Parameter Identification of an Endogenous Production Function

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Abstract

The paper presents an endogenous production function and its identification. A new type of production function is constructed where the lifetime of production capacity is assumed to be limited. An expression for production function was numerically derived under conditions of scientific and technological progress and optimal investment in the best technology. Given the capital–output ratio and the rate of physical degradation of capacity, it is possible to estimate the age structure of production capacities by past production investments. All unknown external parameters are determined indirectly due to verification on historical statistics. Parameter identification of the model is a kind of global optimization problem. Parallel calculations in MPI are used for the identification. To ensure sustainable economic growth it is recommended to reduce the maximum age of capacities.

1 Introduction

The description of production functions represented by the distribution of production capacities by technology (a kind of vintage capital model with putty-clay technology) arose from practical needs in the analysis of specific sectors of the economy [Johansen, 1972]. A mathematical study of this kind of production function of several variables was carried out in [Shananin, 1984]. A new class of production functions represented by the distribution of production capacities by technology was obtained in [Olenev et al., 1986] on the basis of the original micro-description. An analytical expression for the production function in the case when production capacities are not limited by age was obtained there on the balanced growth path.

The endogenous production function, taking into account age-related capacity limitations, was numerically constructed in [Olenev, 2016] to study the real sector of Greek economy. Numerical experiments related to the identification of parameters have shown two significant facts. First, they conduct practical calculations with a limited age of capacity. It is enough to consider the fixed maximum age of capacities. Secondly, when considering the dynamics of the economy for a period of several decades, it is necessary to take into account the change in the coefficient of incremental capital–output ratio (as a rule, it decreases).

In: Yu. G. Evtushenko, M. Yu. Khachay, O. V. Khamisov, Yu. A. Kochetov, V.U. Malkova, M.A. Posypkin (eds.): Proceedings of the OPTIMA-2017 Conference, Petrovac, Montenegro, 02-Oct-2017, published at http://ceur-ws.org

The paper presents analytical construction of an endogenous production function based on distribution of production capacity on technologies and numerical procedure of its parameter identification on data of Russian economy. Production capacity is determined as a maximum of possible output in a year. In this model the capacities has an age limit A on their usage. The boundary brings a new kind of production function that contains maximal age of capacities A as a new parameter.

In this paper, the production function is analytically constructed on a transitional growth path with a decreasing incremental capital–output ratio for capacities that are limited in age. Taking into account the age limit of capacity gives not only additional value to control the economic system, but also makes it possible to practically determine the parameters of this production function in its numerical construction on the basis of statistical data of a specific country. The problem of indirect identification of the parameters of the production function is a special type of the problem of global optimization. The criterion of optimality is the proximity of the calculated and statistical time series of macroeconomic indices. The results of identification of the parameters of this production function are given according to the statistical data of Russia 1970-2015. The identified production function can be used in practice for the analysis of specific economic systems and for scenario forecasts.

2 Production Capacities with Age Limit

Let's consider an economic growth path with the growth rate γ in a closed economy in which output $Y(t)$ (gross domestic product) is divided on consumption $C(t)$ and investment (saving) $b(t)I(t)$ at each current time t:

$$Y(t) = C(t) + b(t)I(t), \tag{1}$$

where $b(t)$ is the incremental capital–output ratio, and $I(t)$ is the volume of newly created production capacities.

If we denote the labor by $L(t)$ and the total capacity by $M(t)$, $Y(t) \leq M(t)$, then the dependence $Y(t)$ on production factors $M(t)$, $L(t)$ defines the production function (see [Olenev et al., 1986])

$$Y(t) = M(t)f(t, x), \tag{2}$$

where $x = L(t)/M(t)$ is an average labor intensity of existing capacities, and $f(t, x)$ is a total capacity utilization. On economic growth path the product balance indeces (output $Y(t)$, total capacity $M(t)$, sum of consumption and investment $C(t) + b(t)I(t)$) increase with the same growth rate γ.

Let's $\nu(t)$ be a smallest labor intensity the dynamics of which are determined by the rate of scientific and technological progress ε [Olenev et al., 1986]:

$$\frac{d\nu(t)}{dt} = -\varepsilon\sigma(t)\nu(t), \tag{3}$$

where $\sigma(t) = I(t)/M(t)$ is a share of new capacities in total capacity.

In contrast to [Olenev et al., 1986], we construct a production function in the case of capacities that are limited in age. For a micro-description of the dynamics of production capacity we use the hypothesis proposed in [Olenev et al., 1986] on its decrease due to aging and that the number of working places on it is fixed since the creation of this capacity, so that here we do not take into account the growth of production capacity through learning by doing [Makarova & Olenev, 2013].

