Computational Technique for Investigating Boundary Value Problems for Functional-Differential Equations of Pointwise Type

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Abstract

The paper considers functional-differential equations of pointwise type (FDEPT) which includes differential equations with delay, that lead to solution and both at once. In the paper, we propose a technology for solving boundary value problems for nonlinear FDEPT systems which is based on the Ritz method and spline collocation approaches. To solve the problem, we discretize system trajectories on the constant step grid and formulate the generalized residual functional, including both weighted residuals of the original differential equation and residuals of boundary conditions. Results of computational experiments for FDEPT make it possible to verify the effectiveness of the proposed technique for investigating problems.

1 Introduction

Functional-differential equations of pointwise type is an extension of a class of ordinary differential equations (ODE). It includes differential equations with retarded argument, and other more complex variants of dynamical systems with deviating argument. Such models evoke significant interest in recent years because of the application requirements: with the use of FDEPT a lot of processes can be successfully simulated in fields of economics, engineering, biology, sociology, medicine and other scientific and technical discipline. Qualitative theory is developed for systems of this type. The number of applied models have been analytically investigated, e.g. the problem of a propagation of traveling waves in homogeneous infinitely long elastic rod, Euler-Lagrange boundary value problem for the calculus of variations and others [Beklaryan L., 2007]. However, the problem of designing reliable numerical methods for solving FDEPT systems in general assumptions remains to be open.

One of the most important approaches to the study of systems of functional-differential operators is N. N. Krasovskii's interpretation of the solutions of equations with retarded argument as an integral line in the extended phase space of functions. The motion in this phase space at each moment of time is determined by the piece of trajectory that depends on the integral line in standard finite-dimensional extended phase space.

Using this approach theorems of existence and uniqueness of the solution for initial value problem are obtained, the structure of solutions space of linear equations is described, bounded and periodic solutions are studied an

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so on.

However, it is not applicable to functional-differential equations of pointwise type, which, in particular, arise as the Euler-Lagrange equations for the optimal control problem with retarded argument. Studying of differential equations of this type one can find two diametrically opposite approach.

In the first case, these equations are considered as a fundamentally new class according to the class of ordinary differential equations. So, all the diversity of new properties of such equations solution are investigated.

The second way is to consider these equations as an extension of ordinary differential equations class. This approach is based on the study of different characteristics, as for equations of advanced-retarded type well-known properties of ordinary differential equations are violated. These properties include: existence and uniqueness of the solution on a given class of functions; continuous solution dependence of initial and boundary conditions; pointwise completeness of the solution; n-parametric of the solution space; "smoothness" of the solution; the "robustness" property of the equation, etc. For equations of advanced-retarded type there are nonimproving conditions under which the extension of this class of problems is possible [Beklaryan L., 2007].

In this paper, we propose an approach to the study of nonlinear functional-differential equations, including equations with deviating argument of various kinds [Beklaryan A., 2013].

2 Initial-boundary Value Problem

Let's consider the basic initial-boundary value problem

$$\dot{x}(t) = f(t, x(q_1(t)), ..., x(q_s(t))),\ t \in B, \tag{1}$$

with boundary conditions

$$\dot{x}(t) = \phi(t),\ t \notin B, \tag{2}$$

and initial condition

$$x(\overline{t}) = \overline{x},\ \overline{x} \in \mathbb{R}^n,\ \overline{t} \in \mathbb{R}, \tag{3}$$

where $B = [t_0, t_1]$, $\mathbb{R} = [t_0, +\infty]$, and function $q_j(t)$, $j = \overline{1, s}$ are homeomorphisms of straight line that maintain the orientation. In general, when the deviation of an argument is arbitrary, we have a problem with nonlocal initial boundary conditions.

All specific FDEPT features emerge because the Lipschitz constant loses duties that it performs for the ODE.

Let L be Lipschitz constant for right side of FDEPT [Beklaryan L., 2007] $f(\cdot)$, $h_i = \max_{t\in\mathbb{R}} |q_i(t)|$, $i = \overline{1, s}$. For $\delta > 0$ define an inequality

$$L \sum_{i=1}^{s} \exp^{\delta |h_i|} < \delta. \tag{4}$$

Obviously, $h_i = 0$, $i = \overline{1, s}$ conditions is true for ODE. For this equation there exists such $\overline{\delta} > 0$, that for all $\delta \in (\overline{\delta}, +\infty)$ inequality (4) is satisfied.

