Numerical Algorithm for Optimal Control of Continuity Equations

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Abstract

An optimal control problem for the continuity equation is considered. The aim of a controller is to maximize the total mass within a target set at a given time moment. An iterative numerical algorithm for solving this problem is presented.

1 Introduction

Consider a *mass* distributed on $\mathbb{R}^n$ that drifts along a controlled vector field $\mathbf{v} = \mathbf{v}(t, x, u)$. The aim of the *controller* is to bring as much mass as possible to a target set A by a time moment T.

Let us give the precise mathematical statement of the problem. Suppose that $\rho = \rho(t, x)$ is the density of the distribution and $u = u(t)$ is a strategy of the *controller*. Then, ρ evolves in time according to the continuity equation

$$\begin{cases} \rho_t + \operatorname{div}_x \left(\mathbf{v} \left(t, x, u(t) \right) \rho \right) = 0, \\ \rho(0, x) = \rho_0(x), \end{cases} \tag{1}$$

where ρ_0 denotes the initial density. Our aim is to find a control u that **maximizes** the following integral

$$J[u] = \int_A \rho(T, x) \, \mathrm{d}x \, . \tag{2}$$

Typically, u belongs to a set $\mathcal{U}$ of admissible controls. Here we take the following one:

$$\mathcal{U} = \left\{ u(\cdot) \text{ is measurable, } u(t) \in U \quad \text{a.e. } \ t \in [0, T] \right\}, \tag{3}$$

where U is a compact subset of $\mathbb{R}^m$.

In this paper we propose an iterative method for solving problem (1)–(3), which is based on the needle linearization algorithm for classical optimal control problems [Srochko, 2000]. Given an initial guess u^0, the algorithm produces a sequence of controls u^k with the property $J[u^{k+1}] \geq J[u^k]$, for all $k \in \mathbb{N}$.

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A different approach for numerical solution of (1)–(3) was proposed by S. Roy and A. Borzì in [Roy, 2017]. The authors used a specific discretization of (1) to produce a finite dimensional optimization problem. It seems difficult to compare the efficiency of both algorithms, because one was tested for 2D and the other for 1D problems.

Finally, let us remark that problem (1)–(3) is equivalent to the following optimal control problem for an ensemble of dynamical systems:

$$\text{Maximize} \quad \int_{\{x\,:\, x=y(T)\}} \rho_0(x)\,\mathrm{d}x \quad \text{subject to} \quad \begin{cases} \dot y = -\mathbf{v}(T-t, y, u(t)), \\ y_0 \in A. \end{cases}$$

Indeed, instead of transporting the mass, one can transport the target A in reverse direction aiming at the region that contains maximal mass.

2 Preliminaries

We begin this section by introducing basic notation and assumptions that will be used throughout the paper. Next, we discuss a necessary optimality condition lying at the core of the algorithm.

Notation

In what follows, $\Phi_{s,t}$ denotes the flow of a time-dependent vector field $\mathbf{w} = \mathbf{w}(t,x)$, i.e., $\Phi_{s,t}(x) = y(t)$, where $y(\cdot)$ is a solution to the Cauchy problem

$$\begin{cases} \dot y(t) = \mathbf{w}\left(t, y(t)\right), \\ y(s) = x. \end{cases}$$

Given a set $A \subset \mathbb{R}^n$ and a time interval $[0,T]$, we use the symbol A^t for the image of A under the map $\Phi_{T,t}$, i.e., $A^t = \Phi_{T,t}(A)$. The Lebesgue measure on $\mathbb{R}$ is denoted by $\mathcal{L}^1$.

Assumptions

- The map $\mathbf{v}\colon [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous.
- The map $x \mapsto \mathbf{v}(t,x,u)$ is twice continuously differentiable, for all $t \in [0,T]$ and $u \in U$.
- There exist positive constants L, C such that $|\mathbf{v}(t,x,u) - \mathbf{v}(t,x',u)| \le L|x-x'|$ and $|\mathbf{v}(t,x,u)| \le C\,(1+|x|)$, for all $t \in [0,T]$, $u \in U$, and $x, x' \in \mathbb{R}^n$.
- The initial density ρ_0 is continuously differentiable.
- The target set $A \subset \mathbb{R}^n$ is a compact tubular neighborhood, i.e., A is a compact set that can be expressed as a union of closed n-dimensional balls of a certain positive radius r.

