Optimality Conditions for Optimal Impulsive Control Problems with Multipoint State Constraints

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Abstract

This paper deals with optimal impulsive control problems whose states are functions of bounded variation and impulsive controls are regular vector measures. The problem under consideration has multipoint state constraints. Examples of such problems may be found in mechanics, quantum electronics, robotics, ecology, economics, etc. Sufficient global optimality conditions involving certain sets of Lyapunov type functions are proposed. These Lyapunov type functions are strongly monotone solutions of the corresponding Hamilton-Jacobi inequalities.

1 Introduction

This paper concerns an optimal impulsive control problem in which the control system is an extension of the control system

$$\dot{x}(t) = f\big(t, x(t), V(t), u(t)\big) + G\big(t, x(t), V(t)\big)v(t), \tag{1}$$

$$u(t) \in U, \quad v(t) \in K \quad \text{a.e. on } T, \tag{2}$$

where $T = [a, b]$ is a fixed time interval, U is a compact set in $\mathbb{R}^r$, K is a convex closed cone in $\mathbb{R}^m$, $x(\cdot) \in W^{1,1}(T, \mathbb{R}^n)$, $u(\cdot) \in L^\infty(T, \mathbb{R}^r)$, $v(\cdot) \in L^\infty(T, \mathbb{R}^m)$. Function $V(\cdot)$ is the total variation on $[a, t]$ for the function $t \to w(t) \doteq \int_a^t v(\tau)d\tau$, that is, $V(t) \doteq \sum_{i=1}^{m} \operatorname*{var}_{[a,t]} w_i(\cdot)$. The symbol "a.e." signifies "almost everywhere with respect to the Lebesgue measure, $\mathcal{L}$".

In general, optimization problems over the control system (1), (2) do not have solutions in the class of absolutely continuous trajectories and Lebesgue measurable controls. This is explained by the fact that the right-hand side of (1) is pointwise unbounded. Thereby minimizing sequences of trajectories may pointwise tend to discontinuous functions. By closing the set of solutions of (1), (2) in the weak* topology in the space of functions of bounded variation, we obtain an impulsive control system which can be formally described as follows

$$dx(t) = f\big(t, x(t), V(t), u(t)\big)dt + G\big(t, x(t), V(t)\big)\mu, \tag{3}$$

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$$u(t) \in U \quad \text{a.e. on } T, \quad \mu(B) \in K \quad \forall\, B \in \mathcal{B}_T. \tag{4}$$

Here, μ is a K-valued bounded Borel measure on T, $x(\cdot)$ is a function of bounded variation, $\mathcal{B}_T$ is the set of all Borel subsets of T, and $V(\cdot)$ is a nonnegative nondecreasing function such that $V(b) \geq |\mu|\big([a,b]\big)$, where the measure $|\mu|$ is the total variation of μ. Let us note that any interpretation of (3), (4) as a measure-driven differential equation cannot provide a concept of solution with well-posedness properties [Bressan & Rampazzo, 1988], [Miller, 1996], [Miller & Rubinovich, 2003], [Motta & Rampazzo, 1995], [Motta & Rampazzo, 1996], [Motta & Rampazzo, 1996], [Sesekin & Zavalishchin, 1997]. There exist many solutions corresponding to given $u(\cdot)$, μ, and an initial point $x(a)$. This is due to the fact that we do not assume any commutativity property of the vector fields generated by the columns of G. Namely, generally the Lie brackets $[G_i, G_j]$, $i,j = \overline{1,m}$, do not vanish identically. To overcome this drawback we extend the notion of impulsive control to a pair $\pi(\mu)$ consisting of μ and an additional component $\gamma(\mu)$ defined below. Such $\gamma(\mu)$ characterizes a way of approximation of μ by some sequences of $\mathcal{L}$-absolutely continuous measures $\mu_k = v_k(t)dt$, where $v_k : T \to K$.

Throughout this paper we assume that the following conditions are satisfied.

