About One Control Problem with Incomplete Information

Boris I. Ananyev
N.N. Krasovskii Institute of Mathematics and Mechanics UB of RAS
Kovalevskaya str. 16,
620990 Yekaterinburg, Russia.
abi@imm.uran.ru

Abstract

A linear control system describing the deviation from a rated trajectory is considered. The state vector is unobserved and the controller can watch only an output with disturbances constrained in LQ-norm. Using the theory of guaranteed estimation, the controller (1st player) builds informational sets and tries to minimize the payoff functional depending of the informational set at the end of observation process. The second player (an opponent) choosing the disturbances intends to prevent the controller and tries to maximize the functional. The situation is reduced to a differential game with complete information where the players are used the positional strategies depending on parameters of informational sets. An example with a flying vehicle is examined.

1 Introduction

We use the theory of guaranteed estimation [Kurzhanski & Varaiya, 2014] and give a generalization for the case with integral constraints. As the result of the solution we obtain the differential equations describing the evolution of the informational sets. With the help of these equations the differential game with complete information is formulated. To solve this we use the theory of positional differential games [Krasovskii, 1985, Lokshin, 1992, Souganidis, 1985, Souganidis, 1999, Subbotin, 1999]. The existence of an optimal strategy is established and a method of construction is suggested. It is established that the differential game has a value and a saddle point. Another approach to the problem under consideration is suggested in [Ananyev, 2017]. Observations' control problems were considered in [Ananiev, 2011, Ananyev, 2012].

The paper is organized as follows. In section 2 we give the background for guaranteed estimation. Section 3 contains a formulation of the problem. In section 4 we consider a solution with the help of Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation and suggest a numerical scheme. Here we define the strategies of players and stepwise solutions of the equations according to [Krasovskii, 1985, Subbotin, 1999]. Unfortunately, we cannot solve the HJBI equation with unbounded functions of the 2nd player. But we avoid this temporarily introducing the constraint on functions of the 2nd player. After that we pass to the limit and obtain the value of the game which computed in [Lokshin, 1992] by other way. Section 5 is devoted to an example.

In: Yu. G. Evtushenko, M. Yu. Khachay, O. V. Khamisov, Yu. A. Kochetov, V.U. Malkova, M.A. Posypkin (eds.): Proceedings of the OPTIMA-2017 Conference, Petrovac, Montenegro, 02-Oct-2017, published at http://ceur-ws.org

2 Guaranteed Estimation

Consider linear non-stationary equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t), \tag{1}$$
$$y(t) = G(t)x(t) + c(t)v(t), \quad t \in [0, T], \tag{2}$$

containing an uncertain function $v(\cdot)$ and a control $u(\cdot)$, where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $G(\cdot)$, $c(\cdot)$ are bounded Borelean matrices. The unobserved state vector $x(t) \in R^n$, the output $y(t) \in R^m$, $u(t) \in R^p$, $v(t) \in R^k$. Suppose that the uncertain function $v(\cdot)$ in (1) and (2) is constrained by the inequality

$$\|v(\cdot)\|^2 = \int_0^T |v(t)|^2 dt \leq 1, \tag{3}$$

where $|\cdot|$ is the Euclidean norm. Besides, the matrix $c(\cdot)$ must satisfy the condition

$$c(t)c'(t) \geq \delta I_m, \ \forall t \in [0, T], \tag{4}$$

where $\delta > 0$ and $I_m \in R^{m \times m}$ is the identity matrix. Hereafter the symbol $'$ means the transposition. According to general theory of guaranteed estimation [Kurzhanski & Varaiya, 2014], we give

Definition 1 The collection $\mathcal{X}_T(u, y)$ of state vectors $\{x(T)\}$ is said to be the **informational set** if for any $x \in \mathcal{X}_T(u, y)$ there exists a function $v(\cdot)$ satisfying (3) such that equality (2) holds a.e.

