x \in \mathbb{R}^q_+,\ \beta \in (1, +\infty)). \tag{5}$$

The properties (4), (5) means that can be defined positively homogeneous of the first degree mapping

$$F_0(x) = \lim_{\alpha \to +0} \alpha^{-1} F(\alpha x), \qquad F_\infty(x) = \lim_{\alpha \to +\infty} \alpha^{-1} F(\alpha x), \tag{6}$$

It was shown [Smirnov, 2016b] that the condition

$$\lambda(F_\infty) < 1 < \lambda(F_0) \tag{7}$$

is sufficient for existence of a nonzero fixed point of the subhomogeneous monotone mapping F. Here $\lambda(H)$ is the dominant eigenvalue of the positive-homogeneous mapping H [Nikaido, 1968]. If, in addition, the primitivity condition of the mapping F on the set K_F is satisfied, then the iterative process (1) is convergent to an element of the set of positive fixed points of the mapping F from any nonzero initial state $x(0)$. Here cone K_F is generated by the set of the nonzero fixed point of the mapping F [Smirnov, 2016b].

We shall assume for simplicity that the non-zero fixed point of the mapping F is unique. If this point is also positive, then the iterative process (1) converges to it regardless of the choice of nonzero initial vector in the absence of primitivity requirement. The positiveness of a non-zero fixed point will be ensured by the requirement that the mapping F be irreducible at zero.

3 The Optimization Problem Formulation and Its Reduction to the Convex Programming Problem

Suppose that the system to be operated in the absence of control is functioning according to law (1), where the mapping F is concave, irreducible at zero, and satisfies condition (7), which, as noted above, guarantees the existence of positive fixed point $\bar{x}_F$ of the mapping F which is assumed to be unique. If the exploitation of this system is understood as the removal of a certain number of its elements, the operation of such a managed system can be specified in the form of an iterative process

$$x_{t+1} = F_u(x_t), \qquad t = 0, 1, 2, \ldots, \tag{8}$$

where $x = (x_1, x_2, \ldots, x_q)$, $u = (u_1, u_2, \ldots, u_q)$, $F_u(x) = (F(x) - u)^+$, $a^+ = \max\{a, 0\}$, $x^+ = (x_i^+)$. The realization of this iterative process with the initial vector x_0 will be denoted by $\{x_t(x_0, u)\}_{t=0}^{+\infty}$, $x_t(x_0, u) = (x_t^i(x_0, u))$.

The purpose of optimization is to obtain the maximum admissible total effect of system exploitation: determine

$$\tilde{c} = \max\{c(u) : u \in \bar{U}\}, \tag{9}$$

where the monotone convex $c(u)$ determine the total effect of using of system elements in the quantities $u_1, u_2, \ldots, u_q$ and the set $\bar{U}$ is the closure of the set

$$U = \{u \in \mathbb{R}_+^q : X_0(u) \neq \varnothing\}, \tag{10}$$

of controls.

The control vector u is considered as admissible if there is at least one initial state x_0 of the iterative process (8), for which all units of the system stably exist for an indefinitely long time:

$$X_0(u) = \{x_0 \in \mathbb{R}_+^q : \inf_t x_t^i(x_0, u) > 0 \ (\forall i \in \overline{1, q})\}, \tag{11}$$

Thus, the size of each unit of the system should not decrease with time to zero. As can be seen from this definition, the set U formalizes the requirement of the indestructability of the managed object.

Unfortunately, the set U defined by (10) is not always closed, so, in the problem (9), its closure is considered as an admissible set.

We will only be interested in the positive realizations of the iterative process (8), when $x_t(x_0, u) > 0$ ($\forall t = 1, 2, \ldots$). In this case $F_u(x) = F(x) - u$.

Denote by $\tilde{u}$ and $\tilde{U}$ the optimal set and optimal vector of problem (9), respectively; and by N_u and N_u^+ the sets of nonzero fixed points and positive fixed points of the mapping F_u , respectively.

We give the following auxiliary assertion characterizing the properties of the set U and the connection between the fixed points of the mappings F and F_u.