H1. The functions $f(t,x,V,u)$, $G(t,x,V)$ are continuous; for any compact set $Q \subset \mathbb{R}^n$ there exist constants L_{1Q}, $L_{2Q} > 0$ such that

$$|f(t,x_1,V,u) - f(t,x_2,V,u)| \leq L_{1Q}|x_1 - x_2|, \quad |G(t,x_1,V) - G(t,x_2,V)| \leq L_{2Q}|x_1 - x_2|$$
$$\text{whenever } (t,x_1,V,u),\ (t,x_2,V,u) \in T \times Q \times \mathbb{R}_+ \times U.$$

Moreover, there exist constants $c_1,\ c_2 > 0$ such that

$$|f(t,x,V,u)| \leq c_1(1+|x|), \qquad |G(t,x,V)| \leq c_2(1+|x|) \quad \text{whenever} \quad (t,x,V,u) \in T \times \mathbb{R}^n \times \mathbb{R}_+ \times U.$$

Here, $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geq 0\}$; $|\cdot|$ denotes a vector norm or a consistent matrix norm.

H2. The set $f(t,x,V,U)$ is a convex set for every $(t,x,V) \in T \times \mathbb{R}^n \times \mathbb{R}_+$.

Denote $K_1 = \{v \in K \mid ||v|| = 1\}$, where $||v|| = \sum\limits_{j=1}^{m} |v_j|$, and let $co\,A$ be the convex hull of a set A. Let μ be a bounded Borel measure on T whose values are from K. Given μ, we denote by $\mu_{\rm c}$, $|\mu_{\rm c}|$, and $S_{\rm d}(\mu)$ the continuous component in the Lebesgue decomposition of the measure μ, total variation of the measure $\mu_{\rm c}$, and the set on which the discrete component of μ is concentrated, that is, $S_{\rm d}(\mu) \doteq \{s \in T \mid \mu(\{s\}) \neq 0\}$, respectively.

Let $\pi(\mu)$ be a pair $\big(\mu, \gamma(\mu)\big)$ whose components satisfy the following conditions:

i) μ is a K-valued bounded Borel measure on T,

ii) $\gamma(\mu)$ is a set $\{d_s, \omega_s(\cdot)\}_{s\in S}$ such that

(a) $S \supseteq S_{\rm d}(\mu)$ and S is at most countable subset of T,

(b) for every $s \in S$, $d_s \in \mathbb{R}_+$ and ω_s is $\mathcal{L}$-measurable function $[0, d_s] \to co\ K_1$ such that

$$d_s \geq ||\mu\big(\{s\}\big)||, \quad \int_0^{d_s} \omega_s(\tau) d\tau = \mu\big(\{s\}\big),$$

(c) $\sum\limits_{s\in S} d_s < \infty$.

The element $\pi(\mu)$ is called an impulsive control. We denote by $\mathcal{W}(T,K)$ the set of all impulsive controls $\pi(\mu)$ satisfying conditions i), ii). The second control $u(\cdot)$ has to be regarded as an ordinary control from $L^\infty(T, \mathbb{R}^r)$ such that $u(t) \in U$ a.e. $t \in T$.

Given controls $u(\cdot)$, $\pi(\mu)$ and an initial condition $x(a)$, consider the system of differential equations with the measure

$$dx(t) = f\big(t, x(t), V(t), u(t)\big)dt + G\big(t, x(t), V(t)\big)\mu_c + \sum_{s\in S,\, s\leq t} \big(z_s(d_s) - x(s-)\big)\delta(t-s), \tag{5}$$

$$dV(t) = |\mu_c| + \sum_{s\in S,\, s\le t} \big(z_{Vs}(d_s) - V(s-)\big)\delta(t-s), \qquad V(a)=0, \qquad t\in T, \tag{6}$$

$$\frac{dz_s(\tau)}{d\tau} = G\big(s, z_s(\tau), z_{Vs}(\tau)\big)\omega_s(\tau), \qquad z_s(0) = x(s-), \tag{7}$$

$$\frac{dz_{Vs}(\tau)}{d\tau} = 1, \qquad z_{Vs}(0) = V(s-), \quad \tau\in[0,d_s], \qquad s\in S. \tag{8}$$

