On Computer-Oriented Algorithms Solving Guaranteed Control Problems under Uncertainty for Stochastic Differential Equations

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Abstract. Two problems of guaranteed closed-loop control under incomplete information are considered for a linear stochastic differential equation (SDE) from the viewpoint of the constructive method of open-loop control packages worked out and realized as a software earlier for the guidance of a linear control system of ordinary differential equations (ODEs) to a convex target set. The problems consist in designing a deterministic control providing (irrespective of a realized initial state from a given finite set) prescribed properties of the solution (being a random process) by a terminal point in time (A) or at this time (B). It is assumed that a linear signal on some number of realizations is observed. By the equations of the method of moments, the problems for the SDE are reduced to equivalent problems for systems of ODEs describing the mathematical expectation and covariance matrix of the original process. The emphasis is on designing computer-oriented solving algorithms based on feasible finite-dimensional optimization procedures. The solvability conditions for the problems are written. An illustrative example is presented.

Keywords: guidance problem, computer-oriented algorithm, linear stochastic differential equation

1 Introduction

In mathematical control theory and its applications, the problem of constructing optimal strategies of guaranteed feedback control under conditions of uncertainty is evidently actual. We follow the theory of closed-loop control developed by N.N. Krasovskii's school [1] and apply the approach based on the so-called method of open-loop control packages originating from the technique of nonanticipating strategies [2] to solving the guidance problem for a linear SDE. The method tested on the guidance problems for linear controlled systems of ODEs consists in reducing the problems of guaranteed control formulated in the class of closed-loop strategies to equivalent problems in the class of open-loop control packages. The latter class contains the families of open-loop controls parameterized by admissible initial states and possessing the property of nonanticipation with respect to the dynamics of observations, see [3], [4], and [5].

This paper is devoted to the study of the problem of guiding (with a probability close to 1) a trajectory of a linear SDE to some target set. The statements mean that we should form a deterministic control providing (irrespective of the realized initial state from a specified finite set) prescribed properties of the solution (being a random process) by a terminal point in time (A) or at this time (B). Here, we observe a linear signal on some number of realizations. Similar problems arise in practical situations, when it is possible to observe the behavior of a large number of identical objects described by a stochastic dynamics. By the equations of the method of moments [6], the problems for the SDE are reduced to equivalent problems for systems of ODEs describing the mathematical expectation and covariance matrix of the original process. The technique of the method of open-loop control packages developed in [3], [4], and [5] is applied to the systems obtained. A similar reduction procedure was used, for example, in [7] for solving the problem of dynamic reconstruction of an unknown disturbance characterizing the level of random noise in a linear SDE.

2 Statement of the problems

Consider a system of linear SDEs of the following form:

$$dx(t,\omega) = (A(t)x(t,\omega) + B_1(t)u_1(t) + f(t))\,dt + B_2(t)U_2(t)\,d\xi(t,\omega). \qquad (1)$$

Here, $x(t_0,\omega) = x_0$, $t \in T = [t_0,\vartheta]$, $x = (x_1,x_2,\dots,x_n) \in \mathbb{R}^n$, $\xi = (\xi_1,\xi_2,\dots,\xi_k) \in \mathbb{R}^k$; $\omega \in \Omega$, (Ω,F,P) is a probability space; $\xi(t,\omega)$ is a standard Wiener process (i.e., a process starting from zero with zero mathematical expectation and covariance matrix equal to It, I is the unit matrix from $\mathbb{R}^{k\times k}$); $f(t)$ is a continuous vector function with values in $\mathbb{R}^n$; $A(t) = \{a_{ij}(t)\}$, $B_1(t) = \{b_{1ij}(t)\}$, and $B_2(t) = \{b_{2ij}(t)\}$ are continuous matrix functions of dimensions $n\times n$, $n\times r$, and $n\times k$, respectively.

Two controls act in the system: a vector $u_1(t) = (u_{11}(t), u_{12}(t),\dots,u_{1r}(t)) \in \mathbb{R}^r$ and a diagonal matrix $U_2(t) = \{u_{21}(t), u_{22}(t),\dots,u_{2k}(t)\} \in \mathbb{R}^{k\times k}$, which are Lebesgue measurable on T and take values from specified instantaneous control resources S_{u1} and S_{u2} being convex compact sets in the corresponding spaces. The control u_1 enters the deterministic component and influences the mathematical expectation of the desired process. Since $U_2 d\xi = (u_{21}d\xi_1, u_{22}d\xi_2,\dots,u_{2k}d\xi_k)$, we can assume that the vector $u_2 = (u_{21},u_{22},\dots,u_{2k})$ characterizes the diffusion of the process (the amplitude of random noises).

