P-Regular Nonlinear Optimization – Calculus and Methods

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Abstract

We present recent advances in nonlinear optimization, which have been obtained based on p-regularity theory, successfully developing for the last years. The main result of this theory gives a detailed description of the structure of the zero set of an irregular nonlinear mapping. We illustrate the theory with an application in different branches of optimization. Amongst the applications, the construction of p-factor operator is used to construct numerical methods for solving degenerate optimization problems and p-order necessary and sufficient optimality conditions are formulated. The reducibility of inequality-constrained optimization problems to the equality constrained optimization problems is proved in the framework of p-regularity theory. Moreover, the connection between singular problems and nonlinear problems is shown.

1 Introduction

This work concerns the problem of solving a nonlinear equation of the form

$$F(x) = 0, \tag{1}$$

or optimization problem

$$\min \varphi(x),\ F(x) = 0, \tag{2}$$

where $F : X \to Y$ is a sufficiently smooth mapping from a Banach space X to a Banach space Y . Of course, the solution to many interesting nonlinear problems can be cast in this form and there have been many works devoted to this problem. The purpose of this paper is to present some of our own work and that of others in this area in a coherent way, which has hitherto been scattered throughout various references, as well as giving a number of new results. This paper is based on [Tretyakov, 1984], [Brezhneva & Tretyakov, 2007] and [Prusinska & Tretyakov, 2016].

We separate nonlinear mappings F and problems of the form (1) into two classes, called regular and irregular. Roughly speaking, regular problems are those to which implicit function theorem arguments can be applied and the irregular ones are those to which it cannot, at least not directly.

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2 Goal of the Present Contribution

In this work, we show how to apply p-regularity theory, also known as *factor-analysis of nonlinear mappings* to the description and investigation of singular mappings and, in addition, to develop methods for finding solutions to related singular problems. In particular, we show how these ideas apply to some specific situations, such as optimization problems.

2.1 The Regular Case

Fix a point $x^* \in X$ and suppose that $F : X \to Y$ is $\mathcal{C}^1$. It is well known that if F is *regular* at x^*, i.e.,

$$\text{Im } F'(x^*) = Y, \tag{3}$$

then the properties of the linear approximation of F locally correspond to the properties of the mapping F, since the mapping F can be locally linearized by a local diffeomorphism; that is, by a nondegenerate transformation of coordinates. Namely, there exist a neighborhood U of the point 0 and a $\mathcal{C}^1$ mapping $\varphi : U \to X$ such that $\varphi(0) = x^*$, $\varphi'(0) = I_X$, (the identity map on X), and

$$F(\varphi(x)) = F(x^*) + F'(x^*)x \tag{4}$$

for all $x \in U$. If the regularity condition (3) is not satisfied, then there is no such correspondence in general.

There exist numerous problems where the linear approximation of F is not enough to describe the properties of the mapping. For example, there are essential nonlinear mappings, i.e., mappings whose local linearization does not give a good approximation. We formalize this as follows.

Definition 1
Let V be a neighborhood of x^* in X. A $\mathcal{C}^2$ mapping $F : V \to Y$ is referred to as *essentially nonlinear at the point* x^*, if there exists a perturbation of the form $\widetilde{F}(x^* + x) = F(x^* + x) + \omega(x)$, where $\|\omega(x)\| = o(\|x\|)$, such that there does not exist any $\mathcal{C}^1$ nondegenerate transformation of coordinates $\varphi(x) : U \to X$ such that $\varphi(0) = x^*$, $\varphi'(0) = I_X$ and (4) holds with φ and $\widetilde{F}$.

Definition 2
We say the mapping F is *singular* (or *degenerate, abnormal*) at x^* if it fails to be regular; that is, its derivative is not onto:

$$\text{Im } F'(x^*) \neq Y. \tag{5}$$

2.2 Essential Nonlinearity and Singular Maps

The following Theorem establishes the relationship between these two notions.