Hypothesis 0: *The number of working places on the production unit is fixed from the moment $\tau \leq t$ it was created, and the production capacity $m(t, \tau)$ is decreased at the constant rate $\mu > 0$.*

Initial capacity $m(t, t) = I(t)$ for each moment t. Then due to hypothesis 0 the capacity will decrease in the future, $m(t, \tau) = I(\tau)exp(-\mu(t - \tau))$. This reduction in capacity requires, in order to maintain the number of jobs, an appropriate increase in labor intensity $\lambda(t, \tau) = \nu(\tau)exp(\mu(t - \tau))$, where $\nu(\tau)$ is the labor intensity on the production unit at moment of its creation τ. Recall [6] that investors at every moment choose the best of existing technologies, that is, the technology with the smallest amount of labor $\nu(t)$, and this lowest labor intensity is decreased with time due to scientific and technological progress with the rate $\varepsilon\sigma(t) > 0$ in accordance with (3).

If the age limit of capacities that can be used in production is $A(t)$, then the total capacity is

$$M(t) = \int_{t-A(t)}^{t} I(\tau)e^{-\mu(t-\tau)}d\tau. \tag{4}$$

The capacity dynamics due to (4) is described by the differential-difference equation

$$\frac{dM(t)}{dt} = I(t) - \mu M(t) - \left(1 - \frac{dA(t)}{dt}\right) I\left(t - A(t)\right) e^{-\mu A(t)}, \tag{5}$$

where $dA/dt \leq 1$. The last condition means that capacities once exceeded $A(t)$, are dismantled and not returned to production, and $A(t)$ is an additional control variable.

In accordance with [3-6] we assume that the total labor (employment) $L(t)$ is used in an optimal way starting from the newly created zero-age capacity with the best technology (with lowest labor intensity) up to the capacity of age $\theta(t, L(t)) \leq A(t)$. So that the system of equations for the output $Y(t)$ and the labor $L(t)$ determines the next expression for the production function:

$$Y(t) = \int_{t-\theta(t,L(t))}^{t} I(\tau) e^{-\mu(t-\tau)} d\tau, \tag{6}$$

$$L(t) = \int_{t-\theta(t,L(t))}^{t} \nu(\tau) I(\tau) d\tau. \tag{7}$$

For transition to the intensive variables in the expression for the production function (6) - (7), we use the notations already introduced: the share of new capacities $\sigma(t) = I(t)/M(t)$, the average labor intensity $x = L(t)/M(t)$, and the capacity utilization $f(t, x) = Y(t)/M(t)$. According to (3) we have $\nu(\tau) = \nu(t) exp\left(\varepsilon \int_{\tau}^{t} \sigma(s) ds\right)$. Then from (6) - (7) we obtain the next parametric expression for the production function:

$$f(t, x) = \frac{1}{M(t)} \int_{t-\theta(t,x)}^{t} M(\tau) e^{-\mu(t-\tau)} d\tau, \tag{8}$$

$$\frac{x}{\nu(t)} = \frac{1}{M(t)} \int_{t-\theta(t,x)}^{t} M(\tau)\sigma(\tau) e^{\varepsilon \int_{\tau}^{t} \sigma(s) ds} d\tau. \tag{9}$$

The differential-difference equation (5) for the total power $M(t)$ in the intensive variables has the next form:

$$\frac{1}{M(t)} \frac{dM(t)}{dt} = \sigma(t) - \mu - \left(1 - \frac{dA(t)}{dt}\right) \sigma\left(t - A(t)\right) \frac{M\left(t - A(t)\right)}{M(t)} e^{-\mu A(t)}.$$

As already mentioned in the introduction, numerical experiments to determine the parameters of the numerical representation of the production function (8) - (9) for a number of countries (see, in particular, [Olenev, 2016]) showed that the largest age of capacities $A(t)$ in the time interval of several decades can be considered fixed for each country on its historically developed level, connected with the international division of labor. Therefore, further calculations will be made on the assumption that $A(t) = A = const$.

Then the differential-difference equation for total capacity with the fixed age limit A will take the form

$$\frac{dM(t)}{dt} = \left(\sigma(t) - \mu\right) M(t) - \sigma(t - A) M(t - A) e^{-\mu A} \tag{10}$$

with the initial condition $M(t) = \psi(t)$ when $-A \leq t \leq 0$.