Denote by $\mathcal{C}(t)$ the matrix $(c(t)c'(t))^{-1}$. Under assumption (4) we have the equalities $v(t) = c'(t)w(t) + \mathcal{C}_1(t)f(t)$ and $\|v(\cdot)\|^2 = \|c'(\cdot)w(\cdot)\|^2 + \|\mathcal{C}_1(\cdot)f(\cdot)\|^2$, where $\mathcal{C}_1(t) = I_k - c'(t)\mathcal{C}(t)c(t)$ is the orthogonal projection on the subspace $\ker c(t)$. Using (2), we may rewrite inequality (3) as

$$\int_0^T \Big\{ |y(t) - G(t)x(t)|^2_{\mathcal{C}(t)} + |f(t)|^2_{\mathcal{C}_1(t)} \Big\} dt \leq 1. \tag{5}$$

From now on, the symbol $|x|^2_Q$ means $x'Qx$, where Q is a symmetrical and non-negatively defined matrix.

It is easily seen that $x \in \mathcal{X}_T(u, y)$ iff there exists a function $f(\cdot)$ satisfying (5) and subjecting to the equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)(\mathcal{C}_1(t)f(t) + c'(t)\mathcal{C}(t)(y(t) - G(t)x(t)))$$

with final condition $x(T) = x$. On the other hand, such a function exists iff the minimum on $f(\cdot)$ of the left-hand side of inequality (5) is less or equal 1. To find this minimum we use the Bellman equation for an equation

$$\dot{x} = f(t, x, v), \quad v(t) \in P, \quad t \in [0, T], \quad x(T) = x_T,$$

with a cost functional

$$J(T, x_T) = \int_0^T L(t, x, v)dt + \phi(x_0).$$

For any end position $\{t, x\}$ we find

$$J(t, x) = \inf_{v(\cdot)} \left\{ \int_0^t L(s, x(s), v(s))ds + \phi(x_0) \right\}, \quad x(t) = x.$$

Bellman's equation in partial derivatives is of the form:

$$J_t = \min_{v \in P} \big\{ L(t, x, v) - J_x f(t, x, v) \big\}, \quad J(0, x) = \phi(x).$$

Let the solution $J(t, x)$ of Bellman's equation be continuous in t, x and differentiable in x. Then it gives the minimum of the functional. This fact is sufficient for us. Note that there is a more advanced theory of viscosity solutions, [Fleming & Soner, 2006]. Trying to solve the Bellman's equation with a quadratic form $|x|^2_{P(t)} - 2d'(t)x + q(t)$, we come to the conclusion.

Lemma 1 *The informational set has the form* $\mathcal{X}_T(u,y) = \{x \in R^n : |x|^2_{P(T)} - 2d'(T)x + q(T) \le 1\}$, *where parameters may be found from equations*

$$\dot{P}(t) = -P(t)A(t) - A'(t)P(t) - P(t)C(t)C'(t)P(t) + (G(t) + c(t)C'(t)P(t))'\mathcal{C}(t)(G(t) + c(t)C'(t)P(t)), \quad P(0) = 0; \tag{6}$$

$$\dot{d}(t) = -\big(A(t) + C(t)C'(t)P(t)\big)'d(t) + (G(t) + c(t)C'(t)P(t))'\mathcal{C}(t)(y(t) + c(t)C'(t)d(t)) + P(t)B(t)u(t), \quad d(0) = 0; \tag{7}$$

$$\dot{q}(t) = \big|y(t)\big|^2_{\mathcal{C}(t)} - \big|C'(t)d(t)\big|^2_{\mathcal{C}_1(t)} + 2d'(t)B(t)u(t), \quad q(0) = 0. \tag{8}$$

If the matrix $P(t)$ is invertible on $(0,T]$, we can introduce the values $\hat{x}(t) = P^{-1}(t)d(t)$ and $h(t) = q(t) - \big|d(t)\big|^2_{P^{-1}(t)}$ which satisfy the equations

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + \big(c(t)C'(t) + G(t)P^{-1}(t)\big)'\mathcal{C}(t)(y(t) - G(t)\hat{x}(t)) + B(t)u(t), \tag{9}$$

$$\dot{h}(t) = \big|y(t) - G(t)\hat{x}(t)\big|^2_{\mathcal{C}(t)}. \tag{10}$$

The value $\hat{x}(T)$ is the center of bounded ellipsoid $\mathcal{X}_T(u,y)$. A simple sufficient condition for invertibility of matrix $P(t)$ on $(0,T]$ is the following.