Theorem 1
Suppose $F : V \to Y$ is $\mathcal{C}^2$ and that x^* is a solution of (1). Then F is essentially nonlinear at the point x^* if and only if F is singular at the point x^*.

Consider the following singular optimization problem

$$\min \phi(x), \tag{6}$$

subject to

$$F(x) = 0, \tag{7}$$

where $F : X \to Y$, X, Y – Banach spaces, and $\phi : X \to R$, $F \in \mathcal{C}^{p+1}(X), \phi \in \mathcal{C}^2(X)$ and at the solution point x^* we have

$$\text{Im } F'(x^*) \neq Y. \tag{8}$$

2.3 Description of the Solution Set. Lyusternik Theorem.

If operator $F'(x^*)$ is nonsingular then $T_1M(x^*) = KerF'(x^*)$, where $T_1M(x^*)$ is a tangent cone to the set $M = \{x \in X : F(x) = F(x^*) = 0\}$ at the point x^*.
If operator $F'(x^*)$ is singular then $T_1M(x^*) \neq \text{Ker } F'(x^*)$. For example for $F(x) = x_1^2 - x_2^2 + o(\|x\|^2)$, the solution of $F(x) = 0$ is $x^* = 0$ and $F'(0) = 0$. Hence Ker $F'(0) \neq R^2$. Moreover $T_1M(0) = \begin{pmatrix} t \\ t \end{pmatrix} \cup \begin{pmatrix} t \\ -t \end{pmatrix}$.

2.4 Optimality Conditions. Lagrange Theorem

If $F'(x^*) \cdot X = Y$ then there exists $\lambda^* \in Y^*$ such that $\phi'(x^*) = F'(x^*)^* \cdot \lambda^*$.

Let us $\phi(x) = x_1^2 + x_3$, $F(x) = \begin{pmatrix} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_3^2 + x_2 x_3 \end{pmatrix}$. Here $x^* = (0,0,0)^T$ and $\phi'(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$F'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ but $\phi'(0) \neq F'(0)^T \cdot \lambda$.

2.5 Newton method for Singular Equations

Consider in general the problem of solving nonlinear equation (1) where $F : X \to Y$, X, Y – B-spaces in general case, and $F \in \mathcal{C}^{p+1}(X), p \in N$. Let x^* solution point to (6), i.e. $F(x^*) = 0$. We will consider the singular case, i.e. $F'(x^*)$ is singular.

For $X = \mathbb{R}^n, Y = \mathbb{R}^n$ it means that matrix $F'(x^*)$ is degenerate.

Example 1.

$$F(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix}, \tag{9}$$

$F : \mathbb{R}^2 \to \mathbb{R}^2$, where $x^* = (0,0)^T$ solution to (8) and

$$F'(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is singular at $x^* = (0,0)^T$.

Newton method

$$x_{k+1} = x_k - \{F'(x_k)\}^{-1} F(x_k) \tag{10}$$

$k = 0, 1, 2, 3, \ldots$.
Let $x_0 = (x_0^1, x_0^2)^T$ and $x_0 \in U_\varepsilon(0), \varepsilon > 0$ sufficiently small. Then we have for $k = 1$

$$x_1 = \frac{1}{x_0^1 - x_0^2} \begin{pmatrix} -x_0^1 x_0^2 \\ x_0^1 x_0^2 \end{pmatrix}. \tag{11}$$

For $x_0^1 = x_0^2$ we obtain $\nexists \{F'(x_0)\}^{-1}$, so it is unapplicable.

But even ever $\exists \{F'(x_0)\}^{-1}$, say for $x_0 = (t + t^3, t)^T$, we have $x_1 = \begin{pmatrix} -\frac{1}{t} - t \\ \frac{1}{t} + t \end{pmatrix}$ and $\|x_1 - 0\| \approx \frac{1}{t} \to \infty, t \to 0$. If $t = 10^{-5}$ then $\|x_1 - 0\| \approx 10^5$ and we have rejecting effect.

