

Algorithm for Solving Stochastic Transportation Problem

Anna V. Zykina
Omsk State Technical University
prospekt Mira 11,
644050 Omsk, Russia
avzykina@mail.ru

Olga N. Kaneva
Omsk State Technical University
prospekt Mira 11,
644050 Omsk, Russia
okaneva@yandex.ru

Abstract

The paper deals with the transportation problem in stochastic formulation. The problem solved by using the two-stage approach. The second stage is formulated as a complementarity problem. The algorithm for transportation problem solving is received. The solution based on stochastic gradient and Monte Carlo method.

1 Introduction

Significant interest is paid to stochastic formulation of the transportation problem with random demand [Williams, 1963] in applications that use a classical transportation problem [Koopmans, 1949]. In this case objective function represents the mathematical expectation of total losses during transportation of the product, damages of poor demand and the costs of excess product storing.

Transportation problem with random demand and continuous function distribution can be turned to deterministic problem of convex programming with linear constraints. In paper [Biswal et al., 2013] stochastic transportation problem is converted to an equivalent deterministic problem and solved by goal programming method. However, such transformation does not always gives an acceptable solution; therefore it is necessary to use other approaches to study the problem.

Another approach for solving the stochastic optimization problems is a two-stage solutions scheme [Judin, 1998]. The process of solving the problem can be divided into two stages: on the first stage we select the preliminary plan from deterministic conditions, on the second stage we implement compensation discrepancies, that have been identified after the implementation of random events. This approach can be used for stochastic problems where a preliminary decision should be taken and put into the implementation before we have know the value of random parameters. The preliminary decision can be determined for the transportation problem with random demand by distribution of materials supplies and the determined reserves of the materials. The algorithm for solving problem which based on the reduction of the original problem to an equivalent mixed-integer linear programming problem after discretization is considered in paper [Kibzun et al., 2016].

In general the difficulties of the two-stage stochastic transportation problems analysis are determined by the choose the best preliminary plan in the original problem, which would guarantee the existence of residual compensation for all implementations of uncertainty parameters.

In this paper we propose an algorithm for solving two-stage stochastic transportation problem which is considered in [Zykina et al., 2016]. In the problem the choice of compensation plan satisfies the conditions which are defined by the linear complementarity problem. The proposed algorithm is based on stochastic gradient and Monte Carlo method [Sakalauskas, 2004].

Copyright © by the paper's authors. Copying permitted for private and academic purposes.

In: Yu. G. Evtushenko, M. Yu. Khachay, O. V. Khamisov, Yu. A. Kochetov, V.U. Malkova, M.A. Posypkin (eds.): Proceedings of the OPTIMA-2017 Conference, Petrovac, Montenegro, 02-Oct-2017, published at <http://ceur-ws.org>

2 Statement of the Linear Complementarity Problem

It is important to consider the linear complementarity problem $LCP(B, q)$ in the form [Bazaraa et al., 1998]:

$$w - By = q, \quad w_j \geq 0, \quad y_j \geq 0, \quad w_j y_j = 0, \quad j = 1, \dots, l.$$

Here B is a given square matrix of order l , (w_j, y_j) is a pair of additional variables.

Condition $w_j y_j = 0, j = 1, \dots, l$, is similar to the condition of complementarity in the duality theory for inequalities $By + q \geq 0$ and $y \geq 0$. That means, in a pair of conjugate inequalities, at least one should be implanted as equality.

Non-negative definiteness of the matrix B ensures the solvability of problem $LCP(B, q)$. When B is positive definite, problem LCP has a unique solution y of problem $LCP(B, q)$.

The algorithm can be used as an additional conversion Lemke for the formulation of LCP problem solution [Bazaraa et al., 1998]. It can be shown as an analogy of the simplex algorithms for linear programming problem.