In addition, to guarantee the existence of an optimal control (see [Pogodaev, 2016] for details), we must assume that

- the vector field $\mathbf{v}$ takes the form

  $$\mathbf{v}(t,x,u) = \mathbf{v}_0(t,x) + \sum_{i=1}^{l} \varphi_i(t,u)\mathbf{v}_i(t,x),$$

  for some real-valued functions φ_i, and the set

  $$\Phi(t,U) = \begin{pmatrix} \varphi_1(t,U) \\ \cdots \\ \varphi_l(t,U) \end{pmatrix} \subset \mathbb{R}^l$$

  is convex.

Necessary Optimality Condition

The necessary optimality condition for problem (1)–(3) looks as follows:

Theorem 2.1 ([Pogodaev, 2016]) *Let u be an optimal control for (1)–(3) and ρ be the corresponding trajectory with $\rho_0 \in \mathbf{C}^1(\mathbb{R}^n)$. Then, for a.e. $t \in [0,T]$, we have*

$$\int_{\partial A^t} \rho(t,x)\, \mathbf{v}\left(t,x,u(t)\right) \cdot \mathbf{n}_{A^t}(x)\, \mathrm{d}\sigma(x) = \min_{\omega \in U} \int_{\partial A^t} \rho(t,x)\, \mathbf{v}(t,x,\omega) \cdot \mathbf{n}_{A^t}(x)\, \mathrm{d}\sigma(x)\,. \tag{4}$$

Here $A^t = \Phi_{t,T}(A)$, where Φ is the phase flow *of the vector field $(t,x) \mapsto \mathbf{v}\left(t,x,u(t)\right)$, $\mathbf{n}_{A^t}(x)$ is the measure theoretic* outer unit normal *to A^t at x, σ is the $(n-1)$-dimensional* Hausdorff measure*.*

The precise definitions of the measure theoretic unit normal and the Hausdorff measure can be found, e.g., in [Evans, 1992]. We remark that whenever ∂A is an $(n-1)$-dimensional surface, $\mathbf{n}_{A^t}$ coincides with the usual outer unit normal to A^t and σ coincides with the usual $(n-1)$-dimensional volume form.

Let $I(\varepsilon) \subseteq [0,T]$ be a measurable set of Lebesgue measure ε. Given two controls u and w, we consider their mixture

$$u_{w,I(\varepsilon)}(t) = \begin{cases} w(t), & t \in I(\varepsilon), \\ u(t), & \text{otherwise.} \end{cases} \tag{5}$$

The proof of Theorem 2.1 gives, as a byproduct, the following increment formula

$$J[u_{w,I(\varepsilon)}] - J[u] = \int_{I(\varepsilon)} \int_{\partial A^t} \rho(t,x)\, \left[\mathbf{v}\left(t,x,u(t)\right) - \mathbf{v}\left(t,x,w(t)\right)\right] \cdot \mathbf{n}_{A^t}(x)\, \mathrm{d}\sigma(x)\, \mathrm{d}t + o(\varepsilon), \tag{6}$$

which will be used in the next section.

3 Numerical Algorithm

In this section we describe the algorithm, prove the improvement property $J[u^{k+1}] \geq J[u^k]$, and discuss a possible implementation.

3.1 Description

1. Let u^k be a current guess. For each t, compute the set ∂A^t and $\rho(t,\cdot)$ on ∂A^t.
2. For each t, find
$$w(t) \in \operatorname{argmin} \left\{ \int_{\partial A^t} \rho(t,x)\, \mathbf{v}\left(t,x,\omega\right) \cdot \mathbf{n}_{A^t}(x)\, \mathrm{d}\sigma(x) \ : \ \omega \in U \right\}. \tag{7}$$
3. Let
$$g(t) = \int_{\partial A^t} \rho(t,x)\, \left[\mathbf{v}\left(t,x,u^k(t)\right) - \mathbf{v}\left(t,x,w(t)\right)\right] \cdot \mathbf{n}_{A^t}(x)\, \mathrm{d}\sigma(x)\,.$$
4. For each $\varepsilon \in (0,T]$, find
$$I(\varepsilon) \in \operatorname{argmax} \left\{ \int_\iota g(t)\, \mathrm{d}t \ : \ \iota \subset [0,T] \text{ is measurable and } \mathcal{L}^1(\iota) = \varepsilon \right\}. \tag{8}$$
5. Construct $u_{w,I(\varepsilon)}$ by (5).
6. Find
$$\varepsilon^* \in \operatorname{argmax} \left\{ J[u_{w,I(\varepsilon)}] \ : \ \varepsilon \in (0,T] \right\}. \tag{9}$$
7. Let $u^{k+1} = u_{w,I(\varepsilon^*)}$.

