Two-Phase Dual Simplex Method for Linear Semidefinite Optimization

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Abstract

The dual simplex method for solving linear semidefinite programming problem is considered. For finding the starting point in this method, which must be an extreme point of the feasible set, two approaches are proposed. The first approach is based on application of reduced gradient technique. The second one is borrowed from the dual affine scaling method for semidefinite optimization.

Introduction

To date the most popular numerical methods for solving linear semidefinite programming problems are interior point techniques. Nevertheless there are some methods which are generalization of primal simplex method for linear programming [Lasserre, 1996], [Pataki, 1996], [Kosolap, 2009], [Zhadan, 2015]. In [Zhadan, 2016a] the dual simplex method had been proposed. The starting point at this dual method must be an extreme point of the feasible set, and all subsequent points are extreme points too. The aim of this paper is to describe two approaches for finding such extreme point in the case where only the feasible point is known. First of all we formulate the primal and dual linear semidefinite programs. In Section 1, we briefly describe the main iteration of dual simplex method. Two approaches for finding starting extreme points are considered in Sections 2 and 3. The first approach is generalization of the procedure, which is well known in linear programming and can be treated as reduced gradient technique. The second approach is based on movement in faces of the feasible set with step-by-step decreasing dimension of these faces. The similar way had been used in dual affine scaling method with steepest descent [Zhadan, 2016b].

Let $\mathbb{S}^n$ denote the space of real symmetric matrices of order n, and let $\mathbb{S}^n_+$ denote the cone of positive semidefinite matrices from $\mathbb{S}^n$. The following inequality $M \succeq 0$ means also that $M \in \mathbb{S}^n_+$. The dimension of $\mathbb{S}^n$ is equal to so-called n-th triangular number $n_\triangle = n(n+1)/2$. The zero n-dimensional vector and the zero $m \times n$ matrix are denoted by 0_n and 0_{nm}, respectively.

Linear semidefinite programming refers to the optimization problem that can be expressed in the form

$$\begin{gathered} \min\ C \bullet X, \\ A_i \bullet X = b^i, \quad 1 \le i \le m, \quad X \succeq 0, \end{gathered} \tag{1}$$

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where $C \in \mathbb{S}^n$ and $A_i \in \mathbb{S}^n$, $1 \le i \le m$, are given. The matrix $X \in \mathbb{S}^n$ is a variable. The notation $C \bullet X$ stands for the inner product of symmetric matrices:

$$C \bullet X = \operatorname{tr} CX = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}.$$

The dual problem to (1) has the form

$$\begin{array}{c} \max\ b^T u, \\ \sum_{i=1}^{m} u^i A_i + V = C, \quad V \succeq 0, \end{array} \tag{2}$$

where $b = [b^1, \dots, b^m]^T$, $V \in \mathbb{S}^n$. It is assumed that matrices A_i, $1 \le i \le m$, are linear independent, and both problems (1) and (2) have solutions. We denote by $\mathcal{F}_D$ the feasible set at problem (2) and by $\mathcal{F}_{D,u}$ the set

$$\mathcal{F}_{D,u} = \left\{ u \in \mathbb{R}^m : \ C - \sum_{i=1}^{m} u^i A_i \succeq 0 \right\}.$$

The set $\mathcal{F}_{D,u}$ is a projection of $\mathcal{F}_D$ at the space of the variable u.

The optimality conditions for problems (1) and (2) are as follows

$$X \bullet V = 0, \qquad A_i \bullet X = b^i, \quad 1 \le i \le m, \qquad V = C - \textstyle\sum_{i=1}^{m} u^i A_i, \qquad X \succeq 0, \ \ V \succeq 0. \tag{3}$$

Below we will use the equivalent form of optimality conditions (3) based on vector representation of matrices.