Lemma 1. *Suppose that the mapping F is concave on $\mathbb{R}_+^q$ and the condition* (7) *is satisfied. Then the following properties hold:*

(1) $u \in U,\ 0 \leqq v \leqq u \Rightarrow v \in U$;

(2) $u \geqq 0 \Rightarrow x_u \leqq \bar{x}_F$; $u > 0 \Rightarrow x_u < \bar{x}_F$ $(\forall x_u \in N_u)$;

(3) $N_u \neq \varnothing,\ 0 \leqq v \leqq u \Rightarrow N_v \neq \varnothing$; $N_u \neq \varnothing,\ 0 \leqq v < u \Rightarrow N_v^+ \neq \varnothing$; $N_u^+ \neq \varnothing,\ 0 \leqq v \leqq u \Rightarrow N_v^+ \neq \varnothing$;

(4) $N_u \neq \varnothing \Rightarrow \exists\, \bar{x}_u \in N_u : x_u \leqq \bar{x}_u$ $(\forall x_u \in N_u)$;

(5) $N_u = \varnothing \Rightarrow \lim_{t\to+\infty} x_t(x_0, u) = 0$; $N_u \neq \varnothing, x_0 \geqq \bar{x}_u \Rightarrow \lim_{t\to+\infty} x_t(x_0, u) = \bar{x}_u$;

(6) $N_u \neq \varnothing,\ 0 \leqq v \leqq u \Rightarrow \bar{x}_v \geqq \bar{x}_u$; $N_u \neq \varnothing,\ 0 \leqq v < u \Rightarrow \bar{x}_v > \bar{x}_u$;

(7) $u \geqq 0 \Rightarrow \bar{x}_u \leqq \bar{x}_F$; $u > 0 \Rightarrow \bar{x}_u < \bar{x}_F$.

As we see, in the case $N_u \neq \varnothing$ among the fixed points of the mapping F_u there exists the largest element - the fixed point $\bar{x}_u$, and the iterative process (8) converges to $\bar{x}_u$, if the initial vector $x_0 \geqq \bar{x}_u$. Thus, this level of system exploitation allows it to stably exist unlimitedly for a long time.

We note that the existence of a positive fixed point of the mapping F ensures that $U \neq \varnothing$, since $0 \in U$. But the question arises whether the set U contains positive vectors. The answer to this question is given by the following statement.

Lemma 2. *Suppose that the mapping F is concave on $\mathbb{R}_+^q$, irreducible at zero and the condition* (7) *is satisfied. Then the set U contains a positive vector. Moreover, the set U together with each positive vector u also contains the segment* $[0, u]$.

Proof. Since the mapping F is irreducible at zero, the condition $\lambda(F_0) > 1$ implies the existence of the vector $x_0 > 0$ for which $F(x_0) > x_0$ [Smirnov, 2016b, Lemma 2.2.5]. Therefore, for sufficiently small $u > 0$ is also true the inequality $F_u(x_0) > x_0$. Because of the monotonicity of the map F, this means that $(F_u)^t(x_0) > x_0$ ($\forall t = 1, 2, \ldots$), i.e. $x_0 \in X_0(u)$, so $X_0(u) \neq \varnothing$. It means that $u \in U$. Further, the segment $[0, u] \subset U$ by virtue of the property (1) of the previous lemma. Proof is complete.

We have the following characterization of the set U.

Lemma 3. *Suppose that the mapping F is concave on $\mathbb{R}^q_+$, globally irreducible and the condition* (7) *is satisfied. Then the following equality hold:*

$$U = \{u \in \mathbb{R}^q_+ : N_u^+ \neq \varnothing\}, \tag{12}$$

Proof. If $U = \varnothing$, then $N_u^+ = \varnothing$, since otherwise $\bar{x}_u \in X_0(u)$. If $U \neq \varnothing$ and $u \in U$, then, according to the definition of the set U, we have $X_0(u) \neq \varnothing$. This means the existence of vectors $x_0 \in X_0(u)$, $e > 0$ satisfying the inequalities $(F_u)^t(x_0) \geqq e$ ($\forall t = 1, 2, \ldots$). By virtue of the inequality $\lambda(F_\infty) < 1$, there exists a vector $\bar{x} \geqq x_0$ that satisfies the inequality $F(\bar{x}) < \bar{x}$. For this vector, we have $(F_u)^t(\bar{x}) \geqq (F_u)^t(x_0) \geqq e > 0$ ($\forall t = 1, 2, \ldots$), so that the sequence $\{(F_u)^t(\bar{x})\}_{t=0}^{+\infty}$ is positive and separated from zero. It follows from the inequalities $F_u(\bar{x}) = F(\bar{x}) - u \leqq (\bar{x}) - u \leqq \bar{x}$ that this sequence is also monotonically decreasing and therefore converges to a certain vector $\bar{y} \geqq e > 0$. This means that $\bar{y} \in N_u^+$, i.e. $N_u^+ \neq \varnothing$. The assertion is proved.