The algorithm produces an infinite sequence of admissible controls. Of course, any its implementation should contain obvious modifications that would cause the algorithm to stop after a finite number of iterations. Note that it may happen that problems (8) and (9) admit no solution. In this case $I(\varepsilon)$ and ε^* must be taken so that the values of the corresponding cost functions lie near the supremums.

3.2 Justification

If u^k satisfies the optimality condition (4) then we obviously get that $u^{k+j} = u^k$, for all $j \in \mathbb{N}$. In particular, this means that $J[u^{k+1}] = J[u^k]$.

If u^k does not satisfy the optimality condition then $\int_{I(\varepsilon)} g(t)\,\mathrm{d}t > 0$, for all small $\varepsilon > 0$. By the increment formula (6), we have

$$J[u_{w,I(\varepsilon)}] - J[u^k] = \int_{I(\varepsilon)} g(t)\,\mathrm{d}t + o(\varepsilon).$$

Since the integral from the right-hand side is positive for all small ε, we conclude that $J[u^{k+1}] = J[u_{w,I(\varepsilon^*)}] > J[u^k]$, as desired.

3.3 Implementation Details

The method was implemented for the case $\dim x = 2$. All ODEs are solved by the Euler method. The set ∂A is approximated by a finite number of points. Below we discuss in details all non-trivial steps of the algorithm.

Step 1

In this step we must compute $\rho(t,x)$ for all t and x satisfying $x \in \partial A^t$. Recall that

$$\rho(t,x) = \frac{\rho_0(y)}{\det D\Phi_{0,t}(y)}, \quad \text{where} \quad y = \Phi_{t,0}(x).$$

Using Jacobi's formula, we may write

$$\frac{d}{dt}\left(\det D\Phi_{0,t}(y)\right) = \left(\det D\Phi_{0,t}(y)\right) \cdot \operatorname{tr}\left[D\Phi_{0,t}(y)^{-1} \frac{d}{dt} D\Phi_{0,t}(y)\right].$$

Meanwhile, by the definition of Φ, we have

$$\frac{d}{dt} D\Phi_{0,t}(y) = D_x \mathbf{v}\left(t, \Phi_{0,t}(y), u(t)\right) \cdot D\Phi_{0,t}(y).$$

Combining the above identities gives

$$\frac{d}{dt}\left(\det D\Phi_{0,t}(y)\right) = \left(\det D\Phi_{0,t}(y)\right) \operatorname{div} \mathbf{v}\left(t, \Phi_{0,t}(y), u(t)\right).$$

Thus, computing of $\rho(t,x)$ requires solving two Cauchy problems, one for finding $\Phi_{0,t}(y)$ and one for finding $\det D\Phi_{0,t}(y)$.

Step 2

In general, the optimization problem (7) is nonlinear, which makes it difficult. On the other hand, in many cases U and $\mathbf{v}$ enjoy the following extra properties:

- the set U is convex and the vector field $\mathbf{v}$ is affine with respect to the control:

$$\mathbf{v}(t,x,u) = \mathbf{v}_0(t,x) + \sum_{i=1}^{m} \mathbf{v}_i(t,x)\, u_i.$$

Now (7) becomes a convex optimization problem, and thus it can be effectively solved.

Step 4

The problem (8) seems difficult at first glance. But note that it is equivalent to the following one:

$$\text{Minimize} \quad l(\lambda) := \left|\mathcal{L}^1\left(\{t : g(t) \geq \lambda\}\right) - \varepsilon\right| \quad \text{subject to} \quad \lambda \in [\min g, \max g]. \tag{10}$$

Indeed, if λ_* solves (10), then the set $I = \{t : g(t) \geq \lambda_*\}$ solves the original problem (8). To find λ_* numerically, we may take a finite mesh on the interval $[\min g, \max g]$ and look for a node that gives minimal value to $l(\cdot)$.

