Polyhedral Complementarity and Fixed Points Problem of Decreasing Regular Mappings on Simplex

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Abstract

A new development of polyhedral complementarity investigation is presented. This consideration extends the author's original approach to the equilibrium problem in a linear exchange model and its variations. Two polyhedral complexes in duality and a cells correspondence are given. The problem is to find a point of intersection of the cells corresponding each other . This is a natural generalization of linear complementarity problem. Now we study arising point-to-set mappings without the original exchange model. The potentiality for a special class of regular mappings is proved. As a result the fixed point problem of mapping reduces to an optimization problem . Two finite algorithms for this problem are considered.

1 Introduction

It is known that the problem of finding an equilibrium in a linear exchange model can be reduced to the linear complementarity problem [Eaves, 1976]. The polyhedral complementarity approach [Shmyrev, 1983] is based on a fundamentally different idea, that reflects more the character of economic equilibrium as a concordance the consumers' preferences with financial balances. In algorithmic aspect it may be treated as a realization of the main idea of the simplex-method of linear programming. It has no analogues and makes it possible to obtain the finite algorithms not only for the linear exchange model [Shmyrev, 1985], but also for some of it's variations [Shmyrev, 2008], (more references one can find in [Shmyrev, 2016]). The simplest algorithms are those for a model with fixed budgets, known more as Fisher's problem. The convex programming reduction of it , given by Eisenberg and Gale [Eisenberg & Gale, 1959], is well known . This result has been used by many authors to study computational aspects of the problem. Some review of that can be found in [Devanur et al., 2008]. The polyhedral complementarity approach has given an alternative reduction of the Fisher's problem to a convex program [Shmyrev, 1983],[Shmyrev, 2006]. Only the well known elements of transportation problem algorithms are used in the procedures obtained by this way [Shmyrev, 2009]. These simple procedures can be used for getting iterative methods for more complicate models [Shmyrev, 1996], [Shmyrev, 2016].

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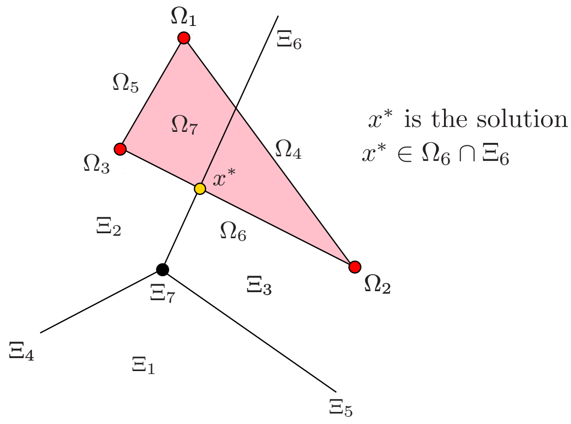

Figure 1: Polyhedral complementarity

By the given approach we try to study a mathematical fundamental principle of the proposed finite algorithms ignoring an original economic model. We consider a class of piecewise constant multivalued mappings on the simplex in R^n , which possess some monotonicity property . The potentiality of these mappings is proved [Shmyrev, 2017]. This makes possible to reduce a fixed point problem to two optimization problems which are in duality similarly to dual linear programming problems. Two finite algorithms presented here are based on the ideas of suboptimization [Rubinstein, 1971].

2 Polyhedral Complementarity Problem

We consider polyhedrons in $\mathbb{R}^n$. Let two polyhedral complexes ω and ξ with the same number of cells r be given . Let $R \subset \omega \times \xi$ be a one-to-one correspondence : $R = \{(\Omega_i, \Xi_i)\}_{i=1}^r$ with $\Omega_i \in \omega$, $\Xi_i \in \xi$.

We say that the complexes ω and ξ are *in duality by R* if the subordination of cells in ω and the subordination of the corresponding cells in ξ are opposite each other:

$$\Omega_i \prec \Omega_j \Longleftrightarrow \Xi_i \succ \Xi_j.$$

The polyhedral complementarity problem is to find a point that belongs to both cells of some pair (Ω_i, Ξ_i):

$$p^* \text{ is the solution } \Longleftrightarrow p^* \in \Omega_i \cap \Xi_i \text{ for some } i.$$

This is natural generalization of linear complementarity, where (in nonsingular case) the complexes are formed by all faces of two simplex cones.