Let $\big(x(\cdot), V(\cdot)\big)$ be a solution of (5)–(8). Then $x(\cdot), V(\cdot)$ are functions of bounded variation. Without loss of generality, suppose that $x(\cdot)$ and $V(\cdot)$ are right continuous on $(a,b]$. From (6), (8) it follows that

$$V(t) = |\mu_c|\big([a,t]\big) + \sum_{s\le t,\, s\in S} d_s, \quad t\in(a,b].$$

Now we consider the impulsive control system $(\mathcal{D})$

$$dx(t) = f\big(t, x(t), V(t), u(t)\big)dt + G\big(t, x(t), V(t)\big)\pi(\mu), \tag{9}$$

$$u(t)\in U \ \text{ a.e. on } T, \quad \pi(\mu)\in\mathcal{W}\big(T,K\big). \tag{10}$$

Let $u(\cdot)$ and $\pi(\mu)$ satisfy (10). The tuple $\sigma = \big(X_V, u(\cdot), \pi(\mu)\big)$ is said to be an impulsive process of $(\mathcal{D})$ if X_V is a set-valued function acting from T to comp$(\mathbb{R}^{n+1})$ such that

i) $\forall\, t\in T/S \qquad X_V(t) = \big\{\big(x(t), V(t)\big)\big\}$,

ii) for every $s\in S \qquad X_V(s) = \big\{\big(z_s(\tau), z_{Vs}(\tau)\big) \mid \tau\in[0,d_s]\big\}$.

Here, functions $\big(x(\cdot), V(\cdot)\big)$, $\big(z_s(\cdot), z_{Vs}(\cdot)\big)$ satisfy (5)–(8) with some initial condition $x(a)$, comp$(\mathbb{R}^{n+1})$ is the set of non-empty compact subsets from $\mathbb{R}^{n+1}$. Set by definition

$$X_V(t-) = \big\{\big(x(t-), V(t-)\big)\big\} \ \ \forall\, t\in(a,b], \quad X_V(a-) = \big\{\big(x(a), 0\big)\big\},$$
$$X_V(t+) = \big\{\big(x(t+), V(t+)\big)\big\} \ \ \forall\, t\in[a,b), \quad X_V(b+) = \big\{\big(x(b), V(b)\big)\big\}.$$

Denote by Σ the set of all impulsive processes $\sigma = \big(X_V, u(\cdot), \pi(\mu)\big)$ of $(\mathcal{D})$.

Let us briefly comment on the relation between $(\mathcal{D})$ and the corresponding conventional system (1), (2).

Given X_V, define its graph on T to be

$$\operatorname*{graph}_T X_V \doteq \big\{(t,x,V) \mid t\in T,\ (x,V)\in X_V(t)\big\}.$$

Let $A, B\in$ comp$(\mathbb{R}^{n+1})$. Denote by $d(A,B)$ the Hausdorff distance between A and B.

Lemma 1 [Samsonyuk, 2015]. 1) *Let* $\sigma = \big(X_V, u(\cdot), \pi(\mu)\big) \in \Sigma$. *Then, there exists a sequence* $\big\{x_k(\cdot), V_k(\cdot), u_k(\cdot), v_k(\cdot)\big\}$ *such that*

i) for every k, *the functions* $x_k(\cdot), V_k(\cdot), u_k(\cdot)$, $v_k(\cdot)$ *satisfy* (1), (2);

ii)

$$d\Big(\operatorname*{graph}_T X_V, \operatorname*{graph}_T\,(x_k, V_k)\Big) \to 0, \qquad k\to\infty. \tag{11}$$

2) *Let* $\big\{x_k(\cdot), V_k(\cdot), u_k(\cdot), v_k(\cdot)\big\}$ *be a sequence of functions such that*

i) $\sup_k ||v_k(\cdot)||_{L_1} < \infty$;

ii) for every k, *the functions* $x_k(\cdot), V_k(\cdot), u_k(\cdot)$, $v_k(\cdot)$ *satisfy* (1), (2);

iii) $\{x_k(a)\}$ *is bounded.*

Then, there exist $\sigma = \big(X_V, u(\cdot), \pi(\mu)\big) \in \Sigma$ *and a subsequence* $\big\{x_{kj}(\cdot), V_{kj}(\cdot), u_{kj}(\cdot), v_{kj}(\cdot)\big\}$ *such that*

$$d\Big(\operatorname*{graph}_T X_V, \operatorname*{graph}_T\,(x_{kj}, V_{kj})\Big) \to 0, \qquad j\to\infty.$$