Step 7

In this step the cost

$$\int_A \rho(T,x)\,\mathrm{d}x = \int_{A^0} \rho_0(x)\,\mathrm{d}x$$

must be computed. To that end, we must know the whole set A^0, while on the other steps of the algorithm we deal only with the boundaries of A^t. It is interesting to note that, under the additional assumption that

- the target set $A \subset \mathbb{R}^n$ is contractible and its boundary ∂A is an $(n-1)$-dimensional smooth surface,

the knowledge of ∂A^0 is enough for computing the cost.

Indeed, since the target $A = A^T$ is contractible, the set A^0 is contractible as well. Any differential form on a contractible set is exact [Tu, 2011]. Hence $\rho_0\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \mathrm{d}a$, for some $(n-1)$-dimensional differential form α. Now the Stokes theorem gives:

$$\int_{A^0} \rho_0\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \int_{\partial A^0} \alpha.$$

Let us compute α in the 2D case to illustrate this approach. We must find a form $\alpha = a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2$ such that $\mathrm{d}\alpha = \rho_0\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$. The latter equation holds when

$$\rho_0 = \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}.$$

Hence, to get the desired α, we may take

$$a_2(x_1,x_2) = \int_0^{x_1} \rho_0(\xi, x_2)\,\mathrm{d}\xi\,, \quad a_1 \equiv 0.$$

4 Examples

This section describes several toy problems, which we used for testing the algorithm.

4.1 Boat

Consider a boat floating in the middle of a river at night. Since it is dark, the boatmen cannot see any landmarks, and therefore are unsure about the boat's position. They want to reach a river island at a certain time with highest probability. How should they act?

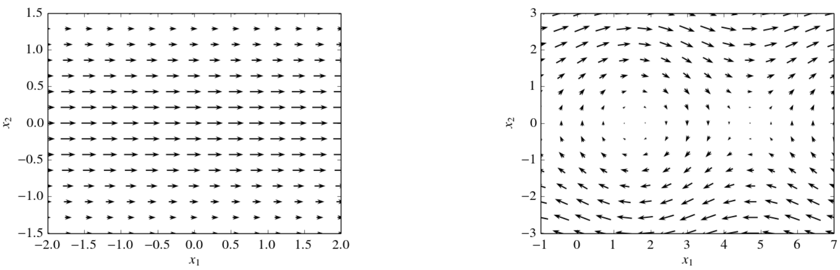

Figure 1: Left: river drift. Right: pendulum drift.

Assume that the speed of the river water is given by

$$\mathbf{v}_0(x) = \begin{pmatrix} \alpha + e^{-\beta x_2^2} \\ 0 \end{pmatrix},$$

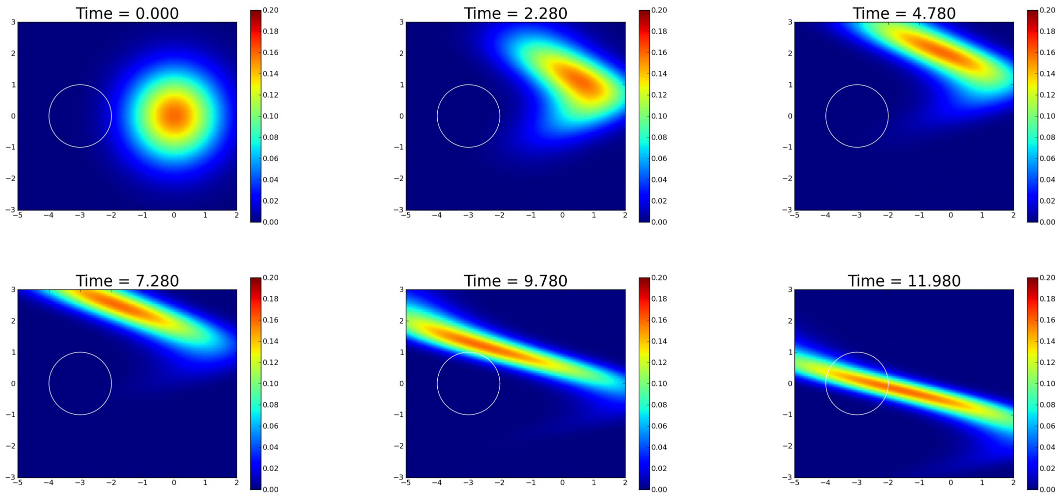

Figure 2: Trajectory for the boat problem computed by the algorithm.

the island is a unit circle centered at x_0, the initial position of the boat is described by the density function

$$\rho_0(x) = \frac{1}{2\pi\sigma^2} e^{-|x|^2/(2\sigma^2)}. \tag{11}$$

Thus, the boat's position $x(t)$ evolves according to the differential equation

$$\dot{x} = \mathbf{v}_0(x) + u,$$

where $u \in \mathbb{R}^2$ is a component of the boat's velocity due to rowing. Here $|u| \le u_{\max}$.