Assumption 1 The solutions $W(t,\tau)$ of matrix differential equation

$$\partial W(t,\tau)/\partial t = A'(t)W(t,\tau) + W(t,\tau)A(t) + G'(t)G(t), \quad W(\tau,\tau) = 0,$$

is positive-definite for all $0 \le \tau < t \le T$.

It is well-known (see [Kurzhanski & Varaiya, 2014]) that Assumption 1 is equivalent to full observability of system (1), (2), where $u = 0$, $v = 0$, on any interval $[\tau, t]$.

Remark

The calculation of value $h(t)$ is extremely unstable. Practically, one can take a small matrix $P_0 > 0$ and solve equations (9), (10) with zero initial data. If $h(t)$ from (10) is grater 1, one needs to take a smaller matrix. Another way to overcome this difficulty is to set $\nu_0 = 0$. This increases a bit the initial set at instant t_0.

3 Problem Formulation

Let the observer (the 1st player) begin their control actions at the instant $t_0 > 0$, $t_0 < T$, and

$$u(t) \in \mathbf{P} \subset R^p,$$

where $\mathbf{P}$ is a compact set. If Assumption 1 holds (from now on, we suppose this) and $u \equiv 0$ on $[0, t_0)$, the observer can build the compact ellipsoidal informational set $\mathcal{X}_0 = \mathcal{X}_{t_0}(0,y) = \Big\{x \in R^n : |x - \hat{x}(t_0)|^2_{P(t_0)} \le 1 - h(t_0)\Big\}$ at the instant t_0 according to equations (6)–(8). The goal of the observer is to minimize the functional

$$\gamma(\mathcal{X}_T(u,y)), \tag{11}$$

where $\gamma(\cdot)$ is a non-negative continuous function defined on all compact sets in R^n. The continuity is understood in the sense of Hausdorff's convergence. For example, $\gamma(X) = \operatorname{diam} X = \max_{x,y\in X}|x-y|$ or $\gamma(X) = \max_{x\in X}|x|$. In our case, $\operatorname{diam}\mathcal{X}_T(u,y) = 2\sqrt{(1-h(T))|P^{-1}(T)|}$, where $|P| = \max_{|x|\le 1}|Px|$, does not depend on control $u(\cdot)$. The value $\max_{x\in\mathcal{X}_T(u,y)}|x| = \max_{|l|\le 1}\Big\{l'\hat{x}(T) + \sqrt{1-h(T)}|l|_{P^{-1}(T)}\Big\}$.

It can turn out that $h(t_0) = 1$. In this case the resource of disturbances is exhausted and we have $v(t) = 0$ for $t \ge t_0$. If so, we obtain $x(t_0) = \hat{x}(t_0)$ and $y(t) \equiv G(t)\hat{x}(t)$ for $t \ge t_0$. Our problem reduces to the ordinary control one for a linear system (9) with complete information on state vector. If $h(t_0) < 1$, we obtain more complicated situation. It is easily seen from equations (9), (10), that the evolution of informational set $\mathcal{X}_t(u,y)$ depends only on the control $u(\cdot)$ and the *innovation function* $w(t) = y(t) - G(t)\hat{x}(t)$ that satisfies the LQ-constraints

$$\int_{t_0}^{T} |w(t)|^2_{\mathcal{C}(t)} \le 1 - h(t_0). \tag{12}$$

We introduce the 2nd player (an opponent) who tries to maximize the functional (11) choosing the function $w(t)$ under constraints (12). Thus, we have a differential game for equations

