Investigation of the time delay difference estimator for FMCW signals

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Abstract. The paper deals with the problem of development of the effective time delay difference estimator of beat signals, obtained by processing the frequency modulated signals. Such beat signals can be formed by the heterodyne scheme of receiver of linear frequency modulated signals with internal coherence. The described problem is actual in many radar applications such as radio direction finding. The paper is carried out the analysis of Cramer-Rao lower bound for delay as signal parameter and its comparison to frequency and initial phase. On the base of the considered approach the estimator of delay difference of beat signals is proposed in the class of smooth parabolic finite difference estimators. Investigation of the proposed method is carried out. Advantages of the developed estimator are shown in comparison with other techniques in such class in the areas of variance and influence of the parasitic signals on the bias.

Keywords: signal processing, FMCW signals, chirp signals, beat signals, delay measurements, radars, delay estimators

1 Introduction

The problem of time delay difference estimation of received signals is actual in many applications of radar, sonar systems and also in ultrasonic, laser and other measurement tasks [1], [2], [3], [4], [5]. The main problem formulation of time delay difference measurement is estimation of $\Delta\tau$ from the resulted samples of two received signals [5]

$$s_1(t) = h_1(t) * s_0(t) + N_1(t); s_2(t) = h_2(t) * s_0(t - \Delta\tau) + N_2(t)), \tag{1}$$

where $\Delta\tau$ is the estimated time delay difference; $s_1(t), s_2(t)$ are two received signals; $s_0(t)$ is the emitted signal; $h_1(t), h_2(t)$ are the impulse responses of media which signals pass; $N_1(t), N_2(t)$ are the noises. In general case, noises of system can be white or colored. The values of $s_i(t), h_i(t), s_0(t), N_i(t)$ can be complex.

The time delay estimation methods can be divided in two groups in the frequency and time domain. Methods of estimation in frequency domain consist in the calculation of difference values of phase spectrum, which correspond to the searched signals. Many authors notice the drawbacks of such methods which are connected with influence of the side lobes [2], [5]. Such influence in the frequency domain leads to the shift of estimated value. This effect can be reduced by using the window functions. Other drawback is the problem of measuring values between frequency grid nodes [5].

The time domain delay difference estimation methods can be separated as follows. Methods based on the zero- or threshold- crossings detection. Methods based on calculation of signal model parameters. Methods that use *a prior* information about phase of signal [6], [7]. Methods based on the cross correlation function analysis [2], and etc. Many authors notice that the phase difference based methods are the most accurate [2], [3].

One of the most effective ways for increasing the accuracy of time delay measurements in the short range radar problem is using the continuous waves with complex modulation. Such signals allow one to provide signal to noise ratio (SNR) higher than pulsed systems with equal power [8].

In radar problems signals with linear frequency modulated continuous waves (FMCW) are often used. One of the main advantages of those one is straight dependence of processed frequency on delay of signals received from each aim. The beat tone obtained by correlation scheme has the following expression [8]:

$$s = A\cos(2\pi[\frac{\Delta f\tau}{T_m}]t+2\pi[f_0\tau+\frac{\tau^2\Delta f}{2T_m}+\Delta\theta_0])+N(t) = A\cos(\omega t+\varphi))+N(t), \quad (2)$$

where A is the amplitude of beat signal; τ is the delay; Δf is the frequency deviation; f_0 is the initial frequency; T_m is the period of modulation; $\Delta\theta_0$ is the initial phase difference between emitted and received signals; ω is the frequency of the beat tone; φ is the initial phase of the beat tone.

The beat tone contains information about the time delay of the received signal in its frequency and phase. As a rule, only a frequency is measured [8]. However, it can be shown that the initial phase can also be considered as the information parameter if measured delay difference is less than one period of the signal. Thus, the measured delay τ can be estimated as a function of the frequency and initial phase. It should be noted that in practice the initial phase of beat signal can be considered as a linear function $\varphi \approx 2\pi[f_0\tau+\varphi_0]$ if $\tau \ll 2T_m$.