If the equation is not an ordinary differential equation, then inequality (4) either does not have a single solution with respect to $\delta > 0$, or it is found $\delta_2 > \delta_1 > 0$ that inequality (4) is satisfied for any $\delta \in (\delta_1, \delta_2)$.

Theorem [Beklaryan L., 2007]. If inequality (4) holds for some $\delta > 0$, then for the initial boundary-value problem (1)–(3) there exists a solution whose asymptotic behavior is upper-majorized by an exponential function. Such solution is unique.

Consequence. Under the hypotheses of the theorem, for any arbitrarily small $\varepsilon > 0$ a solution exists in the class of functions asymptotically majorized by an exponential function $\varepsilon^{\delta_1+\varepsilon}|t|$, and it is unique in the class of functions asymptotically majorized by an exponential function $\varepsilon^{\delta_2-\varepsilon}|t|$.

By the above theorem, if we confine ourselves to solutions with the asymptotic behavior indicated in the theorem, then for them all the properties of the ODE solutions are satisfied: the existence and uniqueness of the initial-boundary value problem; the continuous dependence of the solution on the initial-boundary conditions; n-parametrization of the solution space; pointwise completeness of solutions; assignment of motion along solutions as homeomorphisms of phase space.

In the paper, it is proposed to reduce the initial-boundary problem presented to the optimization problem for the purpose of further investigation of it by applying the developed computing technology.

3 The Statement of Optimization Problem

Dynamics of the system on the main time interval is described by equations

$$0 = F_i(x(g(t)), \dot{x}(g(t))),\ t \in [t_0, t_1],\ i = \overline{1, dim\_n}, \tag{5}$$

where $F_i : \mathbb{R}^{dim\_n} \times \mathbb{R}^{dim\_n} \to \mathbb{R}^1$. On the extended interval of the independent variable variation $t \in [t_N, t_K]$, $t_N \leq t_0$, $t_K \geq t_1$, outside the basic interval, the values of the phase variables derivatives are defined $\dot{x}_i^L = h_i^L(t)$, $t \in [t_N, t_0]$ and $\dot{x}_i^R = h_i^R(t)$, $t \in [t_1, t_K]$, $i = \overline{1, dim\_n}$. Phase variables on the extended time interval must satisfy constraints $xl_i \leq x_i(g(t)) \leq xg_i$, $i = \overline{1, dim\_s}$. Functions of homeomorphisms $g_i(s)$, $i = \overline{1, dim\_s}$ are defined on the interval $s \in [0, 1]$ and must satisfy the monotonicity conditions

$$\frac{dg_i}{ds_i} > 0,\ i = \overline{1, dim\_s}. \tag{6}$$

The boundary conditions are given by the functionals $K_j(x(g(\tau_j)), \dot{x}(g(\tau_j))) = 0$, $j = \overline{1, dim\_l}$, $\tau_j \in [t_0, t_1]$. Thus, the dimensions of a problem are defined by integer variables dim_n, dim_s, dim_r, dim_l. All the elements of the mathematical formulation of the problem must satisfy the corresponding natural conditions of smoothness and consistency.

To solve the presented problem it is formulated the convolution of discrepancies functionals

$$\begin{array}{l} I(x(t)) = \sum_{i=1}^{dim\_n} v_i^N \int_{t_N}^{t_0} [\dot{x}_i(g(t)) - h_i^L(t)]^2 dt + \\ \sum_{i=1}^{dim\_n} v_i^N \int_{t_1}^{t_K} [\dot{x}_i(g(t)) - h_i^R(t)]^2 dt + \\ \sum_{j=1}^{dim\_l} v_j^K K_j^2(x(g(\tau_j)), \dot{x}(g(\tau_j))) + K_0(x(g(\tau_0)), \dot{x}(g(\tau_0))), \end{array} \tag{7}$$

where v_i^N, $i = \overline{1, dim\_n}$ and v_j^K, $i = \overline{1, dim\_l}$ are weighting coefficients for discrepancies and boundary conditions.