2 Statement of the Problem

Let $\theta = (\theta_0, \dots, \theta_k)$ be a vector of fixed points of time such that $a \le \theta_0 < \dots < \theta_k \le b$, $k < \infty$. Given $\sigma = \big(X_V, u(\cdot), \pi(\mu)\big) \in \Sigma$, define the vector q_σ that is composed of the one-sided limits of X_V at the points θ_j, $j = \overline{1,k}$; i.e.,

$$q_\sigma \doteq \big(\{X_V(\theta_j-)\}_{j=\overline{0,k}}, \{X_V(\theta_j+)\}_{j=\overline{0,k}}\big).$$

Let us consider the optimal impulsive control problem $P(\theta)$ with multipoint state constraints

$$\begin{aligned} &\text{minimize} \quad J(\sigma) = l(q_\sigma) \\ &\text{subject to} \quad \sigma \in \Sigma, \quad q_\sigma \in C. \end{aligned}$$

Here, C is a closed set in $\mathbb{R}^{d(q_\sigma)}$, where $d(q_\sigma)$ is the dimension of q_σ, $l : \mathbb{R}^{d(q_\sigma)} \to \mathbb{R}$ is a continuous function. A process $\sigma \in \Sigma$ is said to be a feasible process of $P(\theta)$ if $q_\sigma \in C$.

3 Preliminaries

The aim of this section is to recall some facts about Lyapunov type functions and their monotone property relative to impulsive control problems with trajectories of bounded variation [Samsonyuk, 2010], [Dykhta & Samsonyuk, 2015].

Let φ be a continuous function $T \times \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$.

Let $(t_\alpha, x_\alpha) \in T \times \mathbb{R}^n$, $V_\alpha \ge 0$. Define the sets $\mathcal{T}$, $\mathcal{T}_{V_\alpha}(t_\alpha, x_\alpha)$, and $Q_\varphi(t_\alpha, x_\alpha, V_\alpha)$ as follows

$$\mathcal{T} = \big\{X_V \mid \exists\, \sigma = \big(X_V, u(\cdot), \pi(\mu)\big) \in \Sigma\big\}, \qquad \mathcal{T}_{V_\alpha}(t_\alpha, x_\alpha) = \big\{X_V \in \mathcal{T} \mid X_V(t_\alpha-) = (x_\alpha, V_\alpha)\big\},$$

$$Q_\varphi(t_\alpha, x_\alpha, V_\alpha) = \big\{(t, x, V) \in T \times \mathbb{R}^n \times \mathbb{R}_+ \ \mid \ \varphi(t, x, V) \le \varphi(t_\alpha, x_\alpha, V_\alpha)\big\}.$$

Definition 1. *Function φ is strongly decreasing relative to $(\mathcal{D})$ if for any $(t_\alpha, x_\alpha) \in T \times \mathbb{R}^n$, $V_\alpha \ge 0$ and for any $X_V \in \mathcal{T}_{V_\alpha}(t_\alpha, x_\alpha)$ the inclusion*

$$\operatorname*{graph}_{[t_\alpha, b]} X_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha)$$

holds.

This monotone property may be formulated in term of decreasing φ along of all solutions of $(\mathcal{D})$. Indeed, φ is said to be decreasing along X_V if for any (t_1, x_1, V_1) and (t_2, x_2, V_2), where $(x_1, V_1) \in X_V(t_1)$ and $(x_2, V_2) \in X_V(t_2)$ such that $t_1 < t_2$ or $V_1 < V_2$, the inequality $\varphi(t_1, x_1, V_1) \ge \varphi(t_2, x_2, V_2)$ is fulfilled. Then φ is strongly decreasing if φ decreases along any $X_V \in \mathcal{T}$. Let us note that such functions are usual called Lyapunov type functions.

For optimal impulsive control problem with multipoint state constraints it is natural to consider compound Lyapunov type functions which will be defined below.