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + \mathbb{C}(t)w(t), \quad \dot{\nu}(t) = -\big|w(t)\big|^2_{\mathcal{C}(t)}, \tag{13}$$

where $\nu(t) = 1 - h(t) \geq 0$, $\mathbb{C}(t) = \big(c(t)C'(t) + G(t)P^{-1}(t)\big)'\mathcal{C}(t)$. The initial states $\hat{x}(t_0)$, $\nu(t_0) = 1 - h(t_0)$, for equations (13) are known from (6)–(8). Note that max diam $\mathcal{X}_T(u,y) = 2\sqrt{\nu(t_0)|P^{-1}(T)|}$ is obtained under $w(t) = 0$ if $t \geq t_0$. In the common case, we can consider that

$$\gamma(\mathcal{X}_T(u,y)) = \sigma(\hat{x}(T), \nu(T)), \tag{14}$$

where $\sigma(\cdot,\cdot)$ is a continuous bounded function. Introduce the vector $z(t) = [\hat{x}(t); \nu(t)] \in R^{n+1}$, $t \in [t_0, T]$, $\nu(t) \in [0,1]$. The pair $\{t, z(t)\}$ is called a *position* of the game at the instant t. Let us rewrite system (13) once more:

$$\dot{z}(t) = \mathbf{A}(t)z(t) + \mathbf{B}(t)u(t) + g(t, w(t)), \tag{15}$$

where $\mathbf{A}(t) = [A(t), 0; 0_{1\times n}, 0]$, $\mathbf{B}(t) = [B(t); 0_{1\times p}]$, $g(t,w) = \Big[\mathbb{C}(t)w; -|w|^2_{\mathcal{C}(t)}\Big]$. Hereafter we use the standard notation from Matlab, where $[A_1, \ldots, A_k]$ means the row-concatenation of matrices of appropriate dimensions (sometimes, the comma is replaced by the blank), and $[A_1; \ldots; A_k]$ means the column-concatenation.

Let us remark yet that the function $\sigma(z)$ in terminal functional (14) may be considered Lipschitzean on R^{n+1}. It is fulfilled in many interesting cases.

4 A Solution with HJBI Equation

A strategies of players are defined as arbitrary functions $u(t,z) \in \mathbf{P}$ and $w(t,z,u)$ with the condition $\nu \in [0,1]$. For any admissible position $\{t_*, z_*\}$, where $\nu_* \in (0,1]$, the players independently choose the different partitions Δ_u and $\Delta_w = \{t_* = t_1 < \cdots < t_{k+1} = T\}$. Let any Borelean control $u(\cdot) \in \mathbf{P}$ be already fixed. Then the motion is defined for the interval $[t_i, t_{i+1})$ from the partition Δ_w as the solution of the stepwise differential equation (see eq. (15))

$$\dot{z}(t) = \mathbf{A}(t)z(t) + \mathbf{B}(t)u(t) + g(t, w(t_i, z(t_i), u(t_i))).$$

If for some index $i_* \in 1:k$ there is an instant $t^* \in [t_{i_*}, t_{i_*+1})$ such that $\nu(t^*) = 0$, then the resource of 2nd player is exhausted, and we set $w(t) = 0$ for $t \in [t^*, T]$. On the other hand, if any Borelean function $w(\cdot)$ such that $\nu(t) \geq 0$, $\forall t \in [t_*, T]$, is fixed, then the motion is defined for the interval $[t_i, t_{i+1})$ from the partition Δ_u as the solution of the stepwise differential equation

$$\dot{z}(t) = \mathbf{A}(t)z(t) + \mathbf{B}(t)u(t_i, z(t_i)) + g(t, w(t)).$$

For strategies $u(\cdot)$, $w(\cdot)$, and a position $\{t_*, z_*\}$, introduce the values

$$\mathbf{c}(u(\cdot), t_*, z_*) = \lim_{\delta\to 0} \sup_{\Delta_u} \sup_{w[\cdot]} \sigma(z(T)), \quad \mathbf{c}(w(\cdot), t_*, z_*) = \lim_{\delta\to 0} \inf_{\Delta_w} \inf_{u[\cdot]} \sigma(z(T)), \tag{16}$$

where $\delta = |\Delta_u|$ or $\delta = |\Delta_w|$ and $|\Delta|$ is a diameter of the partition Δ. The supremum in (16) is taken over all admissible realizations $w[\cdot]$ or $u[\cdot]$. Strategies $u^0(\cdot)$, $w^0(\cdot)$ are called optimal if

$$\mathbf{c}(u^0(\cdot), t_*, z_*) = \inf_{u(\cdot)} \mathbf{c}(u(\cdot), t_*, z_*), \quad \mathbf{c}(w^0(\cdot), t_*, z_*) = \sup_{w(\cdot)} \mathbf{c}(w(\cdot), t_*, z_*),$$