Let vec$X$ denote the direct sum of columns X. Let also svec$X$ denote the direct sum of parts of columns of $X \in \mathbb{S}^n$ beginning with the diagonal entry. Moreover, off-diagonal entries of X are multiplied by $\sqrt{2}$ before placing in svec$X$. The dimension of vec$X$ is equal to n^2. Respectfully, the dimension of svec$X$ is equal to $n_\triangle$. With the help of these notations optimality conditions (3) can be rewritten in vector form as

$$\langle \mathrm{svec}X, \mathrm{svec}V \rangle = 0, \qquad \mathcal{A}_{svec} \mathrm{svec}X = b, \qquad \mathrm{svec}V + \mathcal{A}_{svec}^T u = \mathrm{svec}C, \tag{4}$$

where angle brackets indicate the Euclidean inner product in finite-dimensional vector space, and $\mathcal{A}_{svec}$ denotes the $m \times n_\triangle$ matrix with svecA_i as its rows, $1 \le i \le m$. Matrices X and V must be positively semidefinite.

1 Dual Simplex Method

The dual simplex method can be treated as a special way of solving system (4). Assume that the starting point u_0, which is an extreme point of the set $\mathcal{F}_{D,u}$, is given. Assume also that after some iterations we obtain the point $u = u_k$, which is an extreme point of $\mathcal{F}_{D,u}$ too. We define the dual slack $V = V(u)$, where $V(u) = C - \sum_{i=1}^{m} u^i A_i$. Also, we make the following decomposition

$$V = HD(\theta)H^T,$$

where H is an orthogonal matrix, $\theta = \left[\theta^1, \dots, \theta^n\right]^T$ is a vector of eigenvalues, and $D(\theta)$ is a diagonal matrix with the vector θ at its diagonal.

Let a rank of the matrix V be equal $s < n$. The point u is an extreme point of $\mathcal{F}_{D,u}$ if and only if $s_\triangle \le n_\triangle - m$. We call the extreme point u *regular*, if $s_\triangle = n_\triangle - m$. Otherwise, we call it *irregular*. In what follows, we will use the matrices $A_i^H = H^T A_i H$ instead of A_i, $1 \le i \le m$, and we will use the matrix C^H instead of C. We denote also by $\mathcal{A}_{svec}^H$ the $m \times n_\triangle$ matrix with rows svecA_i^H, $1 \le i \le m$.

Suppose that $\theta = \left[\theta^B; \theta^N\right]$, where $\theta^B = 0_{n-s}$ and $\theta^N > 0_s$. Here and in what follows we use punctuation mark [;] in concatenation of vectors for adjoining them in a column. According to decomposition of the vector θ the matrix $\mathcal{A}_{svec}^H$ can be decomposed into two submatrices: $\mathcal{A}_{svec}^H = \left[\mathcal{A}_{svec_B}^H, \mathcal{A}_{svec_N}^H\right]$, where the second matrix $\mathcal{A}_{svec_N}^H$ has the dimension $m \times (n_\triangle - s_\triangle)$. We partition also the vector svecX^H, where $X^H = H^T X H$, on two parts: $\mathrm{svec}X^H = \left[\mathrm{svec}_B X^H;\ \mathrm{svec}_N X^H\right]^T$ with $\mathrm{svec}_B X^H \in \mathbb{R}^{n_\triangle - s_\triangle}$. The same partition will be used for other $n_\triangle$-dimensional vectors.

If we put $\mathrm{svec}_N X^H = 0_{s_\triangle}$, then the second equality from (4) is reduced to

$$\mathcal{A}_{svec_B}^H \mathrm{svec}_B X^H = b. \tag{5}$$

In the case, where the point u is regular and nondegenerate (see [Alizadeh, 1997]), the matrix of system (5) is nonsingular. Therefore, solving this system of linear equations, we obtain

$$\mathrm{svec} X^H = \begin{bmatrix} \left(\mathcal{A}^H_{svec_B}\right)^{-1} b \\ 0_{s_\triangle} \end{bmatrix}.$$

The point $[u, V]$ is the solution of dual problem (2), when the corresponding matrix $X = HX^HH^T$ is positive semidefinite.