Let ρ be some partition of T by points $\{t_0, t_1, \dots, t_N\}$ such that $a = t_0 < t_1 < \dots < t_N = b$. Denote $\Delta_i = (t_{i-1}, t_i)$, $i = \overline{1,N}$, and let $\mathcal{T}_{\Delta_i}$ be the restriction of $\mathcal{T}$ to Δ_i.

First, given Δ_i, we consider the system of proximal Hamilton–Jacobi inequalities

$$p_t + \mathcal{H}_0(t, x, p_x) \le 0 \qquad \forall\, p = (p_t, p_x, p_V) \in \partial_P \varphi(t, x, V), \ \ \forall\, (t, x, V) \in (t_{i-1}, t_i) \times \mathbb{R}^n \times [0, +\infty), \tag{12}$$

$$p_V + \mathcal{H}_1(t, x, p_x) \le 0 \qquad \forall\, p = (p_t, p_x, p_V) \in \partial_P \varphi(t, x, V), \ \ \forall\, (t, x, V) \in [t_{i-1}, t_i] \times \mathbb{R}^n \times (0, +\infty). \tag{13}$$

Here, $\mathcal{H}_0(t, x, \psi) = \max\limits_{u \in U} \langle \psi, f(t, x, u)\rangle$, $\mathcal{H}_1(t, x, \psi) = \max\limits_{\omega \in K_1} \langle \psi, G(t, x)\omega\rangle$, the set $\partial_P \varphi(t, x, V)$ is the proximal subdifferential of φ at the point (t, x, V). Let us recall [Clarke et al., 1998], [Vinter, 2000] that a vector $p \in \mathbb{R}^{d(y)}$ is called a proximal subgradient of a function $y \to \varphi(y)$ at a point y if there exist a neighborhood Q of the point y and a constant $c > 0$ such that

$$\varphi(z) \ge \varphi(y) + \langle p, z - y\rangle - c|z - y|^2 \quad \forall\, z \in Q.$$

This inequality implies that locally (in a neighborhood of y) φ has a quadratic lower support function at the point y with gradient p at this point. The proximal subdifferential $\partial_P \varphi(y)$ consists of all subgradients. In the case $\partial_P \varphi(y)$ is empty, the respective proximal inequalities are assumed to hold automatically at the point y.

Note that $\partial_P \varphi(y) \subset \{\nabla \varphi(y)\}$ if φ is differentiable; moreover, the last inclusion turns into the equality if φ is twice continuously differentiable at y.

Let Φ_{Δ_i} be the set of all continuous solutions of (12), (13). Then Φ_{Δ_i} consists of strongly decreasing on Δ_i functions. Namely, if $\varphi \in \Phi_{\Delta_i}$, then φ decreases along any $X_V \in \mathcal{T}_{\Delta_i}$.

Next, for every t_j, $j = \overline{0,N}$, we consider the so-called limiting system

$$z'(\tau) = G\big(t_j, z(\tau), z_V(\tau)\big)\omega(\tau), \quad z'_V(\tau) = 1, \quad \omega(\tau) \in co\ K_1 \quad \text{a.e.} \quad \tau \geq 0 \tag{14}$$

with Lebesgue measurable controls $\omega(\cdot)$. Let $L\mathcal{T}_{t_j}$ be the set of solutions of (14). Let us recall that a continuous function $\xi(z, z_V)$ strongly decreases relative to (14) if $\xi(z, z_V)$ is a solution of the proximal Hamilton–Jacobi inequality

$$p_{z_V} + \mathcal{H}_1(t_j, z, p_z) \leq 0 \qquad \forall\ (p_z, p_{z_V}) \in \partial_P \xi(z, z_V), \quad \forall\ (z, z_V) \in \mathbb{R}^n \times (0, +\infty) \tag{15}$$

(for more details see [Clarke et al., 1998]). Denote by Ξ_{t_j} the set of all continuous solutions of (15).