Figure 1 shows an example of the polyhedral complementarity problem. Each of two complexes has 7 cells. There is a unique solution of the problem — the point x^* that belongs to Ω_6 and Ξ_6.

3 Polyhedral Complementarity on Simplex

Let σ be the unit simplex in $\mathbb{R}^n$:

$$\sigma = \left\{ p \in \mathbb{R}^n_+ \mid \sum_{j=1}^n p_j = 1 \right\}.$$

We consider on σ two polyhedral complexes in duality $\omega = \{\Omega_i\}$ and $\xi = \{\Xi_i\}$.The cell of full dimension of the complex ω is defined by the condition $p \in \sigma$ and a system of linear inequalities of the form :

$$\sum_{j \in S} h_j p_j + \sum_{k \notin S} h_k p_k \geq \gamma, \tag{1}$$

where $S \neq \emptyset, S \subset J = \{1, ..., n\}$ and

$$h_j > 0, j \in S, \qquad h_k < 0, k \notin S.$$

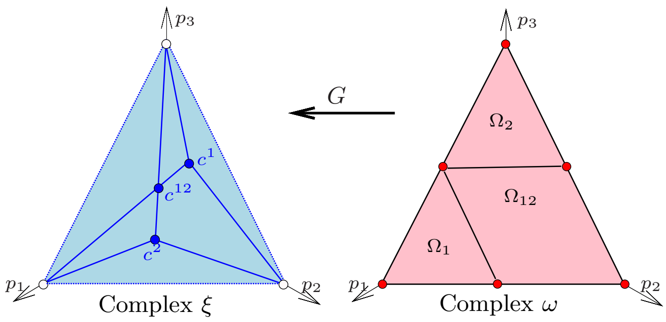

Figure 2: Polyhedral complexes in exchange model

Fig.2 illustrate the polyhedral complexes for a model with 3 commodities and 2 consumers. Each of both complexes has 17 cells. Fig.3 illustrate the arising complementarity problem. The point $c^{12} \in \Omega_{12}$ is it's solution: .

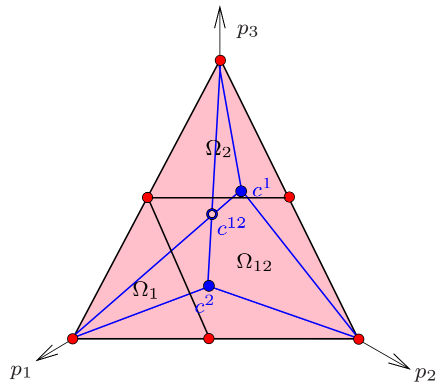

Figure 3: Complementarity problem: c^{12} is the solution.

For the faces points of the cell some of inequalities (1) become equalities. For a face of dimension $(n-2)$ there is only one equality and so we obtain a subdivision J into S and $J \setminus S$.
It is assumed that the cells $\{\Omega_i\}$ form a subdivision of the simplex σ and the cells $\{\Xi_i\}$ form a subdivision of it's interior σ°.
The cells $\{\Xi_i\}$ of full dimension are defined by the inequalities of the form:

$$p_j/p_k \geq \gamma_{jk}. \tag{2}$$

A vertex of ξ will be given by a collection of $(n-1)$ linearly independent equations obtained from inequalities (2). With such a collection we can associate a graph with a set of vertexes J and a set of edges (j,k) corresponding to the selected inequalities. It is easy to see, that to obtain an edge of ξ we have to remove one edge of the graph. In this way we obtain two connected components and also a subdivision J into two subsets Q and $J \setminus Q$.

Concordance condition. *The subdivision for an edge of the complex ξ is the same as that for the corresponding cell of the complex ω .*

4 Monotone Regular Mappings

1°. Monotonicity property.