4 Computational Technology for Investigation of Nonlinear FDEPT Systems

The proposed technology for study boundary value problems is based on the Ritz method and spline-collocation approaches. To solve the problems of the class under consideration, the trajectories of the system are discretized on a grid with a constant step and a error functional is formulated. It includes both the weighted discrepancies of the original differential equation and the discrepancies of the boundary conditions. To evaluate the derivatives of the system trajectories, the "spline differentiation" technique is used, based on two spline approximation methods: using cubic natural splines and a special type of splines whose second derivatives at the edges are also optimized.

To solve the set finite-dimensional optimization problems, in the general non-convex case, a set of local optimization algorithms is implemented (BFGS quasi-Newton method, two versions of the Powell method, Barzilai-Borwain method, the method of trust region, stochastic search methods in subspaces of dimensions 3, 4, and 5 [Anikin, 2011], [Gornov, 2015]) and global optimization algorithms (random multistart method, curvilinear search method, tunnel method, parabolic method and others [Gornov, 2013], [Zarodnyuk, 2013], [Gornov, 2015]). The proposed technique includes an algorithm for the sequential increase in the accuracy of approximation by multiplying the number of the discretization grid nodes (see, for example, [Gornov, 2009]), algorithms for the difference evaluation of the functional's derivatives — from the first to the sixth degree of accuracy inclusive, the method of successively increasing the accuracy of spline differentiation.

The corresponding software OPTCON-F is implemented in C-language under the control of operating systems OS Windows, OS Linux and Mac OS using compiler GCC and is intended for numerical solution of boundary value problems, parametric identification problems and optimal control problems for dynamical systems described by the functional-differential equations of pointwise type.

Algorithmic filling of the complex includes a set of local (unimodal) and global (nonlocal) optimization algorithms. The techniques can be divided also into basic algorithms (optimization from any initial approximation) and algorithms for refinement of the taken result. The duration of the algorithm is limited to either the specified number of iterations, or the achievement of one of the stopping criterions (the minimum gradient norm for the current point is less than a given threshold value or the algorithm (for given values of algorithmic parameters) can not find a better approximation. Among the local algorithms included in OPTCON-F are: Parthan method; the classical Powell-Brent method; the gradient method of trust region; Barzilai-Borwein method; Newton's method with a difference estimate of the Hessian matrix; generalized quasi-Newton method; direct-dual method of gradient search; differential Euler optimization method of the second order; differential optimization method of Adams of the fourth order and others. As algorithms for rectification results obtained at the first stage are adaptive modification of Hooke-Jeeves method, stochastic search methods in random subspaces of the indicated dimension (2, 3, 4, or 5), local version of the curvilinear search method. Nonlocal algorithms in the basis of the OPTCON-F are "parabol method" (a combination of coordinate-wise search with a multistart and a nonlocal one-dimensional search using the parabola algorithm), nonlocal method of curvilinear search, the Luus-Yaakola method, the "forest method" (a multivariant adaptive method of random multi-start) and others.

5 The Results of Computing Experiments

A fragment of the computational experiments for boundary value problems for FDEPT systems using the developed computing technology is presented in this paragraph. In addition to analyzing the stability of the proposed numerical solutions for initial-boundary conditions, their main properties have also been studied, in particular, the qualitative properties of the majorant of such solutions have been investigated.

5.1 Test Problem 01

The initial-boundary test problem 01 is formulated as follows

$$\begin{aligned} &\dot{x}_1(t) = x_2(t^2),\ \dot{x}_2(t) = 4x_1(t+1) - 2x_2(t-1),\ t \in [0,1], \\ &\dot{x}_1(t) = 0,\ \dot{x}_2(t) = 1,\ t \in [-1,0) \cup (1,2], \\ &x_1(0) = 1,\ x_2(0) = 2. \end{aligned} \tag{8}$$

This system consists of equations with argument deviations of complicated structure. The graphs with solution of this test problem obtained by OPTCON-F are shown in Fig. 1.