3 Production Function in Transition Growth Path

In the computational experiments connected with the identification of the parameters of the numerical representation of the production function for a number of countries [Olenev, 2016], a characteristic growth path was observed. In this path, total capacity and output grow at a constant rate, the share of new capacities is constant and the capital–output ratio decreases. In this path, growth is not balanced, the share of total consumption in output increases, and the share of investment decreases. This is a transition path apparently limited in time.

Let's consider this transitional growth path in which, under the influence of scientific, technical and social progress, the coefficient of incremental capital–output ratio decreases over time:

$$b(t) = b_0 e^{-\beta t}, \tag{11}$$

where $\beta > 0$ is a rate of decreasing, and b_0 is an initial value of the capital–output ratio. In this case total capacity and total output grow with constant rate γ,

$$M(t) = M_0 e^{\gamma t}, Y(t) = Y_0 e^{\gamma t}, \tag{12}$$

and share of new capacities and maximal age of capacities remain constants.

$$\sigma = \frac{I(t)}{M(t)} = const, A = const. \tag{13}$$

For the transitional regime of endogenous growth (11)-(13) the following propositions are valid.

Proposition 1. *The new production function with the age limit of capacities A on transition growth path has the form*

$$f(t,x) = \frac{\sigma}{\gamma+\mu}\left\{1 - \left[1 - \frac{(\gamma-\varepsilon\sigma)}{\sigma}\frac{x}{\nu(t)}\right]^{-(\gamma+\mu)/(\gamma-\varepsilon\sigma)}\right\}, \tag{14}$$

where the rate of transition growth $\gamma = \varphi(\sigma,\mu,A)$ is an implicit function of parameters σ, μ, A, determined by

$$\gamma + \mu = \sigma\left(1 - e^{-(\gamma+\mu)A}\right). \tag{15}$$

The function (14) when $A \to \infty$ gives production function with capacities without the age limit received in [Olenev et al., 1986]:

$$f(t,x) = 1 - \left[1 - (1-\varepsilon-\mu/\sigma)\frac{x}{\nu(t)}\right]^{1/(1-\varepsilon-\mu/\sigma)}, \tag{16}$$

where $\sigma = \gamma + \mu$.

Proof: The conditions $\sigma(t) = \sigma = const$ and (10) give a relationship of growth rate γ with parameters σ, μ, A (15). Expressions (8), (9) give expression for the production function (14). Then growth rate is determined from relation $\gamma = \varphi(\sigma, A) - \mu$, where $\varphi(\sigma, A)$ is unique solution of the transcendental equation $1 - \varphi/\sigma = exp(-\varphi A)$ on interval $\varphi \in (0, \sigma)$ under the natural condition of the existence of a solution: $A > 1/\sigma$. The ratio of the average labor intensity to the lowest one $x/\nu(t) = const$. The labor grows: $L(t) = L_0 exp((\gamma - \varepsilon\sigma)t)$.

The relations (14), (15) when $A \to \infty$ give (16). Q.E.D.

Proposition 2. *Let the coefficient of capital–output ratio $b(t)$ in the transition growth path in a closed economy is diminished with constant rate β in accordance with (11), output $Y(t)$ and total capacity $M(t)$ grow with constant rate γ in accordance with (12), share of new capacities σ and age limit of capacities A are constant, and, besides that, scientific and technical progress takes place (3). Then the share of consumption in output $C(t)/Y(t)$ grows, the share of investment $b(t)I(t)/Y(t)$ diminishes, and average consumption in transition growth path $c(t) = C(t)/L(t)$ grows faster than in balanced growth path (with rate greater than the growth rate of scientific and technical progress $\varepsilon\sigma$).*

Proof: Note that from first equality of (13) it follows that the growth rate of new capacities $I(t)$ coincides with the growth rate of the total capacity $M(t)$, and in accordance to (11), (12) the share of investment in the output falls $b(t)I(t)/Y(t) = (b_0 I_0/Y_0)exp(-\beta t)$, accordingly, the share of consumption $C(t)/Y(t) = 1 - b(t)I(t)/Y(t)$ grows. The average consumption in the transition growth path by virtue of the proposition 1 is determined by the relation

$$c(t) = \frac{f(x/\nu(t)) - \sigma b_0 e^{-\beta t}}{\nu_0 x/\nu(t)} e^{\varepsilon\sigma t},$$

That is, the average consumption in the transition growth path grows faster than in balanced growth path. Q.E.D.