3 Mathematical Model of the Transportation Problem

Let us consider the classical transportation problem: minimize the total cost

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \rightarrow \min \quad (1)$$

for specified volumes of homogeneous product transportation $X = \{x_{ij}\} \quad i = \overline{1, m}, \quad j = \overline{1, n}$ with restrictions:

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = \overline{1, m}, \quad (2)$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = \overline{1, n}, \quad (3)$$

$$x_{ij} \geq 0, \quad i = \overline{1, m}, \quad j = \overline{1, n}. \quad (4)$$

Conditions (2) determine the distribution of transportation X in accordance to reserves $a = (a_1, a_2, \dots, a_m)$. Conditions (3) ensure the fulfillment of the demand $b = (b_1, b_2, \dots, b_n)$.

We introduce the matrix of the transportation plan X and matrices C by using vector of the transportation plan x and vector c , and sticking rows of matrix to vector

$$x_{ij} = x_{(i-1)n+j}, \quad c_{ij} = c_{(i-1)n+j}.$$

Then the group of equations (1), (2), (4) is presented as

$$cx \rightarrow \min, \quad Ax = a, \quad x \geq 0,$$

where matrix A corresponds to constraints (2).

4 Two-stage Stochastic Transportation Problem

It is important to consider the stochastic formulation of the transportation problem (1) – (4).

Let $b = b(\omega)$ be the demand which presented as a random variable and $c = c(\omega)$ be the cost of product transportation, which is also random.

Transportation problem with random demand can be presented as the following two-stage stochastic programming problem [Zykina et al., 2016]:

$$F(x) = M \{cx + f(x, b)\} \rightarrow \min_{x \in D}, \quad (5)$$

$$D = \{x \in R^{nm} \mid Ax = a, \quad x \geq 0\}, \quad (6)$$

where

$$f(x, b) = \{p^+ y^+(x, b) + p^- y^-(x, b)\} \rightarrow \min_{y^+, y^-}, \quad (7)$$

$$w^+ - B^+ y^+ = q^+, w^+ \geq 0, y^+ \geq 0, w_j^+ y_j^+ = 0, j \in J^+, \quad (8)$$

$$w^- - B^- y^- = q^-, w^- \geq 0, y^- \geq 0, w_j^- y_j^- = 0, j \in J^-. \quad (9)$$

We show the following meaningful interpretation of the problem by using the terms of the transportation problem:
– the components of vectors q^+ or q^- with coordinates

$$q_j^+ = \sum_{i=1}^m x_{ij} - b_j, \quad j \in J^+,$$

and

$$q_j^- = b_j - \sum_{i=1}^m x_{ij}, \quad j \in J^-,$$

that specify the amount of the deficit or excess of the resource that can occur with certain plan $x \in D$ and in the realization of a random variable b ;

– the components of vectors $y^+ = y^+(x, b)$ or $y^- = y^-(x, b)$ determine as the compensation plan of the deficit for each item of resource consumption;

– element b_{ij} of matrix B^+ or B^- specifies the amount of resource purchasing for i -th consumer under the compensation plan $y = (0, \dots, 1, \dots, 0)$, where the positive unit located on j -th place;

– the components of vectors p^+ or p^- determine fines for conducting corrective actions.

For the solvability of the second stage problem (7)–(9) for all implementations of a random variable b and for any preliminary plan $x \in D$, it is necessary and sufficient that the matrix B^+ and B^- be positive definite. Under such conditions there are a unique solutions $y^+(x, b)$ and $y^-(x, b)$ for each complementarity problem $\text{LCP}(B^+, q^+)$ and $\text{LCP}(B^-, q^-)$. In this case problem (5)–(9) is a nonlinear problem of stochastic programming with the following linear constraints in the form:

$$F(x) = M \{cx + p^+ y^+(x, b) + p^- y^-(x, b)\} \rightarrow \min_{x \in D}, \quad (10)$$

$$D = \{x | Ax = a, x \geq 0\}, \quad (11)$$

here $y^+(x, b)$, $y^-(x, b)$ are unique solutions of the corresponding complementarity problems. The objective function $F(x)$ is the expectation of a random function which depends on a random vector from a certain probability space. The feasible set D , $D \neq \emptyset$, is a bounded convex linear set.