The aim of this paper is investigation of possibility of using the phase information of beat signals for increasing the accuracy of small (less than one period) time delay difference estimation.

2 Problem formulation

The cross correlation based method for the time delay estimation is widely used in practice. In one of it implementations, it consists in phase difference estimation by using of maximum of the corss-correlation function [5]. The additional advantage of this method is the relatively low computational complexity. It can be shown that in the case of single harmonics the method can be considered as follows [2]:

$$\hat{\theta}(t) = \arg\rho(0) = \arg\left(\sum s_2(t)s_1^*(t)\Big/\sqrt{\sum s_2^2(t)\sum s_1^2(t)}\right), \quad (3)$$

where $\rho(0)$ is the normalized cross-correlation coefficient; $*$ is conjugation.

The advantage of estimator (3) is the relatively low computational complexity. However, it is shown that the estimator (3) is not asymptotically effective and has a bias in general case [2].Other drawback of this method is the deterioration in accuracy in areas near 0 and π values of analyzed signals phase difference [1].

The Author of paper [6] has proposed to use *a prior* information of phase to time relation for parameter estimation of processed signals. The initial phase and frequency of single harmonic signal can be calculated by approximation of full phase as follows:

$$\varphi(t) = \hat{\omega} t + \hat{\theta}, \tag{4}$$

where $\varphi(t)$ is the approximation line; $\hat{\omega}$ is the estimated frequency; $\hat{\theta}$ is the estimated initial phase.

By using the least-square method (LSM), the initial phase of (4) can be found as [6]

$$\hat{\theta} = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} \arg s(n), \tag{5}$$

where N is the sample length; *arg* is the operator of obtaining phase of a complex number; $s(n)$ is the analyzed sample.

In the considered problem $s(n) = s_2(n) s_1^*(n)$, where $s_2(n), s_1(n)$ are complex signals, the delay difference between which is measured.

It is noted that solution (5) has the variance which coincides with the Cramer-Rao low bound (CRLB).

The Authors of [7] has performed analysis of phase approximation task with assumption that $|s(n)|$ is a random variable. In this case the solution of least square equation has follow expression:

$$\hat{\theta} = \Psi^{-1}(-\beta[\sum_{n=0}^{N-1} n|s(n)| \arg s(n)] + \eta[\sum_{n=0}^{N-1} |s(n)| \arg s(n)]), \tag{6}$$

where $\Psi = \alpha\eta - \beta^2$; $\alpha = \sum_{n=0}^{N-1} |s(n)|$; $\beta = \sum_{n=0}^{N-1} n|s(n)|$; $\eta = \sum_{n=0}^{N-1} n^2 |s(n)|$. It is noted that if $|s(n)|$ =const, (6) is equivalent to (5).

The model of real-valued harmonic signals of FMCW radars and correlation scheme of processing (2) can be described as follow:

$$s_1(t) = A_1 \cos(\omega_1 t + \theta_1); s_2(t) = A_2 \cos(\omega_2 t + \theta_2), \tag{7}$$

where ω_1, ω_2 are the frequencies of analyzed beat tones; $A_1, A_2, \theta_1, \theta_2$ are the corresponding amplitudes and initial phases.

The result of multiplication of the signals s_1 and s_2 ($s(n) = s_2(n) s_1^*(n)$) without noises $N(t)$ in anaclitic form can be written as

$$s(n) = \sum_{k=1}^{p} A_0 \exp(i2\pi[\frac{\Delta f}{T_m} \frac{n}{f_s} \tau_k + f_0 \tau_k]) = \sum_{k=1}^{p} A_0 \exp(i W_\tau(n) \tau_k), \tag{8}$$

where $s(n)$ is the processed signal; $\Delta\omega$ is the beat frequency difference; $\Delta\theta$ is the initial phase difference; $\Sigma\omega, \Sigma\theta$ are the frequency and initial phase sum respectively; τ_k is the delay of the k-th signals; $W_\tau(n)$ is the weight coefficient which is given as

$$W_\tau(n) = 2\pi[\frac{\Delta f}{T_m}\frac{n}{f_s} + f_0] = 2\pi[n\frac{\Delta f}{N} + f_0]. \quad (9)$$