Parameters for the computation: $\sigma = 1$, $\alpha = \beta = 0.5$, $u_{\max} = 0.75$, $x_0 = (-3, 0)$, $T = 12$.

4.2 Pendulum

Here we want to stop a moving pendulum whose initial position is uncertain. In this case we have

$$\mathbf{v}_0(x) = \begin{pmatrix} x_2 \\ \cos x_1 \end{pmatrix}, \qquad \mathbf{v}_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence the control system takes the form

$$\dot{x} = \mathbf{v}_0(x) + u\,\mathbf{v}_1(x),$$

where $u \in [-u_{\max}, u_{\max}]$ is an external force. The initial position of the pendulum is given by (11). The target is a unit circle centered at $(\pi/2, 0)$.

Parameters for the computation: $\sigma = 1$, $u_{\max} = 0.5$, $x_0 = (\pi/2, 0)$, $T = 6$.

4.3 Sheep

Consider a herd of sheep located near the origin. The sheep are effected by a vector field $v_0(x)$ pushing them away from the origin. To prevent this we can turn on repellers, which are located at the following positions

$$x_k = \left(R\cos\frac{2\pi(k-1)}{m}, R\sin\frac{2\pi(k-1)}{m} \right), \qquad k = 1, \dots, m.$$

Each repeller produces a vector field $\mathbf{v}_k(x)$. So we have

$$\mathbf{v}(x, u) = \mathbf{v}_0(x) + \sum_{k=1}^{m} u_k \mathbf{v}_k(x),$$

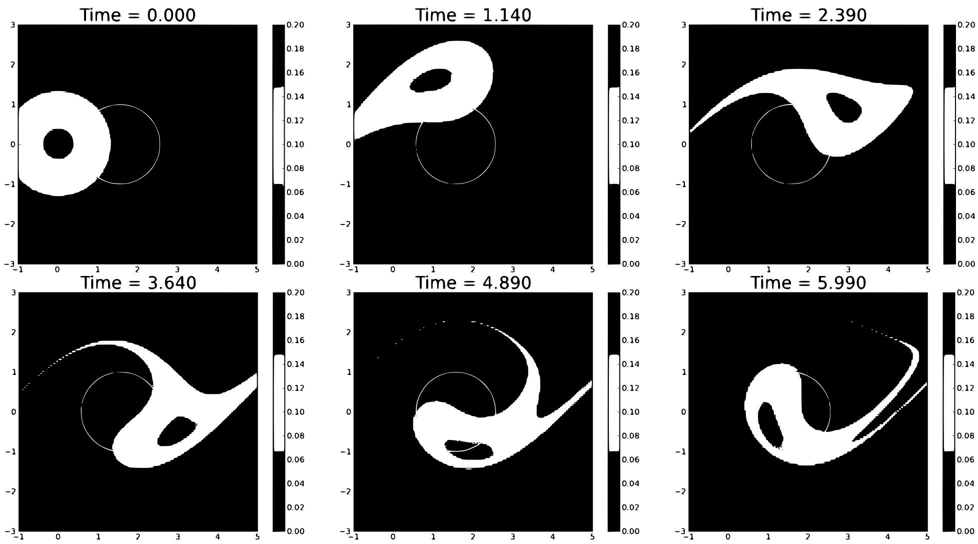

Figure 3: Trajectory for the pendulum problem computed by the algorithm.