To solve the problem (10) – (11), we can use methods of stochastic approximation [Sakalauskas, 2004].

5 Designations

Let us transform the matrix $H_{n \times m}$ into vector $h_{1 \times nm}$:

$$h_{(i-1)n+j} = H_{ij}, \quad i = \overline{1, m}, j = \overline{1, n}. \quad (12)$$

Then we transform the vector $h_{1 \times nm}$ into matrix $H_{n \times m}$:

$$H_{ij} = h_{(i-1)n+j}, \quad i = \overline{1, m}, j = \overline{1, n}. \quad (13)$$

Vectors q^{s+} and q^{s-} can be presented as:

$$q_j^{s+}(X^k) = \sum_{i=1}^m x_{ij}^k - b_j^s, \quad j \in J^{s+}, \quad (14)$$

$$q_j^{s-}(X^k) = b_j^s - \sum_{i=1}^m x_{ij}^k, \quad j \in J^{s-}. \quad (15)$$

Problem $\text{LCP}(B, q)$:

$$w - B_{l \times l} y = q, \quad w_j \geq 0, \quad y_j \geq 0, \quad w_j y_j = 0, \quad j = \overline{1, l}. \quad (16)$$

6 Algorithm

6.1 Starting

1) Input: a matrix M^C of dimension $m \times n$ which contains values of transport costs, a matrix D^C of dimension $m \times n$ of variances, M^b – vector with values of demand in points $j = \overline{1, n}$, D^b – vector of variances, a – warehouse stock vector $i = \overline{1, m}$.

2) Let $\hat{\varepsilon}$, $\hat{\rho}$ and δ be positive defined: $\hat{\varepsilon} > 0$, $\hat{\rho} > 0$ and $\delta > 0$.

3) Define the matrix of corrective measures $B_{n \times n}$, vectors p^+ and p^- (of dimension n) are fines for conducting corrective actions.

4) Select an initial volume N^0 of Monte Carlo estimates samples.

5) Transform the matrix M^C into a vector \bar{c} by the rule (12).

6) Find the initial approximation $x^0 \in R^{nm}$ as a solution of the linear programming problem:

$$\bar{c}x \rightarrow \min, Ax = a, x \geq 0. \quad (17)$$

7) Set $k = 0$ and turn to the main step.

6.2 Main Steps

Step 1.

1. Let x^k , N^k and $\Delta_k > 0$ are given. Transform the vector x^k into the matrix X^k by the rule (13).

2. Generate N^k values of the random vector b : b^1, b^2, \dots, b^{N^k} and the random matrix C : C^1, C^2, \dots, C^{N^k} .

3. For each implementation $s = \overline{1, N^k}$ of a random vector b :

(a) form the index sets J^{s+} and J^{s-} from the elements $j = \overline{1, n}$ according to the following rule:
if

$$\sum_{i=1}^m x_{ij}^k < b_j^s,$$

then the corresponding index j is placed in the set J^{s+} ;

if

$$\sum_{i=1}^m x_{ij}^k > b_j^s,$$

then the corresponding index j is placed in the set J^{s-} ;

(b) form the matrix B^{s+} by the elements of the matrix B at the intersection of rows and columns with numbers from the set J^{s+} , and similarly form the matrix B^{s-} in terms of the index set J^{s-} ;

(c) form vectors:

$$p_j^{s+}, j \in J^{s+},$$

$$p_j^{s-}, j \in J^{s-};$$

(d) form the vectors $q^{s+}(X^k)$ and $q^{s-}(X^k)$ in accordance to (14) and (15) respectively;