The initial state x_0 belongs to a finite set of admissible initial states X_0, which consists of normally distributed random variables with numerical parameters (m_0,D_0), where $m_0 = Mx_0$ is the mathematical expectation, $m_0 \in \mathcal{M}_0 = \{m_0^1, m_0^2,\dots,m_0^{n_1}\}$, $D_0 = M(x_0-m_0)(x_0-m_0)^*$ is the covariance matrix (the asterisk means transposition), $D_0 \in \mathcal{D}_0 = \{D_0^1, D_0^2,\dots,D_0^{n_2}\}$. Thus, the set X_0 contains n_1n_2 elements. We assume that the system's initial state belongs to X_0 but is unknown.

Equation (1) is a symbolic notation for the integral identity (ω is omitted)

$$x(t) = x_0 + \int_{t_0}^{t} (A(s)x(s) + B_1(s)u_1(s) + f(s))\, ds + \int_{t_0}^{t} B_2(s)U_2(s)\, d\xi(s). \quad (2)$$

The latter integral on the right-hand side of equality (2) is stochastic and is understood in the sense of Ito. A solution of equation (1) is defined as a stochastic process satisfying integral identity (2) for any t with probability 1. Under the assumptions above, there exists a unique solution, which is a normal Markov process with continuous realizations [8].

The problems in question consist in the following. Let a nonempty finite set $S_t \subset T$ of admissible guidance times and nonempty convex closed target sets $\mathcal{M}(t) \in \mathbb{R}^n$ and $\mathcal{D}(t) \in \mathbb{R}^{n\times n}$ for any moment $t \in S_t$ as well as a continuous matrix observation function $Q(t)$ of dimension $q \times n$ be given. At any time, it is possible to receive the information on some number N of realizations of the stochastic process $x(t)$. The following signal is available:

$$y(t) = Q(t)x(t). \quad (3)$$

Assume that, for a finite set of some specified times $\tau_i \in T$, $i \in [1 : l]$, we can construct, using N realizations of the process $x(t)$, a statistical estimate m_i^N of the mathematical expectation $m(\tau_i)$ and a statistical estimate D_i^N of the covariance matrix $D(\tau_i)$ such that

$$P\Big(\max_{i\in[1:l]} \left\{ \left\| m_i^N - m(\tau_i) \right\|_{\mathbb{R}^n}, \left\| D_i^N - D(\tau_i) \right\|_{\mathbb{R}^{n\times n}} \right\} \le h(N) \Big) = 1 - g(N), \quad (4)$$

where $h(N)$ and $g(N) \to 0$ as $N \to \infty$. Standard procedures of obtaining the estimates m_i^N and D_i^N admit modifications providing the validity of relation (4) and the specified convergences.

The problems of guaranteed closed-loop ε-guidance consist in forming controls $(u_1(\cdot), u_2(\cdot))$ guaranteeing, whatever the initial state x_0 from the set X_0, prescribed properties of the process x by or at the terminal time ϑ. Here, we mean that, for an arbitrary small (in advance specified) $\varepsilon > 0$, the mathematical expectation $m(\vartheta)$ and the covariance matrix $D(\vartheta)$ reach at some admissible guidance time $t \in S_t$ the ε-neighborhoods of the target sets $\mathcal{M}(t)$ and $\mathcal{D}(t)$, respectively. This is Problem A1. If $S_t = \{\vartheta\}$, we have Problem B1. In the motion process, the sought controls are formed using the information on N realizations of the signal (3). By virtue of estimate (4), it is reasonable to require that the probability of the desired event should be close to 1 for sufficiently large N and algorithm's parameters concordant with N in a special way. For ODEs, Problems A1 and B1 were considered in detail in [4] and [5], respectively.

3 Reduction of the original problems

Let us reduce the guidance problems for the SDE to problems for systems of ODEs. By virtue of the linearity of the original system, the mathematical expectation $m(t)$ depends only on $u_1(t)$; its dynamics is described by the equation

$$\dot{m}(t) = A(t)m(t)+B_1(t)u_1(t)+f(t), \quad t \in T = [t_0, \vartheta], \quad m(t_0) = m_0 \in \mathcal{M}_0. \quad (5)$$

We assume that N ($N > 1$) trajectories $x^r(t)$, $r \in [1 : N]$, of the original SDE are measured; then, we know values of signal (3), i.e., $y^r(t) = Q(t)x^r(t)$.