As the following assertion shows, the set U (and hence the set $\bar{U}$) is convex.

Lemma 4. *Suppose that the mapping F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (7) *is satisfied. Then the set U is bounded and convex.*

Proof. The boundedness of the set follows from assertion (7) of Lemma 1. Indeed, for any $u \in U$ we have: $u \leqq x_u + u = F(x_u) \leqq F(\bar{x}_F) = \bar{x}_F$, i.e. $u \leqq \bar{x}_F$ ($\forall u \in U$).

We now prove the convexity of the set U. If $u^1, u^2 \in U$ and $x^1 \in N_{u^1}^+$, $x^2 \in N_{u^2}^+$, then $x^1 > 0$, $x^2 > 0$, $F_{u^1}(x^1) = x^1 > 0$, $F_{u^2}(x^2) = x^2 > 0$. Let $u = (1-\alpha)u^1 + \alpha u^2$, $x = (1-\alpha)x^1 + \alpha x^2$, where $\alpha \in [0, 1]$. Since F is concave, we have:
$F(x) - u \geqq (1-\alpha)F(x^1) + \alpha F(x^2) - (1-\alpha)u^1 - \alpha u^2 = (1-\alpha)F_{u^1}(x^1) + \alpha F_{u^2}(x^2) = (1-\alpha)x^1 + \alpha x^2 = x > 0.$
Therefore $F(x) - u > 0$ and $F_u(x) = F(x) - u \geqq x$. Hence, since the mapping F is monotone, the sequence $\{(F_u)^t(x)\}_{t=0}^{+\infty}$ is monotonically increasing. It is also bounded as from inequalities $x^1 \leqq \bar{x}_F$, $x^2 \leqq \bar{x}_F$ it follows that $x \leqq \bar{x}_F$, $F_u(x) \leqq F_u(\bar{x}_F) \leqq F(\bar{x}_F) = \bar{x}_F$, so that $(F_u)^t(x) \leqq \bar{x}_F$ ($\forall t = 1, 2, \ldots$). Thus, the sequence $\{(F_u)^t(x)\}_{t=0}^{+\infty}$ is monotonically increasing and bounded and, consequently, converges to a finite limit $\bar{x} \geqq x > 0$ belonging, by virtue of its positivity, to the set N_u^+. This means, in accordance with the representation (12), that the vector $u = (1-\alpha)u^1 + \alpha u^2 \in U$. The proof is complete.

By applying previous statements, we obtain the following consequence.

Lemma 5. *Suppose that the mapping F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (7) *is satisfied. Then the following equality hold:*

$$\bar{U} = \{u \in \mathbb{R}^q_+ : N_u \neq \varnothing\}. \tag{13}$$

Using this result, we can to obtain the following assertion reducing the solution of the problem (9) to the solution of a problem of mathematical programming

Theorem. *Suppose that the mapping F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (7) *is satisfied. Then the problem of mathematical programming*

$$\max\{c(u) : x = F(x) - u,\ x \geqq 0,\ u \geqq 0\} \tag{14}$$

is solvable in the sense that $\tilde{c} < +\infty$ and this value is attained on certain admissible vectors $\tilde{u}$, $\tilde{x}$.

Moreover, $\tilde{c}$, $\tilde{u}$ is a solution of problem (9) *if and only if $\tilde{c}$, $\tilde{u}$, $\tilde{x}$ is a solution of problem* (14).

4 Conclusion

For a wide class of ecosystem models represented by iterative processes with concave mapping as the step operator, we proposed a general approach to formalization of the problem of their stable exploitation. Note that specific varieties of concave mappings are often used in ecosystem modeling, since they reflect the effect of the limited resources of real natural ecosystems.

We note the following important circumstance. From the problem (9), which is transparent from the point of view of interpretation, but does not give constructive approaches to the solution, we turned to the problem of mathematical programming (14), the methods of solving which are well known.