Suppose that X is not a positive semidefinite matrix and decompose $X = QD(\eta)Q^T$. Here Q is an orthogonal matrix, and η is a vector of eigenvalues of X. Since X is not positive semidefinite matrix, there exists the negative eigenvalue η^{i_k} with the corresponding eigenvector q_{i_k}. Then we update the point u by setting

$$u_{k+1} = u_k - \alpha_k \Delta u_k, \tag{6}$$

where $\alpha_k > 0$ is a step-size. The vector Δu_k is satisfied to the following system of linear equations

$$\left(\mathcal{A}^H_{svec_B}\right)^T \Delta u = \mathrm{svec}_B\, Q^H_{i_k}, \qquad Q^H_{i_k} = H^T q_{i_k} q^T_{i_k} H. \tag{7}$$

The value of objective function at the dual problem (2) increases according to the following formula

$$b^T u_{k+1} = b^T u_k - \alpha_k \eta^{i_k} > b^T u_k. \tag{8}$$

The dual slack variable $V^H = H^T V H$ changes in correspondence with (6):

$$V^H_{k+1} = V^H_k + \alpha_k \Delta V^H_k, \quad \Delta V^H_k = \sum_{i=1}^{m} \Delta u^i_k A^H_i.$$

The step-size α_k is chosen as large as possible under condition that the matrix V^H_{k+1} is positive semidefinite. The rank of matrix V^H_{k+1} does not exceed the rank of V^H_k. Therefore, the point u_{k+1} is an extreme point of $\mathcal{F}_{D,u}$ too.

In the case where u_k is an irregular extreme point of $\mathcal{F}_{D,u}$, the system (5) is underdetermined, and we take

$$\mathrm{svec}_B X^H_k = \left(\mathcal{A}^H_{svec_B}\right)^T \left[\mathcal{A}^H_{svec_B} \left(\mathcal{A}^H_{svec_B}\right)^T\right]^{-1} b$$

as its solution. On the contrary, system (7) is overdetermined. In order to overcome this difficulty we pass from the direction Δu_k in $\mathbb{R}^m$ to the direction ΔV_k in the space $\mathbb{S}^n$, for which the following expression is used

$$\Delta V_k = [q_{i_k}\ H_N] \begin{bmatrix} 1 & w^T \\ w & \Delta Z \end{bmatrix} \begin{bmatrix} q^T_{i_k} \\ H^T_N \end{bmatrix}. \tag{9}$$

Here H_N is a submatrix of the matrix H, composed from the last s columns of H, and $w = Wy$, where columns w_j of the matrix W are such that $q^T_{i_k} H_N w_j = 0$. Using relation between Δu_k and ΔV_k from (9), it is possible to find Δu_k, for which formula (8) is preserved.

Theorem 1 *Let solutions X_* and $V_* = V(u_*)$ of problems* (1) *and* (2) *be strictly complementary, i.e. for eigenvalues η^i_* and θ^i_* of X_* and V_* respectively the following inequalities $\eta^i_* + \theta^i_* > 0$, $1 \le i \le n$, hold. Then the sequence $\{u_k\}$, generated by method* (6), *converges to u_*.*

2 Initial Stage of the Method. The First Approach

Consider the problem of finding a starting extreme point in the dual simplex method. In order to obtain such point we apply generalization of the approach using in linear programming. First of all we examine the case, where the feasible point $u \in \mathcal{F}_{D,u}$ can be easily obtained.

Assume that among all equality-type constraints in problem (1) there is the equality $A_{i_1} \bullet X = b^{i_1}$ with the positive definite matrix A_{i_1}. Then the following point

$$u_0 = \left[0, \ldots, 0, u^{i_1}_0, 0, \ldots, 0\right]^T$$

belongs to the feasible set $\mathcal{F}_{D,u}$ and can be taken as starting point. Indeed, if $u_0^{i_1}$ is negative and sufficiently large by absolute value, then the following inequality

$$C - \sum_{i=1}^{m} u_0^i A_i = C - u_0^{i_1} A_{i_1} \succeq 0$$

is fulfilled, i.e. the point u_0 is feasible. Our aim now is to pass from u_0 to another extreme point of $\mathcal{F}_{D,u}$, using the reduced gradient technique.