Taking into account the noise term $N(t)$ in case of the single harmonic signal ($p = 1$) (9) can be given as

$$s = s_2 s_1^* = A_0 \exp iW_\tau(n)\tau + N(n) = A_0[\mu_n + i\nu_n], \quad (10)$$

where $\mu_n = \cos[W_\tau(n)\tau] + Re[N(n)]$ and $\nu_t = \sin[W_\tau(n)\tau] + Im[N(n)]$.

It can be supposed that signal (10) has two random parameters A_0 and τ. The Fisher matrix in this case has rank 2. Each diagonal element of the Fisher Matrix has the following expression [9]

$$J_{\tau\tau} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}[\frac{\partial^2\mu_n}{\partial\tau^2} + \frac{\partial^2\nu_n}{\partial\tau^2}] = \frac{A_0^2}{\sigma^2}\sum_{n=0}^{N-1}W_\tau^2(n), \quad (11)$$

where $J_{\tau\tau}$ is the diagonal element of the Fisher matrix. The Cramer-Rao low bound for estimation of parameter τ of signals in from (10) is given by

$$var[\tau]_\tau = \frac{SNR^{-1}}{\sum_{n=0}^{N-1}W_\tau^2(n)} \approx \frac{SNR^{-1}}{4\pi^2 N(f_0 + \Delta f/2)^2}, \quad (12)$$

where $var[\tau]_\tau$ is the variance of delay estimation by delay as parameter.

The CRLB for estimation of delay by frequency is given by [9]

$$var[\tau]_\omega = (\frac{T_m}{\Delta f})^2(\frac{f_s}{2\pi})^2\frac{12SNR^{-1}}{N^3} \approx \frac{3SNR^{-1}}{\pi^2\Delta f^2 N}, \quad (13)$$

where $var[\tau]_\omega$ is the variance of delay estimation by frequency as parameter of the signal. In (13), it is supposed that the sample length coincides with one period of modulation (T_m), and, hence, $T_m = N/f_s$.

The CRLB for estimation of the delay by the initial phase is given by [9]

$$var[\tau]_\theta = (\frac{1}{2\pi f_0})^2\frac{12SNR^{-1}}{N} \approx \frac{3SNR^{-1}}{\pi^2 f_0^2 N}, \quad (14)$$

where $var[\tau]_\theta$ is the variance of delay estimation by frequency.

From comparing expressions (12)-(14) the following relations can be given

$$var[\tau]_\tau \approx 12\frac{(f_0 + \Delta f/2)^2}{f_0^2}var[\tau]_\theta \approx 12\frac{(f_0 + \Delta f/2)^2}{\Delta f^2}var[\tau]_\omega. \quad (15)$$

Relations (15) show that the proposed approach of using delay as parameter gives the minimal CRLB. For instance, if $f_0 = 2\Delta f$, then expression (15) gives: $var[\tau]_\tau = 18.75 var[\tau]_\theta = 75 var[\tau]_\omega$.

The results mentioned above show the advantages of using full information of the phase of beat tone (that equivalent to use the delay as parameter) for time delay estimation. The main drawback of this approach is restriction, which is connected with the phase uncertainty if its value crosses $\pm\pi$ bound.

Results described above lead to the supposition that it is actual to further investigation of the effective estimator design for the problem of time delay difference measurement of beat signals of FMCW radar systems.

3 The estimator design

The beat tone (10) can be transformed in a Fourier sequence with the follow coefficients

$$C_l(n) = \exp -iW_\tau(n)\tau_l, \tag{16}$$

where τ_l is the value of grid of discreet transform, $k = 0, \ldots, N-1$ are numbers of sample.