2.6 Newton Method for Unconditional Optimization Problems

Consider

$$\min_{x \in \mathbb{R}^n} \phi(x)$$

$$\phi(x) = x_1^2 + x_1^2 x_2 + x_2^4$$

and

$$x_{k+1} = x_k - \{\phi''(x_k)\}^{-1} \phi'(x_k),$$

$x^* = (0,0)^T$, $n = 2$ at the initial point, $x_0 = (x_{01}, x_{02})^T$ where $x_{01} = x_{02}\sqrt{6(1 + x_{02})}$,
$\phi''(x_0) = \begin{pmatrix} 2 + 2x_{02} & 2x_{02}\sqrt{6(1 + x_{02})} \\ 2x_{02}\sqrt{6(1 + x_{02})} & 12x_{02}^2 \end{pmatrix}$, det $\phi''(x_0) = 0$, $\nexists \{\phi''(x_0)\}^{-1}$.

2.7 Modified Lagrange Function Method (Augment Function Method)

Consider the following constrained optimization problem

$$\min \phi(x) \tag{12}$$

$$g_i(x) \le 0, \quad i = \overline{1, m} \tag{13}$$

and modified Lagrange function of the following form

$$L_E(x, \lambda) = \phi(x) + \frac{1}{2}\sum_{i=1}^{m} \lambda_i^2 g_i(x)$$

$w = (x, \lambda)$

$$G(w) = \begin{pmatrix} \nabla\phi(x) + \sum\limits_{i=1}^{m} \lambda_i^2 \nabla g_i(x) \\ D(\lambda)g(x) \end{pmatrix} = 0_{n+m} \tag{14}$$

$$G'(w) = \begin{pmatrix} \nabla^2\phi(x) + \sum\limits_{i=1}^{m} \lambda_i^2 \nabla^2 g_i(x) & (g'(x))^T D(\lambda) \\ D(\lambda)g(x) & D(g(\lambda)) \end{pmatrix},$$

where $D(u) := \operatorname{diag}\{u_i\}$, $i = 1, \ldots, m$, $u \in \mathbb{R}^m$.

If at the solution point of (14) $w^* = (x^*, \lambda^*)$, $g_i(x^*) = 0$ and $\lambda_i^* = 0$ then $G'(w^*)$ is singular.

2.8 Elements of p-regularity Theory

Let us recall the basic constructions of p-regularity theory which is used in solving of singular problems. The construction of the p-factor-operator. Suppose that the space Y is decomposed into a direct sum

$$Y = Y_1 \oplus \ldots \oplus Y_p, \tag{15}$$

where $Y_1 = \overline{\operatorname{Im} F'(x^*)}$, $Z_1 = Y$. Let Z_2 be closed complementary subspace to Y_1 (we assume that such closed complement exists), and let $P_{Z_2} : Y \to Z_2$ be the projection operator onto Z_2 along Y_1. By Y_2 we mean the closed linear span of the image of the quadratic map $P_{Z_2} F^{(2)}(x^*)[\cdot]^2$. More generally, define inductively,

$$Y_i = \overline{\operatorname{span} \operatorname{Im} P_{Z_i} F^{(i)}(x^*)[\cdot]^i} \subseteq Z_i, \;\; i = 2, \ldots, p-1,$$

where Z_i is a chosen closed complementary subspace for $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$ and $P_{Z_i} : Y \to Z_i$ is the projection operator onto Z_i along $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$. Finally, $Y_p = Z_p$.

The order p is chosen as the minimum number for which (15) holds. Let us define the following mappings

$$F_i(x) = P_{Y_i} F(x), \quad F_i : X \to Y_i \;\; i = 1, \ldots, p,$$

where $P_{Y_i} : Y \to Y_i$ is the projection operator onto Y_i along $(Y_1 \oplus \ldots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \ldots \oplus Y_p)$ with respect to Y, $i = 1, \ldots, p$.