Definition 2. *The set* $\big(\{\varphi_i\}_{i=\overline{1,N}}, \{\xi_j\}_{j=\overline{0,N}}\big)$, *where* $\varphi_i \in \Phi_{\Delta_i}$, $i = \overline{1,N}$, $\xi_j \in \Xi_{t_j}$, $j = \overline{0,N}$, *is called a compound strongly decreasing function.*

Given $\Phi^*_{\Delta_i} \subset \Phi_{\Delta_i}$, $i = \overline{1,N}$, and $\Xi^*_{t_j} \subset \Xi_{t_j}$, $j = \overline{0,N}$, we define Φ^*_ρ to be $\Phi^*_\rho = \big\{\{\Phi^*_{\Delta_i}\}_{i=\overline{1,N}},\ \{\Xi^*_{t_j}\}_{j=\overline{0,N}}\big\}$. Then Φ^*_ρ is called a set of compound strongly decreasing functions.

4 Main Results

In this section sufficient global optimality conditions for problem $P(\theta)$ will be formulated.

Let $\sigma = \big(X_V, u(\cdot), \pi(\mu)\big) \in \Sigma$ be a feasible process of $P(\theta)$ and let $\rho = \{t_0, \ldots, t_N\}$, where $a = t_0 < t_1 < \ldots < t_N = b$, be some partition of T including all θ_j, $j = \overline{0,k}$, i.e., $\rho \supseteq \{\theta_0, \ldots, \theta_k\}$. Denote by I the set $\{j \in \{0, \ldots, N\} \mid t_j \in \{\theta_0, \ldots, \theta_r\}\}$. In what follows we use the notation

$$q^j_{\sigma 0} \doteq X_V(t_j-), \qquad q^j_{\sigma 1} \doteq X_V(t_j+), \quad j = \overline{0,N}, \qquad q_{\sigma,\rho} \doteq \Big(\big\{q^j_{\sigma 0}, q^j_{\sigma 1}\big\}_{j=\overline{0,N}}\Big). \tag{16}$$

Given ρ, define the sets

$$\mathcal{X}_{\Delta_i} = \big\{\big((x_{i-1}, V_{i-1}), (x_i, V_i)\big) \ \big|\ \exists\, X_V \in \mathcal{T}: \ \ X_V(t_{i-1}+) = (x_{i-1}, V_{i-1}), \ \ X_V(t_i-) = (x_i, V_i) \ \big\}, \quad i = \overline{1,N},$$

$$\mathcal{Z}_{t_j} = \left\{ \big((z_0, z_{V0}), (z_1, z_{V1})\big) \ \middle|\ \begin{array}{l} \exists\, \big(z(\cdot), z_V(\cdot)\big) \in L\mathcal{T}_{t_j}: \\ z(0) = z_0, \ z(d) = z_1, \ \ z_V(d) - z_V(0) = d, \ d \doteq z_{V1} - z_{V0} \end{array} \right\}, \qquad j = \overline{0,N},$$

$$\mathcal{R}_\rho = \Big\{ q = \big(\{q^j_0, q^j_1\}_{j=\overline{0,N}}\big) \ \big|\ \big(q^{i-1}_1, q^i_0\big) \in \mathcal{X}_{\Delta_i}, \ i = \overline{1,N}, \ \ \big(q^j_0, q^j_1\big) \in \mathcal{Z}E_{t_j}, \ j = \overline{0,N} \ \Big\}.$$

Let us note that the set $\mathcal{R}_\rho$ consists of points connected by trajectories of $(\mathcal{D})$. This set may be interpreted as a reachable set corresponding to ρ. It is easy to see that, for any $\sigma \in \Sigma$ and any ρ, the corresponding $q_{\sigma,\rho}$ belongs to $\mathcal{R}_\rho$. And the contrary, for any $q \in \mathcal{R}_\rho$ there exists $\sigma \in \Sigma$ such that $q_{\sigma,\rho} = q$.