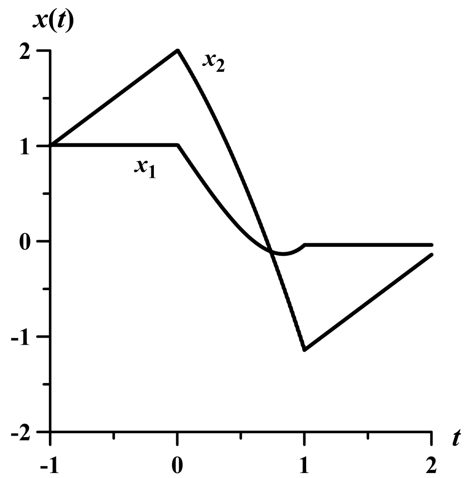

Figure 1: The trajectories of the system in test problem 01

5.2 Test Problem 02

We consider the boundary-value problem in the following formulation

$$\begin{aligned} &\dot{x}_1(t) = \tfrac{1}{2}x_2(t) - \tfrac{1}{2}x_2(t+1),\ \dot{x}_2(t) = \tfrac{1}{2}(t-3)x_1(t-1),\ t \in [0,1], \\ &\dot{x}_1(t) = -4,\ \dot{x}_2(t) = 1,\ t \in [-1,0), \\ &\dot{x}_1(t) = e^t,\ \dot{x}_2(t) = -e^t,\ t \in (1,2], \\ &x_1(0) = -1,\ x_2(0) = -2. \end{aligned} \tag{9}$$

The values of the derivatives of the phase variables are defined by using exponential functions. The graphs of the solution of problem 03 are shown in Fig. 2.

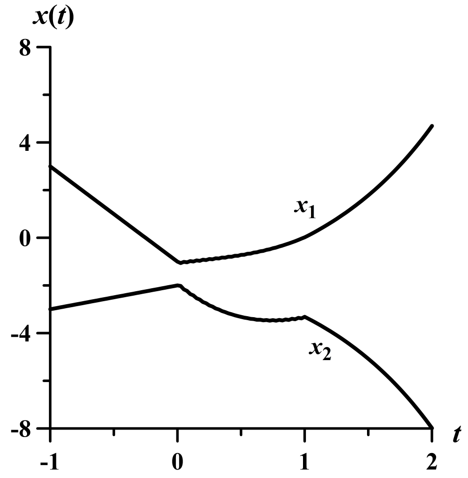

Figure 2: The trajectories of the system in test problem 02

5.3 Test Problem 03

The dynamical process in test problem 03 is described by the following system of differential equations with deviating argument

$$\begin{array}{l} \dot{x}_1(t) = \sin(t),\ \dot{x}_2(t) = x_1(t+1) - (t+1)x_2(t-1),\ t \in [0,1], \\ \dot{x}_1(t) = \sin(t),\ \dot{x}_2(t) = -\sin(t),\ t \in [-1,0), \\ \dot{x}_1(t) = \cos(t),\ \dot{x}_2(t) = \cos(t),\ t \in (1,2], \\ x_1(0) = 2,\ x_2(0) = 1. \end{array} \tag{10}$$

The results of the numerical solution of the test problem 02 are shown in Fig. 3. The minimum value of the convolutional functional corresponding to the system (10) with a grid of 241 nodes is 0.00074.

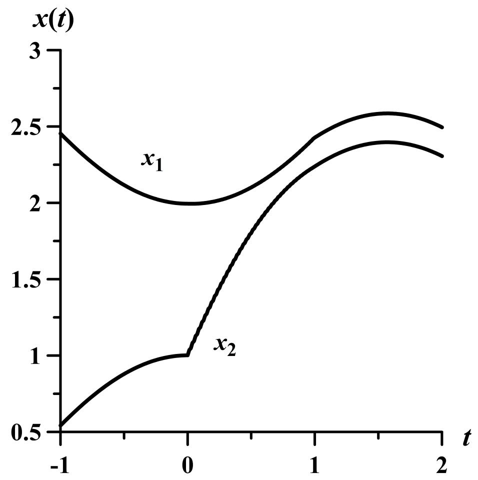

Figure 3: The trajectories of the system in test problem 03

6 Conclusion

The results of computational experiments for FDEPT with deviating argument in the right-hand parts of equations, make it possible to verify the operability of the proposed technology for investigating problems of this type. In the future, it is planned to develop the presented ideas for solving optimal control problems for systems with differential coupling of the FDEPT type.

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