Definition 3
The linear operator $\Psi_p(h) \in \mathcal{L}(X, Y_1 \oplus \ldots \oplus Y_p)$, $h \in X$, $h \neq 0$

$$\Psi_p(h) = F_1'(x^*) + F_2''(x^*)h + \ldots + F_p^{(p)}(x^*)[h]^{p-1},$$

is called the p-factor operator.

Definition 4
We say that the mapping F is p-regular at x^* along an element h, if

$$\operatorname{Im} \Psi_p(h) = Y.$$

Definition 5
We say that the mapping F is p-regular at x^* if it is p-regular along any h from the set

$$H_p(x^*) = \{\bigcap_{k=1}^{p} \mathrm{Ker}^k F_k^{(k)}(x^*)\} \setminus \{\mathbf{0}\},$$

where k-kernel of the k-order mapping $F_k^{(k)}(x^*)$ is as follows

$$\mathrm{Ker}^k F_k^{(k)}(x^*) = \{\xi \in X : F_k^{(k)}(x^*)[\xi]^k = 0\}.$$

For a linear surjective operator $\Psi_p(h) : X \mapsto Y$ between Banach spaces we denote by $\{\Psi_p(h)\}^{-1}$ its *right inverse*. Therefore $\{\Psi_p(h)\}^{-1} : Y \mapsto 2^X$ and we have

$$\{\Psi_p(h)\}^{-1}(y) = \{x \in X \colon \Psi_p(h)x = y\}.$$

We define the *norm* of $\{\Psi_p(h)\}^{-1}$ via the formula

$$\|\{\Psi_p(h)\}^{-1}\| = \sup_{\|y\|=1} \inf\{\|x\| : x \in \{\Psi_p(h)\}^{-1}(y)\}.$$

We say that $\{\Psi_p(h)\}^{-1}$ is *bounded* if $\|\{\Psi_p(h)\}^{-1}\| < \infty$.

The following theorem gives a description of a solution set in degenerate case.

Theorem 2 (Generalized Lyusternik Theorem) Let X and Y be Banach spaces and U be a neighborhood of $x^* \in X$. Assume that $F : X \to Y$, $F \in C^p(U)$ is p-regular at x^*. Then

$$T_1 M(x^*) = H_p(x^*).$$

We now give another version of the theorem.

To state the result, we shall denote by $\mathrm{dist}(x, M)$, the **distance function** from a point $x \in X$ to a set M:

$$\mathrm{dist}(x, M) = \inf_{y \in M} \|x - y\|, \quad x \in X.$$

Theorem 3. Let X and Y be Banach spaces, and U be a neighborhood of a point $x^* \in X$. Assume that $F : X \to Y$ is a p-times continuously Fréchet differentiable mapping in U and satisfies the condition of strong p-regularity at x^*. Then there exist a neighborhood $U' \subseteq U$ of x^*, a mapping $\xi \mapsto x(\xi) : U' \to X$, and constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $F(\xi + x(\xi)) = F(x^*)$,

$$\|x(\xi)\|_X \le \delta_1 \sum_{i=1}^{p} \frac{\|f_i(\xi) - f_i(x^*)\|_{Y_i}}{\|\xi - x^*\|^{i-1}} \tag{16}$$

and $\|x(\xi)\|_X \le \delta_2 \sum_{i=1}^{p} \|f_i(\xi) - f_i(x^*)\|_{Y_i}^{1/i}$ for all $\xi \in U'$.