for any position $\{t_*, z_*\}$. In work [Lokshin, 1992], it is proved the existence of the optimal strategies $u^0(\cdot)$, $w^0(\cdot)$ such that

$$\mathbf{c}(u^0(\cdot), t_*, z_*) = \mathbf{c}(w^0(\cdot), t_*, z_*) = \mathbf{c}^0(t_*, z_*)$$

for any position $\{t_*, z_*\}$. Here $\mathbf{c}^0(t_*, z_*)$ is the value of the game and the pair $\{u^0(\cdot), w^0(\cdot)\}$ is a saddle point. In [Lokshin, 1992], a method of computing of the value $\mathbf{c}^0(t_*, z_*)$ is developed which based on a recurrent construction of concave envelopes for functions in auxiliary designs. In our paper, we use the method of HJBI equation. In addition, note that couterstrategies of 2nd player can be replaced by pure strategies with the same result.

For our problem we need to compute the value of the game $\mathbf{c}(t_0, z_0)$, where from now on, we omit the symbol 0 near $\mathbf{c}$. Fix some $N > 0$ and temporarily impose an additional constraint on the function $w(t)$: $|w(t)| \le N$. The corresponding optimal strategies and the value of the game are denoted by $u_N^0(\cdot)$, $w_N^0(\cdot)$, and $\mathbf{c}_N(t, z)$, respectively. For our temporarily case, in [Subbotin, 1999, Theorem 9.1] it was proved that the value of the game equals $\mathbf{c}_N(t_0, z_0)$, where the function $\mathbf{c}_N : [t_0, T] \times R^n \times [0, \nu_0] \to R$ satisfies (in corresponding minimax formalization) the equation

$$\partial \mathbf{c}_N(t, z)/\partial t + H_N(t, z, D\mathbf{c}_N) = 0 \tag{17}$$

with boundary condition $\mathbf{c}_N(T, z) = \sigma(z)$, $\nu \in [0, \nu_0]$, as $\nu(T) \le \nu_0$. In equation (17) the symbol $D\mathbf{c}$ means the generalized gradient of function $\mathbf{c}(t, z)$ with respect to variables z, and the Hamiltonian H_N is defined by the following way

$$H_N(t, z, l) = \min_{u \in \mathbf{P}} \max_{|\tilde{w}| \le N} h(t, z, u, \tilde{w}, l), \quad l = [l_1; l_2], \quad \tilde{w} = w\chi(\nu), \tag{18}$$

where $h(t, z, u, \tilde{w}, l) = l_1'(A(t)\hat{x} + B(t)u) + \left(l_1'\mathbb{C}(t)\tilde{w} - l_2|\tilde{w}|^2_{\mathcal{C}(t)}\right)$; the Heaviside function $\chi(\nu) = 1$, if $\nu > 0$, and $\chi(\nu) = 0$ otherwise. From now on, the function $\tilde{w}(t)$ is called the *control with switch*. It is easily seen that

$$\lim_{N \to \infty} H_N(t, z, l) = H(t, z, l) = l_1'A(t)\hat{x} + \min_{u \in \mathbf{P}} l_1'B(t)u + |l_1|^2_{\mathbb{C}'(t)\mathcal{C}^{-1}(t)\mathbb{C}(t)}/(4l_2)$$

if $\nu > 0$ and $l_2 > 0$. For equation (17) it is known that its solution in minimax sense coincides with viscosity solution (see [Subbotin, 1999, Fleming & Soner, 2006]). Note that both the solutions are unique. If the function $\mathbf{c}_N(t, z)$ has been built, the optimal strategies of 1-st and 2-nd players are defined as selectors of inclusions

$$u_N^0(t, z) \in \operatorname*{Argmin}_{u \in \mathbf{P}} \max_{|\tilde{w}| \le N} h(t, z, u, \tilde{w}, D\mathbf{c}_N(t, z)),$$
$$w_N^0(t, z, u) \in \operatorname*{Argmax}_{|\tilde{w}| \le N} h(t, z, u, \tilde{w}, D\mathbf{c}_N(t, z)).$$

With the help of this strategies we can built a stepwise approximate trajectory until the instant t^*, where the resource of the 2nd player is exhausted. After this instant the function $w(t) = 0$.