Let $\Phi^*_\rho = \big\{\{\Phi^*_{\Delta_i}\}_{i=\overline{1,N}},\ \{\Xi^*_{t_j}\}_{j=\overline{0,N}}\big\}$ be an arbitrary set of compound strongly decreasing functions. Define the sets

$$\mathcal{A}[\Phi^*_{\Delta_i}] = \bigcap_{\varphi \in \Phi^*_{\Delta_i}} \big\{\big(q_1, q_0\big) \,|\, q_0, q_1 \in \mathbb{R}^n \times \mathbb{R}_+, \, \varphi(t_i, q_0) - \varphi(t_{i-1}, q_1) \leq 0\big\}, \qquad i = \overline{1,N},$$

$$L\mathcal{A}[\Xi^*_{t_j}] = \bigcap_{\xi \in \Xi^*_{t_j}} \big\{\big(q_0, q_1\big) \mid q_0, q_1 \in \mathbb{R}^n \times \mathbb{R}_+, \, \xi(q_1) - \xi(q_0) \leq 0\big\}, \qquad j = \overline{0,N},$$

$$\mathcal{A}[\Phi^*_\rho] = \Big\{ q = \big(\{q^j_0, q^j_1\}_{j=\overline{0,N}}\big) \ \Big|\ \big(q^{i-1}_1, q^i_0\big) \in \mathcal{A}[\Phi^*_{\Delta_i}], \ i = \overline{1,N}, \ \ \big(q^j_0, q^j_1\big) \in L\mathcal{A}[\Xi^*_{t_j}], \ j = \overline{0,N} \ \Big\}.$$

By using the strong decreasing property of functions from $\Phi^*_{\Delta_i}$, $i = \overline{1,N}$, and $\Xi^*_{t_j}$, $j = \overline{0,N}$, one can readily obtain that $\mathcal{A}[\Phi^*_{\Delta_i}]$, $i = \overline{1,N}$, and $L\mathcal{A}[\Xi^*_{t_j}]$, $j = \overline{0,N}$, give outer estimations for $\mathcal{X}_{\Delta_i}$, $i = \overline{1,N}$ and $\mathcal{Z}_{t_j}$, $j = \overline{0,N}$, respectively; i.e.,

$$\mathcal{X}_{\Delta_i} \subseteq \mathcal{A}[\Phi^*_{\Delta_i}], \quad i = \overline{1,N}, \qquad \mathcal{Z}_{t_j} \subseteq L\mathcal{A}[\Xi^*_{t_j}], \quad j = \overline{0,N}. \tag{17}$$

From (17) the next result follows.

Lemma 2. *Let ρ be a partition of T and Φ_ρ^* be an arbitrary set of compound strongly decreasing functions. Then $\mathcal{R}_\rho \subseteq \mathcal{A}[\Phi_\rho^*]$.*

Now let us formulate sufficient optimality conditions.

Denote by $\big(\mathcal{A}P(\theta)\big)$ the finite-dimensional optimization problem

$$l(q_I) \to \min; \quad q_I \in C, \quad q \in \mathcal{A}[\Phi_\rho^*], \quad \text{where } q \doteq \big(\{q_0^j, q_1^j\}_{j=\overline{0,N}}\big), \quad q_I \doteq \big(\{q_0^j, q_1^j\}_{j\in I}\big).$$

Let $\overline{\sigma} = \big(\overline{X}_V, \overline{u}, \overline{\pi}(\overline{\mu})\big)$ be an examining process of $P(\theta)$ and $\overline{q}_{\overline{\sigma}}$ be the corresponding vector of the one-sided limits of $\overline{X}_V$. Let the vector $\overline{q}_{\overline{\sigma},\rho}$ be defined by (16).

The set Φ_ρ^* is said to be resolving for $\overline{\sigma}$ if the vector $\overline{q}_{\overline{\sigma},\rho}$ is a global minimum point in the problem $\big(\mathcal{A}P(\theta)\big)$.

Theorem 1. *Let Φ_ρ^* be resolving for $\overline{\sigma}$. Then $\overline{\sigma}$ yields the global minimum in the problem $P(\theta)$.*

The proof follows from Lemma 2.

In conclusion, let us note that these optimality conditions are in the tradition of modification of Carathéodory and Krotov's type conditions; we refer, for example, to [Clarke et al., 1998], [Krotov, 1996], [Milyutin & Osmolovskii, 1998], [Vinter, 2000], where optimal control problems with absolutely continuous trajectories were considered. Moreover, the optimality conditions stated by Theorem 1 are close to dynamic programming principle developed for impulsive processes in [Fraga & Pereira, 2008], [Motta & Rampazzo, 1996], [Pereira, Matos, & Silva, 2002], [Daryin & Kurzhanski, 2008].