The Fourier transform by coefficients (16) is given by

$$S(l) = S(\tau_l) = \sum_{n=0}^{N-1} s(n)C_l(n) = \sum_{n=0}^{N-1} s(n)e^{-iW_\tau(n)\tau_l}, \tag{17}$$

where $S(l)$ is the specter by τ_l grid. It is obvious that the maximum of (17) corresponds to delay.

It can be supposed that the analogue of the Nyquist theorem for (17) can be expressed as

$$\tau_{N-1} \leq 2\tau_{max}, \tag{18}$$

where τ_{N-1} is the maximum value of the grid τ_l ; τ_{max} is the maximum allowable delay. Condition (18) is equal to the restriction of the phase in range 2π. Thus,

$$\tau_{N-1} \leq 1/(2f_0). \tag{19}$$

The width of peak in transform (17) is proportional to $1/\Delta f$. As it shown in [6], the expression (10) can be rewrite as follows

$$s(n) = s_2(n)s_1^*(n) = \sum_{k=1}^{p} A_0 \exp\left(i[W_\tau(n)\tau_k + \nu_{Ik}(n)]\right) = \sum_{k=1}^{p} |s_k(n)|e^{i\arg s_k(n)}, \tag{20}$$

where $\nu_{Ik}(n)$ are the phase noises $s_k(n)$. The maximum of transform (17) for signal (20) corresponds to the null of its derivative

$$\frac{\partial S(\tau_l)}{\partial \tau_l}|_{\tau_l=\tau_k} = Im[\sum_{n=0}^{N-1} W_\tau(n) \sum_{k=0}^{p} |s_k(n)|e^{(i\arg s_k(n)}e^{-iW_\tau(n)\tau_l}] = 0, \tag{21}$$

where $Im[]$ is the imaginary part of complex expression.

Follow the approach proposed in [10], the multiplication of exponents in (21) at point $\tau_l = \tau_k$ can be transformed in the Taylor series with restriction to its first term as $\exp x_{x \to 0} \approx 1 + x$. Thus, a solution of (21) can be expressed as

$$\sum_{n=0}^{N-1} W_\tau(n) \sum_{k=0}^{p} |s_k(n)| \arg s_k(n) = \sum_{n=0}^{N-1} W_\tau^2(n) \sum_{k=0}^{p} |s_k(n)| \tau_k, \qquad (22)$$

Analysis of equation (22) for $p = 1$ gives the following solution

$$\Delta\hat{\tau} = \frac{\sum_{n=0}^{N-1} W_\tau |s(n)| \arg s(n)}{\sum_{n=0}^{N-1} W_\tau^2 |s(n)|}, \qquad (23)$$

where $\Delta\hat{\tau}$ is the estimated delay difference.

In general case, estimator (23) can be expressed as

$$\Delta\hat{\tau} = [W_\tau^T A W_\tau]^{-1} W_\tau^T A \Psi, \qquad (24)$$

where $W_\tau = [W_\tau(0), \ldots, W_\tau(N-1)]^T$ is the coefficient vector; A is the vector of amplitudes of the sample; Ψ is the vector of arguments of the sample; $\Psi = [\arg(s(0)), \ldots, arg(s(N-1))]^T$; $A = diag[|s(0)|, \ldots, |s(N-1)|]$.

In a special case when $|s(n)| = const$; A =const$\cdot diag[1, \ldots, 1]$ the estimator (24) is given as

$$\Delta\hat{\tau} = [W_\tau^T W_\tau]^{-1} W_\tau^T \Psi. \qquad (25)$$

4 Investigation of the estimator properties

The proposed estimator (24) corresponds to expression of the Gauss-Markov theorem, thus this estimator is a linear, unbiased and asymptotically effective [11]. Expression of the variance of the estimator (24) is given as [11]

$$var[\tau] = \sigma^2 [W_\tau^T A W_\tau]^{-1}. \qquad (26)$$

In case of (25), the variance coincides with CRLB (12) for W_τ (9).