The signal on the trajectory of equation (5) is denoted by $y_m(t) = Q_m(t)m(t)$; its estimate formed by the information on y^r, $r \in [1 : N]$, by $y_m^N(t)$:

$$y_m^N(t) = \frac{1}{N}\sum_{r=1}^{N} y^r(t) = Q(t)m^N(t), \quad m^N(t) = \frac{1}{N}\sum_{r=1}^{N} x^r(t). \quad (6)$$

Obviously, $Q_m(t) = Q(t)$ and, for the finite set of times $\tau_i \in T$, $i \in [1 : l]$, in view of relation (4), it holds that

$$P\big(\forall i \in [1 : l] \ \ \big\|y_m^N(\tau_i) - y_m(\tau_i)\big\|_{\mathbb{R}^q} \le C_1 h(N)\big) = 1 - g(N). \quad (7)$$

Here and below, constants C_i can be written explicitly.

The covariance matrix $D(t)$ depends only on $U_2(t)$; its dynamics is described by the so-called equation of the method of moments [6] in the following form:

$$\dot{D}(t) = A(t)D(t) + D(t)A^*(t) + B_2(t)U_2(t)U_2^*(t)B_2^*(t), \quad t \in T = [t_0, \vartheta],$$
$$D(t_0) = D_0 \in \mathcal{D}_0. \quad (8)$$

For our purposes, matrix equation (8) is conveniently rewritten in the form of a vector equation, which is more traditional for such problems. By virtue of the symmetry of the matrix $D(t)$, its dimension is defined as $n_d = (n^2 + n)/2$. Let us introduce the vector $d(t) = \{d_s(t)\}$, $s \in [1 : n_d]$, consisting of successively written and enumerated elements of the matrix $D(t)$, taken line by line starting with the element located at the main diagonal. Performing standard matrix operations over $A(t)$ and $B_2(t)$, we form continuous matrices $\bar{A}(t): T \to \mathbb{R}^{n_d \times n_d}$ and $\bar{B}(t): T \to \mathbb{R}^{n_d \times k}$ and use them to rewrite system (8) in the form

$$\dot{d}(t) = \bar{A}(t)d(t) + \bar{B}(t)v(t), \quad t \in T = [t_0, \vartheta], \quad d(t_0) = d_0 \in \mathcal{D}_0. \quad (9)$$

The initial state d_0 is obtained from D_0; the notation for the set $\mathcal{D}_0$ is the same. The multiplication of the diagonal matrices $U_2(t)U_2^*(t)$ results in the appearance of the control vector $v(t) = (u_{21}^2(t), u_{22}^2(t), \ldots, u_{2k}^2(t))$ whose elements take values from some convex compact set $S_v \in \mathbb{R}^k$ for all $t \in T$.

The signal on the trajectory of equation (9) is denoted by $y_d(t) = Q_d(t)d(t)$; its estimate formed by the information on y^r, $r \in [1 : N]$, by $y_d^N(t)$. The latter is constructed as follows:

$$\frac{1}{N-1}\sum_{r=1}^{N}(y^r(t) - y_m^N(t))(y^r(t) - y_m^N(t))^* = Q(t)\frac{1}{N-1}$$
$$\sum_{r=1}^{N}(x^r(t) - m^N(t))(x^r(t) - m^N(t))^* Q^*(t) = Q(t)D^N(t)Q^*(t), \quad (10)$$

where $D^N(t) = \{d_{ij}^N(t)\},\ i, j \in [1 : n]$ is the standard estimate of the covariance matrix $D(t)$ for an unknown (estimated by $m^N(t)$) mathematical expectation $m(t)$. By means of algebraic transformations using the symmetry of matrix (10), the expression $Q(t)D^N(t)Q^*(t)$ is transformed into $y_d^N(t) = Q_d(t)d^N(t)$, where $Q_d(t)$ is a continuous matrix of dimension $n_q \times n_d$, $n_q = (q^2 + q)/2$, and $d^N(t)$ is the vector of dimension n_d extracted from $D^N(t)$. Obviously, for the finite set of times $\tau_i \in T$, $i \in [1 : l]$, we have the relation of type (4)

$$P\big(\forall i \in [1 : l]\ \ \big\|y_d^N(\tau_i) - y_d(\tau_i)\big\|_{\mathbb{R}^{n_q}} \le C_2 h(N)\big) = 1 - g(N). \qquad (11)$$