Consider our example

$$F(x) = \begin{pmatrix} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_3^2 + x_2 x_3 \end{pmatrix}$$

here $x^* = (0, 0, 0)^T$, and

$$T_1 M(0) = Span \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \cup Span \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$F'(0) = 0$

$$\mathrm{Ker}^2 F''(0) = Span \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \cup Span \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Let $h = (1, 1, 0)^T$ then $\mathrm{Im} F''(0)h = \mathbb{R}^2$.
It means that the mapping $F(x)$ is 2-regular at $x^* = 0$ and

$$\mathrm{Ker}^2 F''(0) = H_2(0) = T_1 M(0).$$

2.9 Optimality Conditions for p-Regular Optimization Problems

We define p-factor Lagrange function

$$\mathcal{L}_p(x,\lambda,h) = \varphi(x) + \left(\sum_{k=1}^{p} F_k^{(k-1)}(x)[h]^{k-1}, \lambda \right),$$

where $\lambda \in Y^*$ and

$$\bar{\mathcal{L}}_p(x,\lambda,h) = \varphi(x) + \left(\sum_{k=1}^{p} \frac{2}{k(k+1)} F_k^{(k-1)}(x)[h]^{k-1}, \lambda \right).$$

Definition 6
The mapping F is called *strongly p-regular at the point x^** if there exists $\gamma > 0$ such that

$$\sup_{h \in H_\gamma} \left\| \{\Psi_p(h)\}^{-1} \right\| < \infty$$

where

$$H_\gamma = \left\{ h \in X : \left\| F_k^{(k)}(x^*)[h]^k \right\|_{Y_k} \le \gamma, i = \overline{1,p}, \|h\| = 1 \right\}.$$

Let us recall the following basic theorems of the p-regularity theory.

Theorem 4. (Necessary and sufficient conditions for optimality) Let X and Y be Banach spaces, $\varphi \in \mathcal{C}^2(X)$, $F \in \mathcal{C}^{p+1}(X)$, $F : X \to Y$, $\varphi : X \to \mathbb{R}$. Suppose that $h \in H_p(x^*)$ and F is p-regular along h at the point x^*. If x^* is a local solution to the problem (6)–(7) then there exist multipliers, $\lambda^*(h) \in Y^*$ such that

$$\mathcal{L}p'_x(x^*, \lambda^*(h), h) = 0. \tag{17}$$

Moreover, if F is strongly p-regular at x^*, there exist $\alpha > 0$ and a multiplier $\lambda^*(h)$ such that (17) is fulfilled and

$$\bar{\mathcal{L}}p_{xx}(x^*, \lambda^*(h), h)[h]^2 \ge \alpha \|h\|^2. \tag{18}$$

for every $h \in H_p(x^*)$, then x^* is a strict local minimizer to the problem (6)–(7).

Example 2. Consider the problem

$$\begin{array}{l} x_2^2 + x_3 \to \min, \\ F(x) = \begin{pmatrix} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_3^2 + x_2 x_3 \end{pmatrix} = 0. \end{array} \tag{19}$$

It is easy to verify that the point $x^* = 0$ is a local minimum to problem (19).
For $x^* = 0$, we have that $F'(0) = 0$ is singular.

$$\mathrm{Ker}^2 F''(0) = Span \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \cup Span \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Consider the element $h = (1,1,0)^T$.
Since $\mathrm{Im} F''(0)h = R^2$. It means that the mapping $F(x)$ is 2-regular at $x^* = 0$ along h. Consider the 2-factor-Lagrange function with $\lambda_0 = 1$. After some transformations we obtain

$$\mathcal{L}_2(x, \lambda(h), h) = x_2^2 + x_3 + \alpha(x_1 - x_2) + \beta(x_1 - x_2 + x_3),$$

where $\lambda(h) = (\lambda_1(h), \lambda_2(h))$ and $\lambda_1(h) = (0,0)^T$, $\lambda_2(h) = (\alpha, \beta)^T$. Let us calculate the coefficients α and β.
Using the equality $\mathcal{L}'_{2\,x}(x^*, \lambda(h), h) = 0$ we obtain $\alpha = 1$ and $\beta = -1$. Putting the coefficients into we have

$$\bar{\mathcal{L}}_2(x^*, \lambda(h), h) = \frac{2}{3} x_2^2.$$

Therefore,

$$\bar{\mathcal{L}}''_{2\,xx}(x^*, \lambda(h), h)[h]^2 = \frac{4}{3} > 0.$$

It means that x^* is a strict local minimizer to (19).