(e) for each $t = \overline{1, nm}$

i. transform the vector $(x_1^k, x_2^k, \dots, x_t^k + \Delta_k, \dots, x_{nm}^k)$ into the matrix X_t^k by the rule (13);

ii. compute $y^{s+}(X_t^k, b^s)$ as a solution of the complementarity problem LCP $(B^{s+}, q^{s+}(X_t^k))$;

iii. compute $y^{s-}(X_t^k, b^s)$ as a solution of the complementarity problem LCP $(B^{s-}, q^{s-}(X_t^k))$;

iv. compute $y^{s+}(X_t^k, b^s)$ as a solution of the complementarity problem LCP $(B^{s+}, q^{s+}(X_t^k))$;

v. compute $y^{s-}(X_t^k, b^s)$ as a solution of the complementarity problem LCP $(B^{s-}, q^{s-}(X_t^k))$;

vi. calculate the coordinate t of the stochastic gradient $g^s(X^k, b^s)$ according to the following formula:

$$g_t^s(X^k, b^s) = \bar{c}_t + p^{s+} \frac{y^{s+}(X_t^k, b^s) - y^{s+}(X^k, b^s)}{\Delta_k} + p^{s-} \frac{y^{s-}(X_t^k, b^s) - y^{s-}(X^k, b^s)}{\Delta_k};$$

(f) calculate the values:

$$f(X^k, C^s, b^s) = \sum_{i=1}^m \sum_{j=1}^n c_{ij}^s x_{ij}^k + p^{s+} y^{s+}(X^k, b^s) + p^{s-} y^{s-}(X^k, b^s);$$

4. after calculating all N^k values of the objective function $f(X^k, C^s, b^s)$ and the stochastic gradients $g^s(X^k, b^s)$ we find the following estimates:

$$\begin{aligned}\tilde{F}(X^k) &= \frac{1}{N^k} \sum_{s=1}^{N^k} f(X^k, C^s, b^s), \\ \tilde{D}^2(X^k) &= \frac{1}{N^k - 1} \sum_{s=1}^{N^k} (f(X^k, C^s, b^s) - \tilde{F}(X^k))^2,\end{aligned}$$

5. find the gradient of the objective function:

$$\tilde{\nabla} F(X^k) = \frac{1}{N^k} \sum_{s=1}^{N^k} g^s(X^k, b^s),$$

6. find the sample covariance matrix:

$$Q(X^k) = \frac{1}{N^k} \sum_{s=1}^{N^k} (g^s(X^k, b^s) - \tilde{\nabla} F(X^k))(g^s(X^k, b^s) - \tilde{\nabla} F(X^k))^T$$

7. determine Φ_γ as γ -quantile of the Fisher distribution with degrees of freedom $(N^k - n, n)$. Determine η_β as β -quantile of the standard normal distribution;

8. if $(N^k - n)(\tilde{\nabla} F(X^k))^T(Q(X^k))^{-1}\tilde{\nabla} F(X^k) \leq \Phi_\gamma$ and $2\eta_\beta \frac{\tilde{D}(X^k)}{\sqrt{N^k}} \leq \delta$ then STOP. Otherwise go to step 2.

Step 2.

1. Calculate the value

$$\varepsilon_{x^k}(\tilde{\nabla} F(X^k)) = \hat{\varepsilon} \max_{\substack{\tilde{\nabla} F_t(X^k) \geq 0, \\ 1 \leq t \leq nm}} \{\min\{x_t^k, \hat{\rho} \tilde{\nabla} F_t(X^k)\}\}.$$

2. Find the ε -possible direction d^k by solving the following problem:

$$\begin{aligned}\|d - \tilde{\nabla} F(X^k)\| &\rightarrow \min \\ Ad &= 0, \\ d_j &\leq 0, \quad j \in J^k,\end{aligned}$$

here J^k is the set of indices j for which inequality

$$0 \leq x_j^k \leq \varepsilon_{x^k}(\tilde{\nabla} F(X^k))$$

is specified.