5 An Example

Let us consider the optimal impulsive control problem

$$J = V(t_1+) - y(t_1+) \to \min, \tag{18}$$

$$dy = (px_1 + qx_2)dt, \quad dx_1 = a\,(1-x_1)\,\mu_1, \quad dx_2 = c\,(1 - x_2/x_1)\,\mu_2, \tag{19}$$

$$y(0) = 0, \quad x_1(0-) = x_{10} \in (0,1), \quad x_2(0-) = x_{20} \in (0, x_{10}), \quad V(\theta-) \le R. \tag{20}$$

Here, parameters a, c, p, q are nonnegative, θ is a fixed point from $(0, t_1)$, $\mu = (\mu_1, \mu_2)$ is a nonnegative Borel measure on $[0, t_1]$. The vector fields generated by columns $G_1 = \begin{pmatrix} 0 & a(1-x_1) & 0 \end{pmatrix}^T$ and $G_2 = \begin{pmatrix} 0 & 0 & c(1-x_2/x_1) \end{pmatrix}^T$ do not commutative. So, we need to use $\pi(\mu) = \big(\mu, \gamma(\mu)\big)$, where $\gamma(\mu) = \{d_s, \omega_s(\cdot)\}_{s\in S}$, instead of only μ. As usual $V(t) = |\mu_c|\big([0,t]\big) + \sum\limits_{s\le t,\, s\in S} d_s, \quad t \in (0, t_1]$.

Let us note that this problem may be interpreted as a model of the advertising expense optimization for two mutually complementary products in which an aggressive advertising campaign is possible. In [Dykhta & Samsonyuk, 2009] this problem was studied by using a maximum principle for impulsive processes.

Let us consider one partial case of this problem. We assume that parameters satisfy the following conditions

$$x_1^* < x_1^{**}, \quad cq(t_1-\theta)\big(1 - x_{20}/x_1^{**}\big) < 1, \quad cqt_1\big(1 - x_{20}/x_1^*\big) - \beta < 1,$$

where $x_1^* = 1 - (1-x_{10})e^{-aR}$, $x_1^{**} = 1 - 1/\big(ap(t_1-\theta)\big)$, $\beta = p\theta a(1-x_1^*)$. Consider the impulsive process $\overline{\sigma}$ consisting of the control $\overline{\pi}(\overline{\mu})$ with the components

$$\overline{\mu}_1 = R\delta(t) + \Big((1/a)\ln\big(ap(t_1-\theta)(1-x_{10})\big) - R\Big)\delta(t-\theta), \qquad \overline{\mu}_2 = 0,$$

$$\overline{S} = \{0;\theta\}, \quad \overline{d}_{s=0} = \overline{\mu}_1\big(\{0\}\big), \quad \overline{d}_{s=\theta} = \overline{\mu}_1\big(\{\theta\}\big), \quad \overline{\omega}_{s=0}(\tau) \equiv (1,0), \quad \overline{\omega}_{s=\theta}(\tau) \equiv (1,0)$$

and the corresponding trajectory

$$\bar{x}_1(0-) = x_{10}, \qquad \bar{x}_1(t) = x_1^*, \quad t \in (0,\theta), \qquad \bar{x}_1(t) = x_1^{**}, \quad t \in [\theta, t_1], \qquad \bar{x}_2(t) \equiv x_{20}.$$

Then the optimality of $\overline{\sigma}$ is stated by using the strongly decreasing functions

$$\varphi_1(t, x_1, x_2, V, y) = -(1-x_1)e^{aV}, \qquad t \in [0,\theta),$$

$$\varphi_2(t, x_1, x_2, V, y) = y - (1+\beta)V - \frac{1+\beta}{a}\ln(1-x_1) - ptx_1 + q(t_1-t)x_2, \qquad t \in [0,\theta),$$

$$\varphi_3(t, x_1, x_2, V, y) = y - V - \frac{1}{a}\ln(1-x_1) + p(\theta-t)x_1 + q(t_1-t)x_2, \qquad t \in [\theta, t_1].$$

One can prove that the set $\{ \varphi_1, \varphi_2, \varphi_3\}$ is resolving for $\overline{\sigma}$.

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