Let now u_0 be a feasible point from $\mathcal{F}_{D,u}$. We take the corresponding dual slack $V_0 = V(u_0) = C - \sum_{i=1}^{m} A_i u_0^i$ and decompose it

$$V_0 = H_0 D(\theta_0) H_0^T,$$

where H_0 is an orthogonal matrix, and θ_0 is a vector of eigenvalues.

Let also V_0 be the matrix of rank s, and the following representation $\theta_0 = [\theta_0^B; \theta_0^N]$ be valid for the vector of eigenvalues, in which $\theta_0^B = 0_{n-s}$, $\theta_0^N > 0_s$. As in the previous section, we pass from V_0 to the matrix $V_0^{H_0} = H_0^T V_0 H_0$. The matrix $V_0^{H_0}$ is a representation of V_0 at the basis defined by columns of the orthogonal matrix H_0. We have

$$V_0^{H_0} = \begin{bmatrix} 0 & 0 \\ 0 & D(\theta_0^N) \end{bmatrix}, \qquad \mathrm{svec} V_0^{H_0} = \begin{bmatrix} 0_{n_\triangle - s_\triangle} \\ \mathrm{svec}\, D(\theta_0^N) \end{bmatrix}.$$

Denote $C^{H_0} = H_0^T C Y_0$. Denote also $l = n_\triangle - s_\triangle$, and split vectors $\mathrm{svec}\, V_0^{H_0}$ and $\mathrm{svec}\, C^{H_0}$ onto two parts

$$\mathrm{svec}\, V_0^{H_0} = \left[\mathrm{svec}_B V_0^{H_0};\ \mathrm{svec}_N V_0^{H_0}\right], \quad \mathrm{svec}\, C^{H_0} = \left[\mathrm{svec}_B C^{H_0};\ \mathrm{svec}_N C^{H_0}\right],$$

where $\mathrm{svec}_B V_0^{H_0} = 0_l$, $\mathrm{svec}_N V_0^{H_0} = \mathrm{svec} D(\theta_0^N)$.

Suppose that u_0 is not an extreme point of the set $\mathcal{F}_{D,u}$, i.e. the rank s does not satisfy to the inequality: $s_\triangle \leq n_\triangle - m$. Then $l < m$. In this case the system of linear equations

$$\left(\mathcal{A}_{svec_B}^{H_0}\right)^T u = \mathrm{svec}_B C^{H_0}, \tag{10}$$

where the matrix $\mathcal{A}_{svec_B}^{H_0}$ consists of the first l columns of $\mathcal{A}_{svec}^{H_0}$, is undetermined. The point u_0 is satisfied to this system.

Let a rank of system (10) be equal l. We partition the vector u_0 onto two subvectors: $u_0 = [u_0^P; u_0^Z]$, where u_0^P is a l-dimensional vector, and u_0^Z is a $(m-l)$-dimensional vector. According to partition of u_0 the matrix $\mathcal{A}_{svec_B}^{H_0}$ can be also to decomposed into two submatrices

$$\mathcal{A}_{svec_B}^{H_0} = \begin{bmatrix} \mathcal{A}_{svec_B}^{H_0,P} \\ \mathcal{A}_{svec_B}^{H_0,Z} \end{bmatrix}.$$

We assume that the first quadratic matrix $\mathcal{A}_{svec_B}^{H_0,P}$ of order l is nonsingular. Then system (10) can be rewritten as

$$\left(\mathcal{A}_{svec_B}^{H_0,P}\right)^T u^P = \mathrm{svec}_B C^{H_0} - \left(\mathcal{A}_{svec_B}^{H_0,Z}\right)^T u^Z.$$

Solving this system with respect to u^P, we obtain

$$u^P = u^P(u^Z) = \left[\left(\mathcal{A}_{svec_B}^{H_0,P}\right)^T\right]^{-1} \left[\mathrm{svec}_B C^{H_0} - \left(\mathcal{A}_{svec_B}^{H_0,Z}\right)^T u^Z\right]. \tag{11}$$

In particular, for $u = u_0$ we derive that $u_0^P = u^P(u_0^Z)$.