Unfortunately, we cannot assert the uniqueness and existence of solution in any sense if the function H_N in (17) is replaced with H. But in [Lokshin, 1992] it is shown that the limit

$$\lim_{N \to \infty} \mathbf{c}_N(t, z) = \mathbf{c}(t, z) \tag{19}$$

exists for any position $\{t, z\} \in [t_0, T] \times R^n \times [0, 1]$ uniformly on $\{t, z\}$. Thus, equations (17), (18) and relation (19) gives a method of computing of the value of the game.

4.1 A Numerical Solution

A numerical procedure can be built on the base of [Souganidis, 1985, Souganidis, 1999, Fleming & Soner, 2006]. For technical reason the all the matrices in (1), (2) will be considered Lipschitzean in t. Then the left-hand side of equation (15) $F(t, z, u, w) = \mathbf{A}(t)z + \mathbf{B}(t)u + g(t, w)$, where $u \in \mathbf{P}$, $|w| \le N$, satisfies the uniform Lipschitz condition $|F(t_1, z_1, u, w) - F(t_2, z_2, u, w)| \le C_1(|z_1 - z_2| + |t_1 - t_2|)$. In addition, the function $\sigma(z)$ from (14) is Lipshitzean, i.e. $|\sigma(z_1) - \sigma(z_2)| \le C_2|z_1 - z_2|$.

For approximation of $\mathbf{c}_N(t, z)$, we consider the partition $\Delta = \{t_0 < t_1 < \cdots < t_{K(\Delta)+1} = T\}$ of the interval $[t_0, T]$. The diameter $\max_i |t_{i+1} - t_i|$ of the partition is denoted by $|\Delta|$ as before. Define the function $\mathbf{c}_\Delta : [0, T] \times R^n \times [0, \nu_0] \to R$ as

$$\begin{gathered}\mathbf{c}_\Delta(T, z) = \sigma(z) \text{ on } R^n \times [0, \nu_0], \quad \mathbf{c}_\Delta(t, z) = \min_{u \in \mathbf{P}} \max_{|\tilde{w}| \le N} \left\{\mathbf{c}_\Delta(t_{i+1}, z + (t_{i+1} - t)F(t, z, u, \tilde{w})\right\}, \\ \text{if } t \in [t_i, t_{i+1}), \text{ and } i \in 0 : K(\Delta).\end{gathered} \tag{20}$$

We do the computation backward from right to the left. For $i = K(\Delta)$ we have

$$\mathbf{c}_\Delta(t, z) = \min_{u \in \mathbf{P}} \max_{|\tilde{w}| \le N} \left\{\sigma(z + (T - t)F(t, z, u, \tilde{w})\right\}, t \in [t_{K(\Delta)}, T).$$

Using [Souganidis, 1999, Theorem 4.4], we obtain

Theorem 1 *Under $|\Delta| \to 0$, the function* (20) *converges to $\mathbf{c}_N(t,z)$ locally uniformly on $[t_0,T] \times R^n \times [0,\nu_0]$. The function $\mathbf{c}_N(t,z)$ is the unique solution of equation* (17) *in minimax or viscosity sense. Besides, there exists a constant $L(C_2, ||\sigma||, ||\sigma_z||)$, such that $|\mathbf{c}_\Delta(t,z) - \mathbf{c}_N(t,z)| \le L|\Delta|^{1/2}$ for all $(t,z) \in [t_0,T] \times R^n \times [0,\nu_0]$. Here $||\cdot||$ is the sup-norm of corresponding function.*

Let us introduce the attainability set $\mathcal{W}(t)$ of system (13) with support function

$$\rho(l|\mathcal{W}(t)) = \max_{x \in \mathcal{W}(t)} l'x = \rho(l|X(t,t_0)\mathcal{X}_0) + \int_{t_0}^{t} \rho(l|B(s)\mathbf{P})ds + \sqrt{\nu_0 \int_{t_0}^{t} |\mathbb{C}'(s)l|^2 ds},$$

and the union $\mathcal{W} = \cup_{t \in [t_0,T]} \mathcal{W}(t)$. The set $\mathcal{W}$ is compact. We can suggest the following