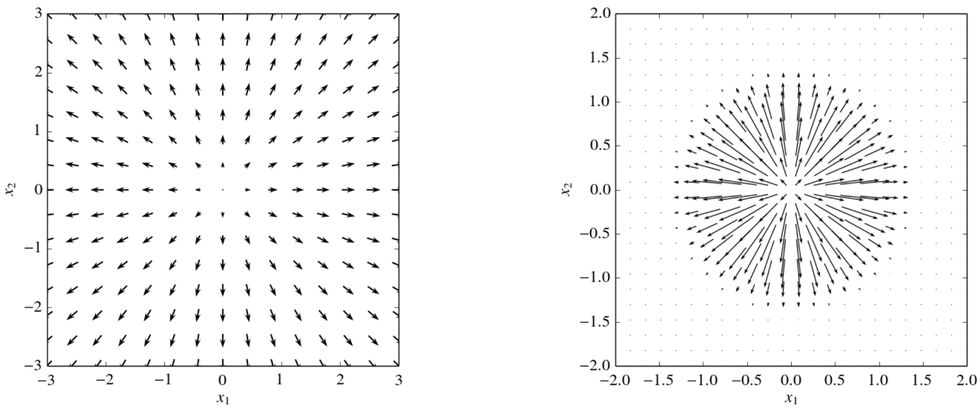

Figure 4: Left: sheep drift. Right: repeller's force field.

where u_k is an intensity of k-th repeller. The control $u = (u_1, \dots, u_m)$ belongs to the simplex

$$U = \left\{ (u_1, \dots, u_m) : \sum_{k=1}^{m} u_k = 1,\ u_k \in [0,1],\ k = 1, \dots, m \right\}.$$

In what follows we set

$$\mathbf{v}_0(x) = \alpha \, \frac{x - x_0}{\sqrt{1 + |x - x_0|^2}},$$

where x_0 is a certain point not far from the origin, and

$$\mathbf{v}_k(x) = \beta \, e^{-|x - x_k|^4} (x - x_k), \qquad k = 1, \dots, m.$$

Suppose that the initial distribution is given by (11), the target is an ellipse centered at x_0 whose major and minor semi-axes are a and b.

Parameters for the computation: $\sigma = 1$, $x_0 = (0, 0)$, $T = 3$, $m = 6$, $a = 2$, $b = 1.2$.

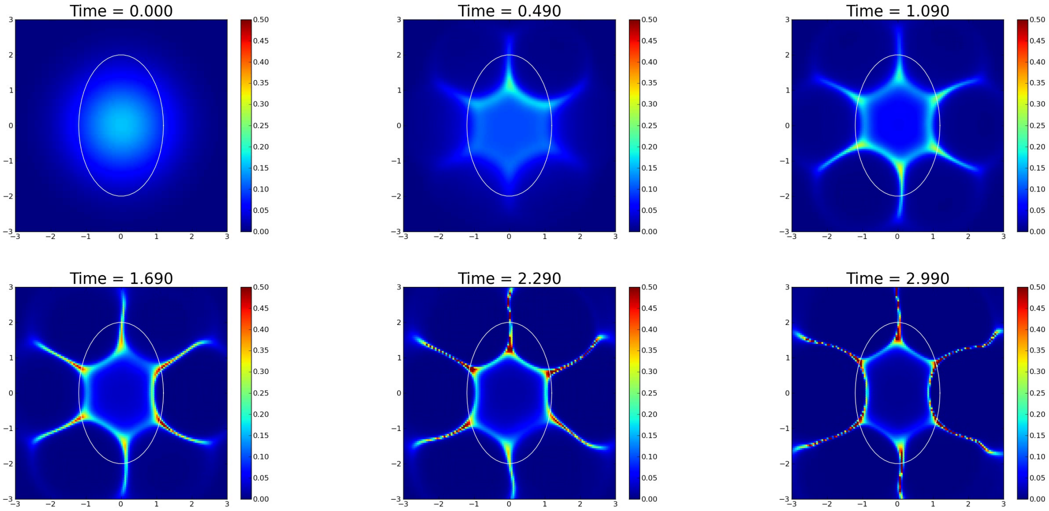

Figure 5: Trajectory for the sheep problem computed by the algorithm.

Remark 4.1 *The answer to the minimization problem*

$$\sum_{i=1}^{m} c_i \omega_i \to \min, \quad \omega \in U,$$

arising in the second step of the algorithm, is very simple. Let j be such that

$$c_j \leq c_i \quad \textit{for all } i = 1, \dots, m;$$

then an optimal solution is given by $\bar{\omega} = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is located at the j-th position. In particular, this means that at every time moment t only one repeller is turned on. Hence instead of repellers, we may think of a dog that jumps from one place to another.

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