Furthermore, splitting the vector $b = [b^P; b^Z]$ and substituting the partition of the vector u onto two subvectors at the goal function of problem (2), we come to conclusion, that values of this function depends on only the second variable u^Z, namely:

$$\langle b, u \rangle = \langle b^P, u^P \rangle + \langle b^Z, u^Z \rangle = \langle b^P, u^P(u^Z) \rangle + \langle b^Z, u^Z \rangle = f(u^Z).$$

We rewrite this function as

$$f(u^Z) = \langle \bar{b}^Z, u^Z \rangle + \bar{c}, \tag{12}$$

where

$$\bar{b}^Z = b^Z - \mathcal{A}^{H_0,Z}_{svec_B} \left(\mathcal{A}^{H_0,P}_{svec_B}\right)^{-1} b^P, \qquad \bar{c} = \left[\left(\mathcal{A}^{H_0,P}_{svec_B}\right)^T\right]^{-1} \mathrm{svec}_B C^{H_0}.$$

The function (12) is linear by u^Z.

Let us pass to the updated point

$$u^Z = u^Z(\alpha) = u^Z_0 + \alpha \bar{b}^Z.$$

Here $\alpha > 0$. Then, taking $u(\alpha) = \left[u^P(\alpha); u^Z(\alpha)\right]$, where $u^P(\alpha) = u^P(u^Z(\alpha))$, and using (11), we come to conclusion that $u(\alpha)$ can be written in the form

$$u(\alpha) = u_0 + \alpha \Delta u, \qquad \Delta u = \left[\begin{array}{c} -\left[\left(\mathcal{A}^{H_0,P}_{svec_B}\right)^T\right]^{-1} \left(\mathcal{A}^{H_0,Z}_{svec_B}\right)^T \\ I_{m-l} \end{array}\right] \bar{b}^Z.$$

Therefore the dual slack $V^{H_0}(u(\alpha))$ changes by the following manner

$$V^{H_0}(u(\alpha)) = \left[\begin{array}{cc} 0 & 0 \\ 0 & D(\theta^N_0) - \alpha \bar{\Omega}_{NN} \end{array}\right].$$

Here $\bar{\Omega}_{NN} = \sum_{i=1}^m A^{H_0}_{i,N} \Delta u^i$, and $A^{H_0}_{i,N}$ is the lower right block of $A^{H_0}_i$ of order s.

The matrix $D(\theta_0) - \alpha\bar{\Omega}_{NN}$ is positive definite, if $\alpha = 0$. It remains positive definite, when α is positive and small enough. The maximal possible step length α_* is determined by the case, when among all eigenvalues of $D(\theta_0) - \alpha\bar{\Omega}_{NN}$ the null eigenvalue appears at the first time. Then the rank of the matrix $V^{H_0}(u(\alpha_*))$ becomes less with respect to the rank of V^{H_0}. At the same time the value of the goal function increases: $\langle b, u(\alpha_*)\rangle > \langle b, u_0\rangle$. If $u(\alpha_*)$ is not an extreme point, then we must proceed calculations. Theoretically there exists possibility that the matrix $D(\theta_0) - \alpha\bar{\Omega}_{NN}$ stays positive definite at any $\alpha > 0$. If $\bar{b}^Z \neq 0$, it means that problem (2) has not solution.