Proposition 3. *If the parameter β in (11) is set to zero, then the transition growth path will move into a balanced growth path. Sotwo variables $C(t)$ and $I(t)$ with exponential growth will add in (12):*

$$C(t) = C_0 e^{\gamma t}, I(t) = I_0 e^{\gamma t}.$$

The form of the production function (14), (15) will not change in this case.

If the parameter ε in (14) is set to zero, then the production function will remain non-linear:

$$f(t,x) = \frac{\sigma}{\gamma+\mu}\left\{1 - \left[1 - \frac{\gamma x}{\sigma\nu}\right]^{-(1+\mu/\gamma)}\right\},$$

and labor $L(t)$ on the balanced growth path will grow with the same rate γ, as the other scalable variables, $L(t) = L_0 exp(\gamma t)$.

The proof is immediate from the direct substitution of $\beta = 0$, and $\varepsilon = 0$ in (1)-(14).

4 Production Function Parameter Identification for Russian Economy

In general case when the share of new capacities in the total capacity $\sigma(t)$ is not constant the analytical form of the production function is not available. In this case it is possible to construct the production function numerically by using the micro relations following from hypothesis 0 directly: $m(t, t) = I(t)$, $m(t, \tau) = I(\tau)exp(-\mu(t - \tau))$, $\lambda(t, \tau) = \nu(\tau)exp(\mu(t - \tau))$, plus equation (3).

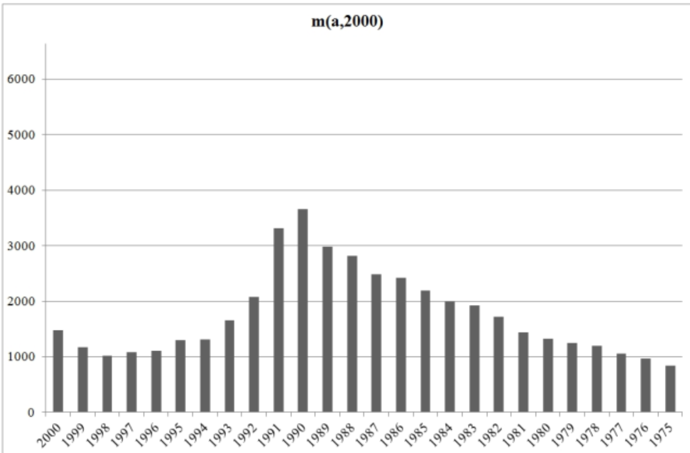

Figure 1: Age distribution of production capacities in 2000 in constant prices of 2005, Ruble billions

Identification of the parameters of the numerical representation of the production function in real conditions was carried out according to the data of Russian economy.

Here, an annual gross domestic product (GDP) at constant 2005 prices in billions Rubles is used as the output Y(t) of Russian economy. Consider the distribution of production capacity on age (see. Figure 1-2) for the distribution of production capacity on age in 2000, and 2009, respectively).

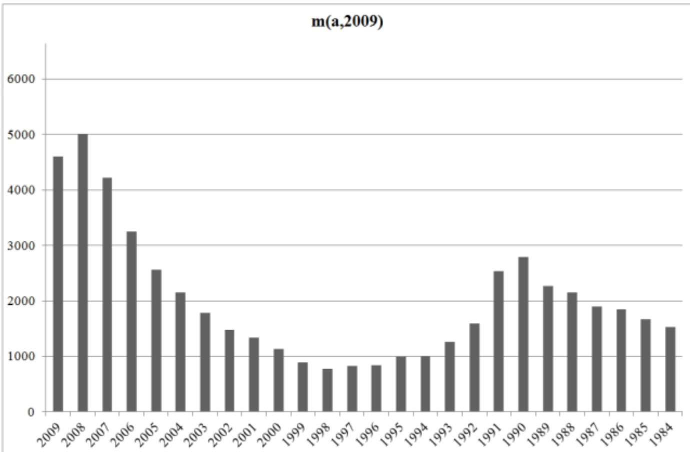

Figure 2: Age distribution of production capacities in 2009 in constant prices of 2005, Ruble billions

Let us evaluate the parameters $b(0), \beta, \mu, \nu(0), \varepsilon, m_0(\lambda)$ of Russian economy 1970-2015 by the model described above from some natural conditions. One of the condition is that the production capacities are utilized an average on 70% approximately, implying an existence of a normal reserve of capacities on the level of approximately 30% [Olenev, 1995].