2.10 P-Factor Method

Based on p-factor operator construction we give new method for solving nonlinear equations

$$F(x) = 0, \quad F : \mathbb{R}^n \to \mathbb{R}^n \tag{20}$$

where matrix $F'(x^*)$ is singular at the solution point x^*. Let $Y_1 = \mathrm{Im} F'(x^*)$, $\bar{P}_1 = P_{Y_1^\perp}$,
$Y_2 = \mathrm{Im}\left(F'(x^*) + \bar{P}_1 F''(x^*)h\right)$, $\bar{P}_2 = P_{Y_2^\perp}$,

$$Y_{k+1} = \mathrm{Im}\left(F'(x^*) + \sum_{i=1}^{k} \bar{P}_i F''(x^*)h + \sum_{\substack{i_2 > i_1 \\ i_1, i_2 \in \{\overline{1,k}\}}} \bar{P}_{i_2}\bar{P}_{i_1} F^{(3)}(x^*)[h]^2 + \ldots + \sum_{\substack{i_k > \ldots > i_1 \\ i_1, \ldots, i_k \in \{\overline{1,k}\}}} \bar{P}_{i_k} \ldots \bar{P}_{i_1} F^{(k)}(x^*)[h]^{(k-1)} \right),$$

$\bar{P}_{k+1} = P_{Y_{k+1}^\perp}$, $k = \overline{2, p-1}$.

Then the principal scheme of p-factor method the following

$$x_{k+1} = x_k - \{F'(x_k) + P_1 F''(x_k)h + \ldots + P_{p-1} F^{(p)}(x_k)h^{p-1}\}^{-1} \cdot$$

$$\cdot (F(x_k) + P_1 F'(x_k)h + \ldots + P_{p-1} F^{(p-1)}(x_k)h^{p-1}), \tag{21}$$

where $P_1 = \sum_{i=1}^{p-1} \bar{P}_i$, $\quad P_2 = \sum_{\substack{i_2 > i_1 \\ i_1, i_2 \in \{\overline{1,p-1}\}}} \bar{P}_{i_2}\bar{P}_{i_1}$, $P_{k+1} = \sum_{\substack{i_k > \ldots > i_1 \\ i_1, \ldots, i_k \in \{\overline{1,p-1}\}}} \bar{P}_{i_k} \ldots \bar{P}_{i_1}$, $k = \overline{2, p-1}$ and h some fixed element, $\|h\| = 1$ and $P_i, i = \overline{1, p-1}$ matrices of orthoprojection such that in solution point x^*.

$$(F(x^*) + P_1 F'(x^*)h + \ldots + P_{p-1} F^{(p-1)}(x^*)h^{p-1}) = 0 \tag{22}$$

and p-factor matrix

$$F'(x^*) + P_1 F''(x^*)h + \ldots + P_{p-1} F^{(p)}(x^*)h^{p-1} \tag{23}$$

is nonsingular (p-regular along h). It means that $\bar{P}_p = 0$, $Y_p = \mathbb{R}^n$.

Consider the case $p = 2$ for our example

$$x_{k+1} = x_k - \{F'(x_k) + P_1 F''(x_k)h\}^{-1} \cdot (F(x_k) + P_1 F'(x_k)h) \tag{24}$$

where P_1 is ortoprojection on to $\mathrm{Im}(F'(x^*))^\perp$ and element $h, (\|h\| = 1)$, such that 2-factor matrix

$$F'(x^*) + P_1 F''(x^*)h \tag{25}$$

nonsingular (2-regular along h). Then at the solution point will be hold

$$F(x^*) + P_1 F'(x^*)h = 0$$

and we can solve the following equation

$$F(x) + P_1 F'(x)h = 0 \tag{26}$$

where by virtue of (25) x^* will be locally unique solution.