For the problem under consideration it is naturally to introduce piecewise constant point-to-set mapping G, which for every point of the relative interior of a cell $\Omega \in \omega$ assigns the corresponding cell $\Xi \in \xi$: $G(p) = \Xi$ for all $p \in \Omega^\circ$. So the polyhedral complementarity problem becomes the fixed point one: we have to find $p \in G(p)$.
It is clear, that the fixed point of the mapping G will be also the fixed point of it's restriction G° on σ°.
A key feature of the considered fixed point problem is a specific monotonicity property of arising mappings.

Definition 1. We say that cells $\Omega_1, \Omega_2 \in \omega$ are *adjacent*, if they have common $(n-2)$-dimensional face.

Let the cells Ω_1, Ω_2 be adjacent and q^1, q^2 are the corresponding vertexes of ξ. Let h be a vector, for which the inequality $(h, \Omega_2^\circ - \Omega_1^\circ) \geq 0$ holds.

Definition 2. The mapping G is *locally decreasing*, if for each two adjacent Ω_1, Ω_2 the inequality $(h, q^2 - q^1) \leq 0$ is valid.

For a positive vector $q = (q_1, ..., q_n)$ we introduce the vector $\ln q = (\ln q_1, ..., \ln q_n)$.

Definition 3. The mapping G is *locally logarithmically decreasing*, if

$$(p^2 - p^1, \ln q^2 - \ln q^1) \leq 0, \qquad \forall p^1 \in \Omega_1, p^2 \in \Omega_2.$$

In what follows we consider a narrower class of *regular mappings* for which in the inequalities (1) we have:

$$h_j = 1, \qquad j \in S,$$

$$h_k = -1, \qquad k \notin S,$$

and $-1 \leq \gamma \leq 1$. So (1) becomes

$$\sum_{j \in S} p_j - \sum_{k \notin S} p_k \geq \gamma.$$

It can be proofed, that for regular mappings the subclass of locally decreasing mappings coincides with the subclass of locally logarithmically decreasing mappings.

2°. **Reduction to the optimization problem.**

Definition 4. A mapping G is named *potential* if there exists piecewise linear concave function f on σ such that

$$\forall p \in \sigma \quad \partial f(p) = \{\ln q + t\theta | q \in G(p), t \in R^1, \}$$

where $\theta = (1, ..., 1)$ and $\partial f(p)$ is the subdifferential of the function f at the point p.

The main feature of the considered fixed point problem is the fact that logarithmically decreasing mappings are potential [Shmyrev, 2017]. We have as a corollary that locally logarithmically decreasing mappings are logarithmically decreasing in the large :

$$(p^2 - p^1, \ln q^2 - \ln q^1) \leq 0, \qquad \forall p^1, p^2 \in \sigma, \quad \forall q^1 \in G(p^1), q^2 \in G(p^2).$$

This allows us to reduce the fixed point problem to the optimization one.
For $p > 0$ we introduce the function $h(p) = (p, \ln p)$ and consider the function

$$\varphi(p) = h(p) - f(p),$$

where $f(p)$ is the potential function of the mapping G°.

Theorem 1. *The fixed point of G° coincides with the minimum point of the convex function $\varphi(p)$ on σ°*

The function φ is very simple and the suboptimization approach [Rubinstein, 1971] can be used to minimize it . In this way we obtain the finite algorithm for the fixed point searching.

Another algorithm for the problem can be obtained if we take into account that the mapping G and the inverse mapping G^{-1} have the same fixed points. For the introduced concave function f we can consider the conjugate function f^* :

$$f^*(y) = \inf_z \{(y, z) - f(z)\}$$

Theorem 2. *The fixed point of G° is the maximum point of the concave function $\psi(q) = f^*(\ln q)$ on σ°.*

It can be shown that for the functions $\varphi(p)$ and $\psi(q)$ there is a duality relation as for dual programs of linear programming:

Proposition. *For all* $p, q \in \sigma^\circ$ *the inequality*

$$\varphi(p) \geq \psi(q)$$

holds . If this inequality turns into equality then $p = q$ *.*

Corollary. $\varphi(r) = \psi(r)$ *if the point* r *is the fixed point of the mapping* G

3°. **Algorithms.**

The mentioned theorems allow us to propose two finite algorithms for searching fixed points.
Algorithmically they are based on the ideas of suboptimization [Rubinstein, 1971], which were used for minimization quasiconvex functions on a polyhedron. In considered case we exploit the fact that the complexes ω and ξ define the cells structure on σ° similarly to the faces structure of a polyhedron.
For implementation of the algorithms one does not need to have function $\varphi(p)$ and $f^*(y)$ explicit. We just need to be able to verify the inequality defining cells $\Omega \in \omega$ and $\Xi \in \xi$.