Figure 1 shows the relation of standard deviation (STD) in logarithmic scale to SNR for proposed estimator (prop.τ) and compared ones (Tretφ (3); FKφ (4) and MaxCorr (3)) . Also in Fig. 1 the results are shown for the maximum likelihood (ML) estimator and CRLB (12); (13) and (14). The ML was performed as the procedure of calculation of frequency difference which corresponds to the maximum of spectrums. The spectrum calculation was carried out by the Fast Fourier transform (FFT) with zero padding to 2^{22} sample size.

The values that are shown in Fig. 1 were calculated for signals (7) with delays 1 and 1.0001. The beat signals was simulated for the follow configuration of FMCW: initial frequency 100; frequency deviation 50; period of modulation 0.05; sample frequency 10000 (samples size 500 points). All values above are given in relative units. It should be noted that the selected configuration does not interfere with the generality of carried investigation.

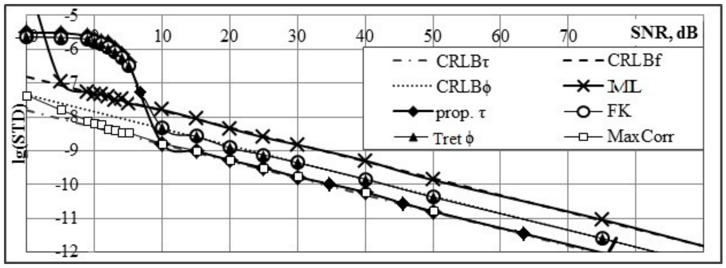

Fig. 1. Relation of STD in log-scale to SNR for proposed estimator and compared ones

The obtained results (see Fig. 1) shows that the proposed estimator has the threshold value of STD in range 6 - 10 dB, which is coincides with the theoretical estimations [12]. The results for other estimators also match with the theoretical suppositions. The values of STD for (24) attain CRLB, which confirms the supposition of it asymptotic effectiveness.

Investigation of the influence of a parasitic signal in the analyzed sample on the bias of estimation was performed for proposed estimators (24) and (25) and compared ones. The following model of superposition of fundamental and parasitic beat tones was used:

$$s_1(n) = A_1 \exp\left(i[W_\tau(n)\tau_1]\right) + A_2 \exp\left(i[W_\tau(n)(\tau_1 + \tau_{21}]\right), \quad (27)$$

$$s_2(n) = A_1 \exp\left(i[W_\tau(n)(\tau_1 + \Delta\tau_1]\right) + A_2 \exp\left(i[W_\tau(n)(\tau_1 + \Delta\tau_1 + \tau_{21} + \Delta\tau_{21})]\right),$$

where $\tau_{21} = \tau_2 - \tau_1$; $\Delta\tau_{21} = \Delta\tau_2 - \Delta\tau_1$; τ_1, τ_2 are the delays for the first fundamental and parasitic signals respectively (for $s_1(n)$); $\Delta\tau_1, \Delta\tau_2$ are the delay differences for second and first fundamental and parasitic signals respectively (for $s_2(n)$).

In Figure 2 relations are shown for the relative bias δ in % to the difference of delays (τ_{21}) normalized on the τ_1. Obtained values were calculated for the proposed estimators prop. τ_1 (24) and prop. τ_{1a} (25) and compared ones for $A_2 = 0.1A_1$ and $\Delta\tau_{12} = 0.00015$. The conditions of numerical experiment are similar to those ones that was used for obtaining Fig. 1. The Figure 3 is shown the relations for the most accurate (unbiased) estimators from fig. 2 for δ in range of $\pm 0.2\%$.

The Analysis of the performed results (see Fig. 2) show the advantages of the proposed estimator prop.τ_a (25). It is also should be noted that ML method has the biggest bias with compared ones. The results of estimators: prop.τ, FK and MaxCorr have similar bias.