In the case, where not a matrix A_i is positive definite, but it is known by some reasons, that the solution X_* of problem (1) is restricted, for example $\mathrm{tr} X_* < b^{m+1}$ for some sufficiently large $b^{m+1} > 0$, then additional inequality type constraint can be introduced in problem (1)

$$\begin{array}{c} \min\ C \bullet X, \\ A_i \bullet X = b^i, \quad 1 \le i \le m, \quad I_n \bullet X \le b^{m+1}, \quad X \succeq 0. \end{array} \tag{13}$$

Solving (13), we obtain the same solution as in problem (1).

The dual problem to (13) has the form

$$\begin{array}{c} \max\ \langle b, u\rangle - u^{m+1} b^{m+1}, \\ C - \sum_{i=1}^m u^i A_i + u^{m+1} I_n \succeq 0, \quad u^{m+1} \ge 0. \end{array} \tag{14}$$

Now we can take $\bar{u}_0 = [u_0; u^{m+1}_0]$ as the starting point. The number $u^{m+1}_0 > 0$ must be sufficiently large in order that the matrix inequality $C + u^{m+1} I_n \succeq 0$ holds.

Further calculations for finding a feasible extreme point are the same as in the previous case. Moreover, the inequality $u^{m+1} \ge 0$ must be taken into account, when we choose the step length α. When we obtain, that $u^{m+1} = 0$, then the additional inequality in (14) is deleted, and we pass to solve the initial dual problem (2). As a starting point we take the first component from two-component point $[u; u^{m+1}]$, in which $u^{m+1} = 0$.

3 Initial Stage of the Method. The Second Approach

Consider the other approach for finding a starting extreme point. Suppose that we have the feasible point u, which is not an extreme point of $\mathcal{F}_{D,u}$. If the corresponding matrix $V = V(u)$ has the rank $s < n$, then u is a boundary point. Thus, it belongs to some face of $\mathcal{F}_{D,u}$ with non-zero dimension. Suppose additionally, that the decomposition $V = HD(\theta)H^T$ is valid, where H is an orthogonal matrix, and $\theta = [\theta_B; \theta_N]$ is the vector of eigenvalues with $\theta_B = 0_{n-s}$, $\theta_N > 0_s$. The matrix H can also be decomposed into two submatrices $H = [H_B, H_N]$.

According to this decomposition of H, we partition the space $\mathbb{S}^n$ into two linear subspaces $\mathbb{S}^n_B$ and $\mathbb{S}^n_N$, where $\mathbb{S}^n_B$ consists of matrices $M \in \mathbb{S}^n$ with the zero lower right block of order s. On the contrary, the subspace $\mathbb{S}^n_N$

consists of matrices $M \in \mathbb{S}^n$ in which only the lower right block of order s can contain nonzero elements. These two subspaces are mutually orthogonal, and any matrix $M \in \mathbb{S}^n$ can be represented as the sum of two matrices, one of which is in $\mathbb{S}^n_B$ and other — in $\mathbb{S}^n_N$. For example, $V = V_B + V_N$, where

$$V_B = \begin{bmatrix} V_{BB} & V_{BN} \\ V_{NB} & 0_{ss} \end{bmatrix}, \quad V_B = \begin{bmatrix} 0_{rr} & 0_{rs} \\ 0_{sr} & V_{NN} \end{bmatrix}, \qquad r = n - s.$$

In what follows we are interested more in matrices $V^H = H^T V H$, $X^H = H^T X H$ and $A_i^H = H^T A_i H$, rather then V, X and A_i, $1 \le i \le m$.

The first two equalities from optimality conditions (3) can be rewritten in the form

$$X_B^H \bullet V_B^H + X_N^H \bullet V_N^H = 0, \tag{15}$$

$$A_{i,B}^H \bullet X_B^H + A_{i,N}^H \bullet X_N^H = b^i, \quad 1 \le i \le m. \tag{16}$$

Moreover, if we require $X_B^H \succeq 0$ and $X_N^H \succeq 0$ and take into account, that $V_N^H \succeq 0$, then equality (15) splits into two equalities

$$X_B^H \bullet V_B^H = 0, \quad X_N^H \bullet V_N^H = 0. \tag{17}$$

Since V_B^H is zero matrix the first equality from (17) holds true for any matrix X_B^H.