3. Compute ρ^k . If $\exists t$ for which $d_t^k > 0$, then

$$\rho^k = \min\{\hat{\rho}, \min_{\substack{d_t^k > 0, \\ 1 \leq t \leq nm}} \left(\frac{x_t^k}{d_t^k}\right)\},$$

else $\rho^k = \hat{\rho}$.

4. Find

$$x^{k+1} = x^k - \rho^k d^k$$

and

$$N^{k+1} = \frac{\hat{\rho} \Phi_\gamma}{\rho^k (d^k)^T (Q(X^k))^{-1} d^k}. \quad (18)$$

5. Replace k by $k + 1$ and go to step 1.

7 Convergence

Convergence of the proposed algorithm provided [Sakalauskas, 2004] by determining the sample size from the conditions (18). In this case the objective function (10) must be differentiable. And its gradient must satisfy the Lipschitz condition. As mentioned above, when choosing the positive definite matrixs B^+ and B^- there is unique solutions $y^+(x, b)$ and $y^-(x, b)$ of the corresponding complementarity problem. In this case it is possible to obtain a linear dependence of the functions $y^+(x, b)$ and $y^-(x, b)$ [Kaneva, 2005]. This fact ensures differentiability of the objective function (10) and the Lipschitz condition for the gradient.

8 Conclusion

The effectiveness of the proposed numerical method for solving stochastic transportation problem stipulated by the following cases:

- We use two-step scheme where the choice of compensation plan of the second stage satisfy the conditions of the linear complementarity problem. This fact allows us compensate not only in the case when the demand is not satisfied, but when the supply exceed the demand. In addition, using the complementarity problem provides compensation on the limit of compatibility, which allows us to reduce costs of compensation.
- Under defined conditions of the compensation scheme choice (the positive definiteness of the matrix in the complementarity problem) ensures the uniqueness of the complementarity problem solution. This fact allows us use the solution of the complementarity problem to construct the stochastic gradient.

Acknowledgements

This work was supported by Russian Foundation for Basic Research, Project 15-41-04436-a-Siberia and by Ministry of education and science of Russia, Project 2.9314.2017

References

- [Koopmans, 1949] Koopmans, T.C. (1949) Uptimum utilization of the transportation system. *Econometrica*, 3-4.
- [Williams, 1963] Williams, A. C. (1963). A stochastic transportation problem. *Operations Research*, 759 – 770.
- [Biswal et al., 2013] Biswal, M. P. and Samal, H. K. (2013). Stochastic Transportation Problem with Cauchy Random Variables and Multi Choice Parameters. *Journal of Physical Sciences*, 17, 117 - 130.
- [Judin, 2010] Judin, D.B. (2010). *Mathematical methods of control under incomplete information. Objectives and methods of stochastic programming*. Moscow: Krasand.
- [Kibzun et al., 2016] Kibzun, A.I. and Khromova, O.M. (2016). Mathematical modelling of a transport system with minimal maintenance costs. *Vestn. YUUrGU. Ser. Matem. Modelirovanie i Programirovanie*, 9:3, 41 - 54. DOI: 10.14529/mmp160304
- [Zykina et al., 2016] Zykina, A.V.& Kaneva, O.N. (2016). Stochastic optimization models in transportation logistics. *Sovremennye informacionnye tekhnologii i IT-obrazovanie*, 12(1), 251 – 255.
- [Sakalauskas, 2004] Sakalauskas, L. (2004). Application of the Monte-Carlo Method to Nonlinear Stochastic Optimization with Linear Constraints. *Informatika*, 2, 271 - 282.
- [Bazaraa et al., 1998] Bazaraa, M. S. & Shetty, C. (1998). *Nonlinear Programming. Theory and Algorithms*. New York: Wiley.
- [Kaneva, 2005] Kaneva, O.N. (2005). Non-linear two-stage problem of stochastic programming with a deterministic matrix correction. *Mathematical structures and modeling* , 25 - 33.