For specific classes of models that are in demand at present in the study of natural ecosystems, the solution of the problem of mathematical programming (14) can be substantially simplified. This is true, in particular, for the generalization of the classical Leslie model of the age structure of the population proposed and investigated in [Smirnov, 2010]. This generalization takes place in two directions: firstly, a population with a binary structure is considered (i.e., there is a partition of one more feature besides age), and secondly, the dependence of the numbers of the initial age classes of these additional population subdivisions on the population density is described by subhomogeneous function. In the case of using this generalization, the problem (14) with a linear objective function can be considered as a set of linear programming problems that depend on a numerical parameter. This allows us to obtain a fairly simple algorithm for solving it, based on the transition to the dual linear programming problem.

We note that in this paper additive control is considered, in contrast to the above-mentioned research on the exploitation of populations using multiplicative control, when a certain fraction of each of the subdivision of the population is removed. It seems to us that the use of additive control for problems of stable exploitation is more natural and realistic.

Unfortunately, passing from the set U of positive trajectories of the iterative process (8) to its closure in the problem (14), we can no longer guarantee the positivity of the optimal vector x. Nevertheless, for certain classes of mappings F, one can guarantee the existence of positive optimal solutions x of this problem. In particular, this is true for the generalization of the Leslie model [Smirnov, 2010] mentioned above with the concave map as the step operator.

All of the above allows us to hope that the approach we proposed to formalize the notion of rational exploitation of populations will be useful both for the development of theoretical models of ecological systems and for the solving specific problems of nature management.

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References

[Beddington & Taylor, 1973] Beddington, J. R. and Taylor, D. B. (1973). Optimum age specific harvesting of a population. *Biometrics*, (29):367–378.

[Caswell, 2001] Caswell, H. (2001). *Matrix Population Models: Construction, Analysis, and Interpretation. Second edition.* Sinauer Associates, Sunderland MA.

[Doubleday, 1975] Doubleday, W. G. (1975). Harvesting in matrix population model. *Biometrics*, (31):367–378.

[Dunkel, 1970] Dunkel, G. (1970). Maximum sustainable yields. *SIAM J. Appl. Math.*, (2):367–378.

[Getz & Haight, 1989] Getz, W. M. and Haight, R. G. (1989). *Population Harvesting: Demographic Models of Fish, Forest, and Animal Resources.* Princeton: Princeton University Press.

[Lemmens & Nussbaum, 2012] Lemmens, B. and Nussbaum, R. D. (2012). *Nonlinear Perron-Frobebius Theory. Cambridge Tracts in Mathematics. Vol. 189.* Cambridge: Cambridge Univ. Press.

[Mazurov & Smirnov, 2016a] Mazurov, V. D. and Smirnov, A. I. (2016) (in russian). The conditions of irreducibility and primitivity monotone subhomogeneous mappings. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 22(3):169–177.

[Nikaido, 1968] Nikaido, H. (1968). *Convex Structures and Economic Theory.* Academic Press, New York.

[Reed, 1980] Reed, W. J. (1980). Optimal harvesting policy for an age specific population. *Biometrics*, 36(4):579–593.

[Rorres & Fair, 1975] Rorres, C. and Fair, W. (1975). Optimal harvesting policy for an age specific population. *Math. Biosci.*, 24(1/2):31–47.

[Smirnov, 1980] Smirnov, A. I. (1980) (in russian). The ultimate-sustainable-yield problem in nonlinear age-structured populations. *Optimisation and Pattern Recognition methods in planning problems. Trans of the Inst of Math and Mech Ural Scie Centre USSR Acad. of Scie*, 48:106–112.

[Smirnov, 2010] Smirnov, A. I. (2010) (in russian). On some nonlinear generalizations of the leslie model, taking into account the saturation effect. *Bulletin of the Ural Institute of Economics, Management and Law*, 4(13):98–101.

[Smirnov, 2016b] Smirnov, A. I. (2016) (in russian). *Equilibrium and stability subhomogeneous monotone discrete dynamical systems*. Ekaterinburg: Ural Institute of Economics, Management and Law Press.

[Smirnov, 2016c] Smirnov, A. I. (2016) (in russian). On some weakenings of the concept of irreducibility. *Bulletin of the Ural Institute of Economics, Management and Law*, 2(35):26–30.