Numerical Algorithm

1. Choose a finite set (a grid) $\mathcal{N} = \{z_k\} \subset \mathcal{W} \times [0,\nu_0]$, $k \in 1:K_1$.
2. Select a partition $\Delta = \{t_0 < t_1 < \cdots < t_{K+1} = T\}$ of $[0,T]$.
3. Form and remember the function $\sigma_K(z) = \min\limits_{u \in \mathbf{P}} \max\limits_{|\tilde{w}| \le N} \sigma\big(z + (t_{K+1} - t_K)F(t_{K+1}, z, u, \tilde{w})\big)$ and corresponding optimal controls u_K^* and $\tilde{w}_K^*(u)$, where $z \in \mathcal{N}$.
4. On subsequent steps the grid function is formed: $\sigma_i(z) = \min\limits_{u \in \mathbf{P}} \max\limits_{|\tilde{w}| \le N} \sigma_{i+1}\big(z + (t_{i+1} - t_i)F(t_{i+1}, z, u, \tilde{w})\big)$ and corresponding optimal controls u_i^* and $\tilde{w}_i^*(u)$. If the value $z + (t_{i+1} - t_i)F(t_{i+1}, z, u, \tilde{w})$ does not lie in the grid, then this value is changed for the nearest element from $\mathcal{N}$.
5. The value $\sigma_0(z)$ gives an approximate value of the game.

Before the calculation of $\sigma_0(z)$ we need to choose the sufficiently large number N.

5 An Example

Let us consider the rectilinear movement of an airplane in the vertical plane at h height: $x_{1(nom)} = L + Vt$, $x_{2(nom)} = h$. The real initial state $[\tilde{x}_0; \tilde{y}_0]$ may be differ from $[L; h]$ and unknown. The deviation from the basic movement is described by the system

$$\dot{x}_1 = x_3, \quad \dot{x}_3 = u_1, \quad \dot{x}_2 = x_4, \quad \dot{x}_4 = u_2, \quad t \in [0,T],$$

where $x_1 = \tilde{x}_1 - x_{1(nom)}$, $x_2 = \tilde{x}_2 - x_{2(nom)}$, $x_3 = \tilde{x}_3 - V$, $x_4 = \tilde{x}_4$. The control accelerations are limited by the constraint $u_1^2 + u_2^2 \le 10$. The model of measurements is of the form $\tilde{y} = \sqrt{(L - \tilde{x}_1)^2 + \tilde{x}_2^2}$. Linearizing this with respect to basic movement, we obtain

$$y(t) = g_1(t)x_1 + g_2(t)x_2 + v(t), \quad g_1(t) = Vt/\sqrt{V^2t^2 + h^2}, \quad g_2(t) = h/\sqrt{V^2t^2 + h^2},$$

where $v(t)$ is a disturbance with the constraint $\int_0^T v^2(t)dt \le 1$. It is possible to check that the system is completely observable. Therefore, for any $0 < t_0 < T$ the initial informational set $\mathcal{X}_0$ is bounded. Given numerical data: $L = 2500\,\mathrm{m}$, $h = 10000\,\mathrm{m}$, $V = 1200\,\mathrm{m/s}$, $T = 30\,\mathrm{s}$, we calculate the set $\mathcal{X}_0$ for $t_0 = 10\,\mathrm{s}$, $y(t) = g_1(t)(-500 + 200t) + g_2(t)(-1000 + 10t) + \sin t/\sqrt{20}$.