Let use for estimation the fitting of time series for labour $L(t)$ and output $Y(t)$. Namely, we iterate through all valid sets of parameters using high-performance computing on supercomputer MVS-1000M of Joint Supercomputer Centre of Russian Academy of Sciences and select a combination of the parameters in which statistics and calculations for GDP Yt) coincide and fitting of statistics and calculations for labor $L(t)$ will be the best one. Assuming that all the capacities were used up to the age determined by the actual output from statistical data, the maximum age of used capacities determines the labor force. The corresponding volume of labor was calculated and compared with statistics (Figure 3).

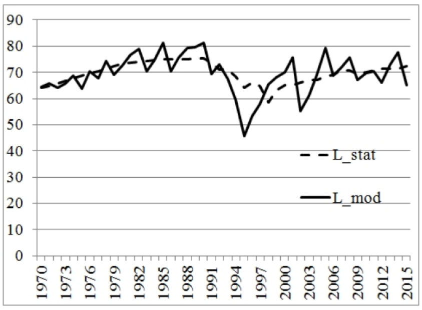

Figure 3: Identification of the model by the employment fitting to Russian economy data. Time series for employment: L_{mod} – estimation by vintage capacity model, L_{stat} – statistical data, millions people

The parameters were chosen so that the deviation of the calculation from the statistics was minimal. In this case, parallel calculations in MPI were used to speed up the calculation.

The real sector of Russian economy in 1970-2015, as shown by numerical experiments, is close to the transition regime studied above analytically with a share of new capacities $\sigma \approx 0.11$ (Figure 4) on the intervals of 1970-1990, 2007-2015. In addition from Figure 4 it is seen that the share of new capacities $\sigma(t)$ in the interval of 1991-2006 falls to the "bottom" $\sigma \approx \mu$, determined by the rate of capacity diminished due to wear and tear, and then growth to the "top" determined by some natural level of capacity utilization.

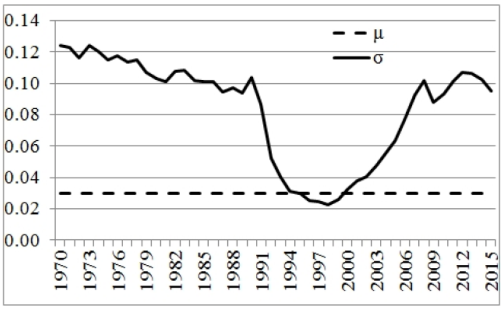

Figure 4: Time series for macroeconomic indices: $\sigma(t)$ — share of investments in total capacity, μ — rate of depreciation

As a result of research we have the next estimations for Russian economy of 1970-2015: capital–output ratio in 1970 $b_0 = 5.625$, rate of its decline $\beta = 0.040$, the best norm of labour intensity in 1970 $\nu_0 = 0.004725$, the rate of scientific and technical progress in Russian economy $\varepsilon = 0.2525$ (it was big enough), the rate of capacity depreciation $\mu = 0.030$, the age limit of production capacities $A = 25$. Initial distribution $m_0(\lambda)$ is specified parametrically as $m_0(a) = I_0 exp(-0.1275a)$, $\lambda(a) = \nu_0 exp(0.1275a)$, where the age of capacities $a \in [0, A]$.

The optimality criterion is the Theil index T of inequality [Theil, 1966] for calculated $L(t)$ and statistical $L_S(t)$ time series of labor, its minimum value is $T = 0.0580$. The Theil index for for calculated and statistical time series of output $Y(t)$ equals 0 in accordance to the chosen algorithm.

Value of $b(t)$ differs in different countries, the value characterizes the level of technology in new capacities and partially the level of corruption. A value of rate μ characterizes an average level of capacities depreciation as a result of technical ageing in the given country (it depends on the production culture and the climate conditions). The inverse value $1/\mu$ gives an evaluation of the maximum possible lifetime (age limit) of capacities by technical performance. Taking into account the moral depreciation the lifetime drops below $1/\mu$ increasing the competitiveness of the economy.

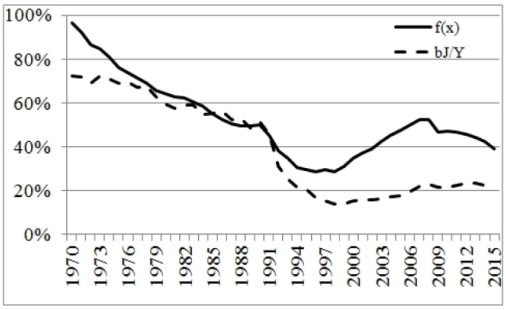

Figure 5: Time series for macroeconomic indices: $f(x)$ – capacity utilization, bI/Y – ratio of investment product to GDP

Figure 5 shows the dynamics of production capacity utilization by labor employed in Russian economy. It is seen that after 1991 the capacity utilization has fallen sharply below the economically reasonable level of 60-70%. The figure 5 shows also a ratio of investment product to GDP. It also falls down after 1991. Figure 6 shows that in 2000-2015 maximal age of capacities used in production is fell down below the average age.