Denote by $X \circ V$ the symmetrized product of two symmetric matrices X and V, i.e. $X \circ V = (XV + VX)/2$. Under $X_N^H \succeq 0$ and $V_N^H \succeq 0$ the equality $X_N^H \bullet V_N^H = 0$ takes place iff $X_N^H \circ V_N^H = 0_{nn}$. Therefore, the second equality from optimality conditions (17) can be replaced by the latter matrix equality. We rewrite it in vector form as

$$(\tilde{V}_N^H)^{\otimes} \mathrm{svec} X = 0_{n_\triangle}. \tag{18}$$

Here $(\tilde{V}_N^H)^{\otimes} = \tilde{\mathcal{L}}_n (V_N^H)^{\otimes} \tilde{\mathcal{D}}_n$, and $(V_N^H)^{\otimes} = \left(I_n \otimes V_N^H + V_N^H \otimes I_n\right)/2$ is the Kronecker sum of matrix V_N^H. The matrix $\tilde{\mathcal{L}}_n$ is *the elimination matrix*, which performs the transformation $\tilde{\mathcal{L}}_n \mathrm{vec} M = \mathrm{svec} M$ for any matrix $M \in \mathbb{S}^n$ (see [Magnus, 1980]). Similarly, $\tilde{\mathcal{D}}_n$ is *the duplication matrix.* For each matrix $M \in \mathbb{S}^n$ it performs the inverse transformation $\tilde{\mathcal{D}}_n \mathrm{svec} M = \mathrm{vec} M$.

We denote by $\tilde{\Theta}_N$ the lower right submatrix of order $s_\triangle$ of the matrix $(\tilde{V}_N^H)^{\otimes}$. The matrix $\tilde{\Theta}_N$ is diagonal with nonzero diagonal entries. In addition, we partition vectors $\mathrm{svec} X_B^H$ and $\mathrm{svec} X_N^H$ onto two parts, namely:

$$\mathrm{svec} X_B^H = \left[\mathrm{svec}_B X_B^H;\ \mathrm{svec}_N X_B^H\right], \quad \mathrm{svec} X_N^H = \left[\mathrm{svec}_B X_N^H;\ \mathrm{svec}_N X_N^H\right],$$

where $\mathrm{svec}_N X_B^H \in \mathbb{R}^{s_\triangle}$ and $\mathrm{svec}_N X_N^H \in \mathbb{R}^{s_\triangle}$. This partition we will use for other $n_\triangle$-dimensial vectors. Then equality (18) is reduced to

$$\tilde{\Theta}_N \ \mathrm{svec}_N X_N^H = 0_{s_\triangle}. \tag{19}$$

We denote also by $\mathcal{A}^H_{svec_B}$ the $m \times (n_\triangle - s_\triangle)$ matrix with rows $\mathrm{svec}_B A_i^H$, $1 \le i \le m$. Similarly, let $\mathcal{A}^H_{svec_N}$ denote the $m \times s_\triangle$ matrix formed by the vectors $\mathrm{svec}_N A_i^H$. With the help of these matrices the equality (16) can be written as

$$\mathcal{A}^H_{svec_B} \mathrm{svec}_B X_B^H + \mathcal{A}^H_{svec_N} \mathrm{svec}_N X_N^H = b. \tag{20}$$