We now describe the general scheme of the algorithm that is based on the theorem 1. The other one using the theorem 2 is quite similar.

Consider a couple of two cells $\Omega \in \omega$, $\Xi \in \xi$ corresponding each other. Let L, M be their affine hulls respectively. It can be shown that $L \cap M$ is singleton. Let r be the point of this intersection.

Theorem 3. *The point* r *is the minimum point of the function* $\varphi(p)$ *on* L *and the maximum point of the function* $\psi(q)$ *on* M.

On the current k-step of the process there are two cells $\Omega_k \in \omega$, $\Xi_k \in \xi$ corresponding each other and two points $p^k \in \Omega_k$, $q^k \in \Xi_k$. We consider affine hulls $L_k \supset \Omega_k$, $M_k \supset \Xi_k$ and obtain the point of their intersection r^k . For this we need descriptions of these sets.

As it was mentioned before, with an edge of ξ we associate a graph with two connected components and a subdivision J into two subsets Q and $J \setminus Q$. For a cell of higher dimension the associated graph will have more components, that will entail an increase of the sets number in the subdivision of J. Let τ be the number of connected components of the associated graph for the cell Ξ_k and $J = Q_1 \cup Q_2, \cup ..., Q_\tau$ is the obtained subdivision of J. It is easy to verify that the linear system for L_k is going to be equivalent to this one :

$$\sum_{j \in Q_\nu} p_j = \alpha_\nu, \qquad \nu = 1, ..., \tau. \tag{3}$$

The conditions for the cell Ξ_k define coordinates q_j on each connected component up to a positive multiplier:

$$q_j = t_\nu q_j^k, \qquad j \in Q_\nu.$$

To obtain the coordinates of the point r^k we need to put $p_j = q_j$ in corresponding equation (3), which gives the multiplier t_ν.

For the obtained point r^k can be realized two cases.

(i) $r^k \notin \Omega_k$. Since r^k is a minimum point on L_k for the strictly convex function $\varphi(p)$, the value of the function will diminish for the moving point $p(t) = (1-t)p^k + tr^k)$ when t increases in [0,1]. In considered case this point reaches a face of Ω_k at $t = t^* < 1$. This face we take as Ω_{k+1}, that determines the cell Ξ_{k+1}. We accept $p^{k+1} = p(t^*), q^{k+1} = q^k$ and pass to the next step.

It should be noted that the dimension of the cell Ω reduces. It will certainly be $r^k \in \Omega_k$ when the current cell Ω_k degenerates into a point and we have $r^k = p^k$. But it can occur earlier.

(ii) $r^k \in \Omega_k$. In this case we can assume $p^k = r^k$. Otherwise, we can simply replace p^k by r^k with a decrease of the function's $\varphi(p)$ value. If $r^k \in \Xi_k$, then r^k is the required fixed point. Otherwise, we are looking for the maximum t^*, at which point $q(t) = (1-t)q^k + tr^k$ is still in the Ξ_k. At $t = t^*$ the point $q(t)$ reaches a face of the cell Ξ_k, which is accepted as Ξ_{k+1}. The corresponding cell of the complex ω will be Ω_{k+1}. We accept $p^{k+1} = p^k, q^{k+1} = q(t^*)$ and pass to the next step.

Nondegeneracy condition. *The dimension of the current cells* Ω_k, Ξ_k *at each step of the process changes per unit.*

Under this condition the value of the difference $\varphi(p^k) - \psi(q^k)$ decreases at each step of the process and we use this to prove the finiteness of the process.

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