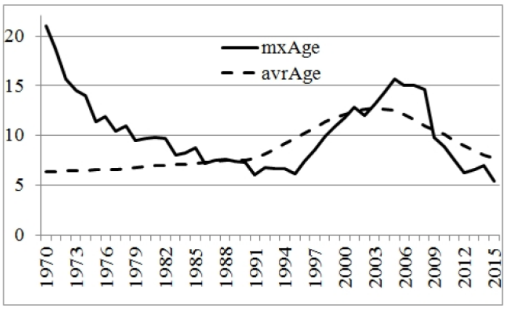

Figure 6: Time series for macroeconomic indices: $mxAge$ – maximal age of capacities which are used in production of output, $avrAge$ – average age of production capacities

The model allows conducting a comparative analysis of Russian economy with other countries. For example, Greek economy gives the next estimations for the model: capital–output ratio $b_0 = 2.755$ in 1970, rate of its decline $\beta = 0.009$, the best norm of labour intensity in 1970 $\nu_0 = 0.01576$, the rate of scientific and technical progress realized in production technologies is only $\varepsilon = 0.051$, the rate of capacity depreciation $\mu = 0.0375$.

Figure 7 illustrates the dynamics of the capital–output ratio for Russia (new) and for Greece [Olenev, 2016]. So that b(2015)=0.99 for Russia, and b(2015)=1.85 for Greece. This means particularly that the nowadays level of corruption (if we exclude the difference in technology and economic structure) in Greece is on 85% more greater than in Russia. It can be shown the smaller the age limit is, the more stable the economic system be.

The total capacity in each of compared countries was calculated by a condition that average usage of capacities is approximately equaled 70%. The condition gives a value for age limit of capacities which included in total capacity. It was determined that for Russian economy the age limit is 25 years old, and for Greece it is 30 years old. The smaller is the tail of the capacity distribution, the more sustainable is the growth rate of the economy.

5 Conclusions

Identification of classical production functions based on the data of a particular economy in the 60s of the XX century showed that the growth of the production factors used in them does not explain the growth of the economy [Barro et al., 2003]. As a result, exogenous values were used to explain economic growth or they consider a two-sector model with sectors of production and education taking into account the role of human capital [Barro et al., 2003].

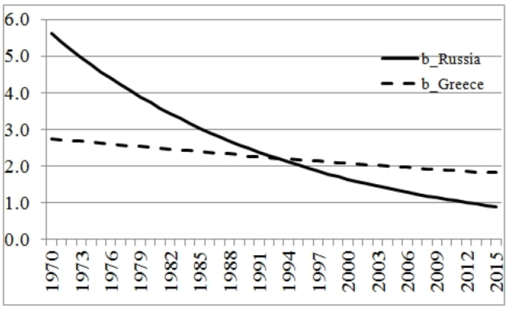

Figure 7: Time series for capital–output ratio $b(t)$ for economy of Russia (solid curve) and for economy of Greece (dashed curve)

In this paper we consider another possibility of explaining economic growth by taking into account three exogenous mechanisms of technological progress: 1) growth of labor productivity on new capacities, 2) the age limit in use of capacities, and 3) reducing the capital–output ratio of creating new production capacities.

An analytical expression is obtained here for the production function in the transition growth path at a fixed age limit of production capacities A. It coincides with the form of production function on balanced growth path.

The new transition growth path proposed in this work has been studied analytically and numerically. In this path the incremental capital–output ratio $b(t)$ decreases at a constant rate, and the share of consumption in the output $c(t)$ increases. The decrease in the coefficient of incremental capital–output ratio $b(t)$, found in numerical experiments when identifying the parameters of the production function from different countries, can be explain due to various reasons: 1) scientific and technological progress, 2) reducing corruption, 3) increasing access to financial resources, 4) structural changes in the real sector. This requires further research.

Results of identifying the parameters of the production function for Russian economy were presented.

Acknowledgements

This work was supported by the Russian Science Foundation, Project 14-11-00432.

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