Multiplying equality (20) by matrices $(\mathcal{A}^H_{svec_B})^T$ and $(\mathcal{A}^H_{svec_N})^T$ respectively and summing them with equality (19), we obtain the system of linear equations with respect $\mathrm{svec}_B X_B^H$ and $\mathrm{svec}_N X_N^H$,

$$\Phi \left[\mathrm{svec}_B X_B^H;\ \mathrm{svec}_N X_N^H\right] = \left[(\mathcal{A}^H_{svec_B})^T b;\ (\mathcal{A}^H_{svec_N})^T b\right], \tag{21}$$

where the matrix Φ has the form

$$\Phi = \begin{bmatrix} (\mathcal{A}^H_{svec_B})^T \mathcal{A}^H_{svec_B} & (\mathcal{A}^H_{svec_B})^T \mathcal{A}^H_{svec_N} \\ (\mathcal{A}^H_{svec_N})^T \mathcal{A}^H_{svec_B} & (\mathcal{A}^H_{svec_N})^T \mathcal{A}^H_{svec_N} + \tilde{\Theta}_N \end{bmatrix}.$$

We say that the point $u \in \mathcal{F}_{D,u}$ is *strongly nondegenerate*, if columns of the matrix $\mathcal{A}^H_{svec_B}$ are linear independent. In [Zhadan, 2016b] the following result had been proved.

Proposition 1 *Let the point $u \in \mathcal{F}_{D,u}$ be strongly nondegenerate. Then the matrix Φ is nonsingular.*

Below we assume that any point $u \in \mathcal{F}_{D,u}$ is strongly nondegenerate. Introduce notations

$$G_N = \mathcal{A}^H_{svec_N} \tilde{\Theta}_N^{-1} (\mathcal{A}^H_{svec_N})^T, \quad W_N = (I_m + G_N)^{-1},$$

$$G_B = (\mathcal{A}^H_{svec_B})^T W_N \mathcal{A}^H_{svec_B}, \quad W_B = \mathcal{A}^H_{svec_B} G_B^{-1} (\mathcal{A}^H_{svec_B})^T.$$

Then solving the system (21), we obtain

$$\mathrm{svec}_B X_B^H = G_B^{-1} \left(\mathcal{A}^H_{svec_B}\right)^T W_N b, \quad \mathrm{svec}_N X_N^H = \tilde{\Theta}_N^{-1} (\mathcal{A}^H_{svec_N})^T \left[W_N - W_N W_B W_N\right] b.$$

After substitution these expressions at the left hand side of (20) we get that (20) can be written as

$$\left[W_N - W_N W_B W_N\right] b = 0_m. \tag{22}$$

This is a system of m nonlinear equations with respect of m unknowns, which are components of the vector u.

Let $\Delta u = \left[W_N - W_N W_B W_N\right] b$. It can be shown that $\langle b, \Delta u \rangle \geq 0$. The direction Δu in the u-space determines the direction $\Delta V^H = -\sum_{i=1}^m (\Delta u)^i A_i^H$ in the V-space. Moreover, considering the point $\bar{u}(\alpha) = u + \alpha \Delta u$, which depends on $\alpha > 0$, we have the corresponding point $\bar{V}^H(\alpha) = V + \alpha \Delta V^H$. Since $(\mathcal{A}^H_{svec_B})^T \Delta u = 0_{n_\triangle - s_\triangle}$, the following formula

$$\bar{V}^H(\alpha) = \left[\begin{array}{cc} 0 & 0 \\ 0 & D(\theta_N) - \alpha \Omega_N \end{array} \right]$$

holds, where Ω_N is a symmetric matrix having the vector representation $\mathrm{svec}\, \Omega_N = \left(\mathcal{A}^H_{svec_N}\right)^T \Delta u$.

The matrix $\bar{V}^H(\alpha)$ remains positive semidefinite if α is sufficiently small. The maximal possible step α_* can be found from the condition that some eigenvalue of the matrix $D(\theta_N) - \alpha_* \Omega_N$ becomes zero. In this case the point $\bar{u}(\alpha_*) \in \mathcal{F}_{D,u}$ is such that the rank of the matrix $V(\bar{u}(\alpha_*))$ is less than the rank of $V(u)$. If the point $\bar{u}(\alpha_*)$ is not extreme, it is necessary to repeat the procedure.

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