The real part of the considered model of signals (27) can be transformed in the following manner:

$$s = A_1 \cos\left(\varphi_1(t)\right) + A_2 \cos\left(\varphi_2(t)\right) = A \cos\left(\varphi_1(t) + b\right), \quad (28)$$

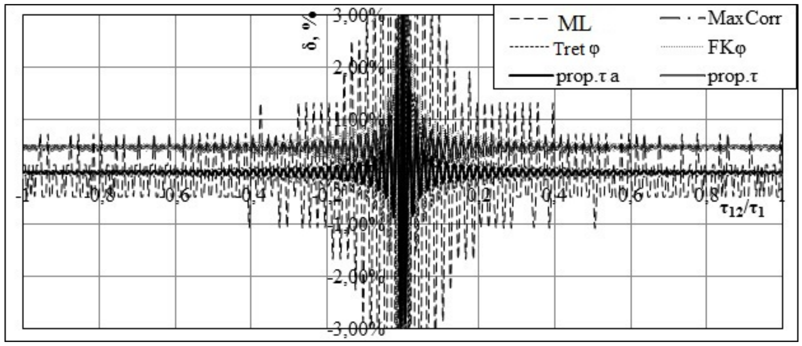

Fig. 2. The relation of bias δ in % to delay difference τ_{12} normalized on the τ_1 for investigated estimators

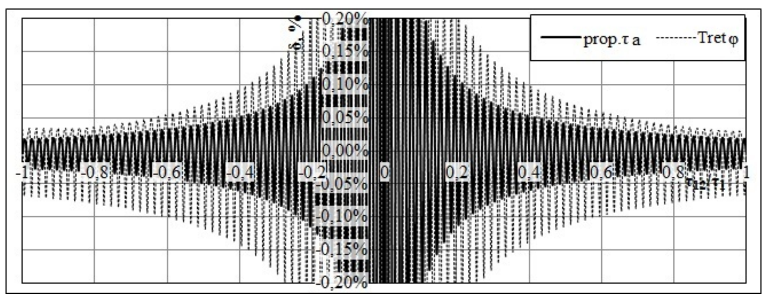

Fig. 3. The relation of bias δ in % to delay difference τ_{12} normalized on the τ_1 for investigated estimators

where $A = \sqrt{(A_1 + A_2 \cos \varDelta\varphi(t))^2 + A_2^2 \sin^2 \varDelta\varphi(t)}$ and $b = \arcsin(A_2 \sin \varDelta\varphi(t)/A)$.

If $A_1 \gg A_2$ and b is sufficiently small than it can be made the assumption that $\sin b \approx b$ then $b \approx A_2 \sin \varDelta\varphi(t)/A_1$. In this case, the analytic form of the (28) is given as

$$s = A_1 \exp(i[\varphi_1(t) + \frac{A_2}{A_1} \sin \varDelta\varphi(t)]). \tag{29}$$

Using (29) and (27) in (9) leads to the following expression for $s(n)$:

$$s = s_2(n)s_1(n)^* = |s(n)| \exp(i[W_\tau(n)\varDelta\tau_1 + \frac{A_2}{A_1}\varDelta_s]). \tag{30}$$

where $\varDelta_s = \sin(\tau_{21} + \varDelta\tau_{21})W_\tau(n) - \sin \tau_{21} W_\tau(n)$. After simple transforms $\varDelta_s$ can be expressed as

$$\varDelta_s = 2 \sin(\varDelta\tau_{21} W_\tau(n)/2) \cos(\varDelta\tau_{21} W_\tau(n)/2 + \tau_{21} W_\tau(n)). \tag{31}$$

It can be supposed that for the most part of applications $\Delta\tau_{21} \ll 1$ and $\Delta\tau_{21}W_\tau(n) \ll 1$, and, hence,

$$\Delta_s = \Delta\tau_{21}W_\tau(n)\cos\left(\Delta\tau_{21}W_\tau(n)/2 + \tau_{21}W_\tau(n)\right). \tag{32}$$