Here $A = [0,0,1,0;0,0,0,1;0,0,0,0;0,0,0,0]$, $B = [0,0;0,0;1,0;0,1]$, $C(t) = 0_{4\times1}$, $G(t) = [g_1(t), g_2(t), 0, 0]$, $c(t) = 1$. Let us write system (6)–(8):

$$\dot{P}(t) = -P(t)A - A'P(t) + G'(t)G(t), \quad P(0) = 0;$$
$$\dot{d}(t) = -A'd(t) + G'(t)y(t) + P(t)Bu(t), \quad d(0) = 0;$$
$$\dot{q}(t) = y^2(t) + 2d'(t)Bu(t), \quad q(0) = 0.$$

For $t_0 = 10$ we get $\hat{x}_0 \approx [2331; -1898; 200; -89]$, $h(t_0) = 0.3574$. The diameter of initial set equals 29.0228. To the end of time interval it increases to 143.2507 and does not depend of the control of 1st player. Of course, the control of 2nd player must be zero. Let $\sigma(z) = \max_{|l| \le 1} \Big\{ l'\hat{x}(T) + \sqrt{1 - h(T)}|l|_{P^{-1}(T)} \Big\}$, i.e. we minimize the maximal deviation. We have $P^{-1}(T) = [1003, -3594, 1, -120; -3594, 12902, -2, 431; 1, -2, 0, 0; -120, 431, 0, 14]$. Using Numerical Algorithm and setting $\nu_0 = 0$, we obtain the value of the game as $\mathbf{c}(t_0, z_0) = 71.3471$. More detail consideration of this example will be done in subsequent works.

6 Conclusion

A control problem for non-stationary linear systems is considered. These systems as a rule describe the deviation from a rated trajectory. According to the output a controller builds the informational set containing the real state vector and tries to minimize the functional of final informational set. A 2nd player using disturbances tries to disturb the 1st player and maximizes the functional. The situation is reduced to a differential game with complete information for linear differential equation describing the evolution of center of informational ellipsoid. To solve this problem we use the theory of differential games. An approximation of the value of the game may be found by integration of corresponding HJBI equation, the solution of which is understood in a generalized sense. The optimal strategies are also defined due to this solution. The numerical approximation is specified and the estimation of the rate of convergence is given for the approximating scheme. An example with a rectilinear movement of an airplane is examined.

Acknowledgements

This work was supported by Russian Science Foundation (RSF) under Project No. 16-11-10146.

References

[Kurzhanski & Varaiya, 2014] Kurzhanski, A., & Varaiya, P. (2014). *Dynamics and Control of Trajectory Tubes: Theory and Computation.* SCFA. Birkhäuser, Boston.

[Krasovskii, 1985] Krasovskii, N.N. (1985). *Control of Dynamic Systems.* Nauka, Moscow. (in Russian)

[Lokshin, 1992] Lokshin, M.D. (1992). Differential games with integral constraints on the control actions. *Differential Equations, 28(11)*, 1602-1610. (translation from Russian)

[Souganidis, 1985] Souganidis, P.E. (1985). Approximation Schemes for Viscosity Solutions of Hamilton-Jacobi Equations. *J. Differen. Equat.*, **vol. 59**, 1-43.

[Souganidis, 1999] Souganidis, P.E. (1999). Two-Player, Zero-Sum Differential Games and Viscosity Solutions. In *Annals of the Intern. Society of Dynamic Games (ed. Tamer Basar)*, **vol. 4**. Birkhüser, Boston, pp. 69–104.

[Subbotin, 1999] Subbotin, A.I. (1999). Constructive Theory of Positional Differential Games and Generalized Solutions to Hamilton-Jacobi Equations. In *Annals of the Intern. Society of Dynamic Games (ed. Tamer Basar)*, **vol. 4**. Birkhüser, Boston, pp. 3–67.

[Ananyev, 2017] Ananyev, B.I. (2017). Parameter Optimization for Estimation of Linear Non-Stationary Systems. In *8th Intern. Scientific Conf. on Physics and Control. Physcon'2017.* Florence, Italy. July 17-19, 2017. Internet publ. (to be published)

[Ananiev, 2011] Ananiev, B.I. (2011). Some Problems of Observations' Control. *American Institute of Physics. Conference Proceedings.* Vol. **1404**, No. 1, pp. 235-241. Melville, NY 11747-4502, USA.

[Ananyev, 2012] Ananyev, B.I. (2012). *American Institute of Physics. Conference Proceedings.* Vol. **1487**, No. 1, pp. 143-150.

[Fleming & Soner, 2006] Fleming, W.H. & Soner, H.M. (2006). *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, New York.