Theorem 5. Let $F \in C^p(\mathbb{R}^n)$ and there exists $h, \|h\| = 1$ such that p-factor matrix (23) is nonsingular. Then for any $x_0 \in U_\varepsilon(x^*)$ ($\varepsilon > 0$ sufficiently small) will be fulfilled for scheme (21)

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^2, k = 0, 1, 2, \ldots. \tag{27}$$

where $c > 0$ – constant. ▲

Example 3.
$F(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix}$, $x^* = (0,0)^T$ and $F'(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is singular at $x^* = (0,0)^T$. The scheme of 2-factor method is the following

$$x_{k+1} = x_k - \{F'(x_k) + P_1 F''(x_k)h\}^{-1} \cdot (F(x_k) + P_1 F'(x_k)h) \tag{28}$$

where $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $h = (1, -1)^T$. Then

$$F'(x_k) + P_1 F''(x_k)h = \begin{pmatrix} 1 & 1 \\ x_k^2 - 1 & x_k^1 + 1 \end{pmatrix}$$

and

$$x_{k+1} = x_k - \begin{pmatrix} 1 & 1 \\ x_k^2 - 1 & x_k^1 + 1 \end{pmatrix}^{-1} \begin{pmatrix} x_k^1 + x_k^2 \\ x_k^1 x_k^2 + x_k^2 - x_k^1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ x_k^2 - 1 & x_k^1 + 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ x_k^1 x_k^2 \end{pmatrix}.$$

It means, that $\|x_{k+1} - 0\| \le c\|x_k - 0\|^2$.

Example 4

$$\min_{x \in \mathbb{R}^2} x_1^2 + x_1^2 x_2 + x_2^4$$

$F(x) = \varphi'(x) = \begin{pmatrix} 2x_1 + 2x_1 x_2 \\ x_1^2 + 4x_2^3 \end{pmatrix}$, $x^* = (0,0)^T$, F is 3-regular at x^* along $h = (1,1)^T$

$$F'(0) + P_1 F''(0)h + P_2 F^{(3)}(0)[h]^2 = \varphi''(0) + P_1 \varphi^{(3)}(0)h + P_2 \varphi^{(4)}(0)[h]^2 = \begin{pmatrix} 2 & -11 \\ 2 & 11 \end{pmatrix} \text{ is non singular!}$$

Here $\bar{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $P_1 = \bar{P}_1 + \bar{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$ $P_2 = \bar{P}_2 \bar{P}_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

Consider the 3-factor scheme

$$x_{k+1} = x_k - \left(\varphi''(0) + P_1 \varphi^{(3)}(0)[h] + P_2 \varphi^{(4)}(0)[h]^2\right)^{-1} \cdot \left(\varphi'(x_k) + P_1 \varphi''(x_k)[h] + P_2 \varphi^{(3)}(x_k)[h]^2\right).$$

Let us denote $x_k = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$\begin{aligned} \|x_{k+1} - 0\| &= \left\| x_k - \begin{pmatrix} 2 & -11 \\ 2 & 11 \end{pmatrix}^{-1} \begin{pmatrix} 2x_1 - 11x_2 + 2x_1x_2 - 6x_2^2 \\ 2x_1 + 11x_2 + x_1^2 + 18x_2^2 + 4x_2^3 \end{pmatrix} \right\| = \\ &= \frac{1}{44} \left\| \begin{pmatrix} 11x_1^2 + 132x_2^2 + 22x_1x_2 + 44x_2^3 \\ 2x_1^2 + 48x_2^2 - 4x_1x_2 + 8x_2^3 \end{pmatrix} \right\| \le 10\|x_k - 0\|^2. \end{aligned}$$

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