Thus, estimated $\delta = \tau - \hat{\tau}$ has the following expression

$$\delta = \frac{A_2}{A_1}\frac{\sum_{n=0}^{N-1} W_\tau|s(n)|\Delta_s}{\sum_{n=0}^{N-1} W_\tau^2|s(n)|} \approx \frac{A_2}{A_1}\Delta\tau_{21}\frac{\sum_{n=0}^{N-1} W_\tau^2(n)\cos\left(\Delta\tau_{21}W_\tau/2 + \tau_{21}W_\tau\right)}{\sum_{n=0}^{N-1} W_\tau^2}. \tag{33}$$

The sum in the numerator of (33) can be replaced by the corresponding integral which solution is propositional to the $N/(\tau_{21}^3\Delta f)$. In the case when $\tau_{12} \gg \Delta\tau_{12}$, the investigation of (33) leads to the follow relation

$$\delta \sim \frac{A_2}{A_1}\Delta\tau_{12}\frac{\sin\tau_{12}}{\tau_{12}^2} \sim \frac{1}{\tau_{12}^2};\ \lim_{N\to\infty}\delta \sim 1/\Delta f^2 f_0. \tag{34}$$

The results of the bias relation is the frequency deviation that is normalized by the initial frequency are shown in the Fig. 4. The presented results are obtained for δ_Σ (33), δ_{int} the integral replacement (see 34) and calculated biases for prop τ_a and prop τ. The relation $\Delta f/f_0$ changes in range [0-1]. The beat signals configuration is the same as used for 2 obtaining for $\tau_1 = 1$; $\tau_{21} = 0.1$; $\Delta\tau_1 = 0.0001$ and $\Delta\tau_{21} = 0.00015$. All values are given in relative units. The obtained results confirm ones presented in the Fig. 2 and assumptions (34).

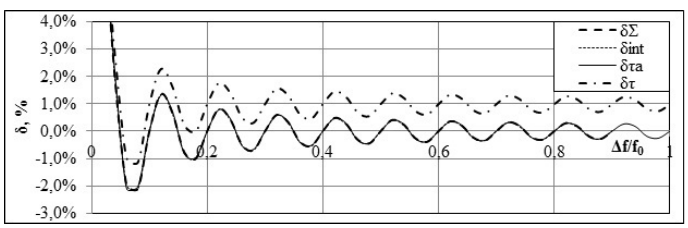

Fig. 4. The relation of bias δ in % to frequency deviation normalized on the initial frequency for investigated expressions and proposed estimators.

5 Discussion and conclusion

The paper propose a new approach for time delay difference estimation of beat signals, which are formed by heterodyne scheme of FMCW signals processing. The method is based on the idea of using delay as a parameter of received beat tones. The Cramer-Rao low bound analysis for this case shows the advantages compared to the traditional parameters: frequency and initial phase. For instance, if $f_0 = 2\Delta f$, then the variance of considered method is by 19 times smaller then for phase and in 75 times smaller then for frequency as parameters.

Based on the proposed method, a new estimator has been designed in class of the smooth parabolic finite difference estimators. It was shown that the considered estimator is a linear, consistent, asymptotically effective, and, also corresponds to the Gauss-Markov theorem. However, the proposed approach has restrictions, which are connected with cyclic nature of phase. The SNR threshold is about 6 dB.

Advantages of the developed estimator are shown in comparison with other techniques in such class in the influence of the parasitic signals on the bias of obtained value. The analysis of bias expression shows its dependence on FMCW signals configuration and on time delay difference between the fundamental and parasitic beat tones. The increasing of the last parameter value leads to the bias reduction as square of its value.

Results of the carried investigation shows the advantages of the proposed approach, particulary, of estimator prop. τ_a for the tasks of the time delay difference measurement of beat signals in comparison with the considered traditional techniques in the areas of the variance and influence of the parasitic signals on the bias.

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