Original problems of guaranteed closed-loop ε-guidance for the SDE can be reformulated as follows. For an arbitrary small (in advance specified) $\varepsilon > 0$, it is required to choose controls $u_1(\cdot)$ in equation (5) and $v(\cdot)$ in equation (9) such that, whatever the initial states $m_0 \in \mathcal{M}_0$ and $d_0 \in \mathcal{D}_0$, the trajectories of (5) and (9) reach at some admissible guidance time $t \in S_t$ the ε-neighborhoods of the target sets $\mathcal{M}(t)$ and $\mathcal{D}(t)$, respectively (Problem A2). If $S_t = \{\vartheta\}$, we have Problem B2. It is important that the probability of the desired event should be close to 1. The required controls are formed through the estimates of the signals y_m and y_d satisfying relations (7) and (11); actually, these controls define the control in SDE (1). The dependence of the number N of measurable trajectories on the value ε is given below. The next theorem follows from the aforesaid.

Theorem 1. *Problems A*1 *(B*1*) and A*2 *(B*2*) are equivalent.*

Thus, to solve the original problems, one should establish some conditions of consistent solvability of the problems of ε-guidance for ODEs (5) and (9) and should find the form of concordance of parameters N and ε as well.

4 The method of open-loop control packages: a brief review of results for ODE

Let us present briefly the approach by A.V. Kryazhimskii and Yu.S. Osipov to solving the problem of closed-loop guidance for a linear ODE [3], [5].

Consider a dynamical control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \quad t \in T = [t_0, \vartheta], \quad x(t_0) = x_0 \in X_0,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in P \subset \mathbb{R}^m$ (P is a convex compact set); $A(\cdot)$, $B(\cdot)$, and $f(\cdot)$ are continuous matrix functions of dimensions $n \times n$, $n \times m$, and $n \times 1$, respectively; X_0 is a finite set of possible initial states. The real initial state of the system is assumed to be unknown. A nonempty finite set $S_t \subset T$ of admissible guidance times and nonempty convex closed target sets $\mathcal{M}(t) \in \mathbb{R}^n$, $t \in S_t$, as well as a continuous observation function $Q(t)$ of dimension $q \times n$ are given.

The problem of guaranteed closed-loop ε-guidance consists in forming by the signal $y(t) = Q(t)x(t)$ a control guaranteeing that the system's state $x(t)$ reaches at some admissible guidance time $t \in S_t$ the ε-neighborhood of the target set $\mathcal{M}(t)$. This is Problem A. If $S_t = \{\vartheta\}$, we have Problem B. The solution of the problem is sought in the class of closed-loop control strategies with memory. The correction of the values of a control $u(\cdot)$ is possible at in advance specified times. In [3], the equivalence of the formulated problem of closed-loop control to the so-called problem of package guidance is established. Let us briefly present basic notions of the latter problem. Consider the homogeneous system $\dot{x}(t) = A(t)x(t)$, $t \in T = [t_0, \vartheta]$, $x(t_0) = x_0 \in X_0$; its fundamental matrix is denoted by $F(\cdot, \cdot)$. For any $x_0 \in X_0$, the *homogeneous signal* corresponding to x_0 is the function $g_{x_0}(t) = Q(t)F(t, t_0)x_0$, $t \in [t_0, \vartheta]$. The set of all admissible initial states x_0 corresponding to a homogeneous signal $g(\cdot)$ till a time τ is denoted by $X_0(\tau|g(\cdot)) = \{x_0 \in X_0 \colon g_{x_0}(\cdot)|_{[t_0,\tau]} = g(\cdot)|_{[t_0,\tau]}\}$, where $g(\cdot)|_{[t_0,\tau]}$ is the restriction of the homogeneous signal $g(\cdot)$ onto the interval $[t_0, \tau]$.

A family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ of open-loop controls is called an *open-loop control package* if it satisfies the condition of nonanticipation: for any homogeneous signal $g(\cdot)$, time $\tau \in (t_0, \vartheta]$, and admissible initial states x_0', $x_0'' \in X_0(\tau|g(\cdot))$, the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for all $t \in [t_0, \tau]$. Any family $s_t = (s_{x_0})_{x_0 \in X_0}$ of elements of S_t is called a *family of admissible guidance times*. An open-loop control package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is *guiding with a family of admissible guidance times* s_t if for any $x_0 \in X_0$, the motion from x_0 corresponding to $u_{x_0}(\cdot)$ takes a value exactly in the target set $\mathcal{M}(s_{x_0})$. If there exists an open-loop control package that is guiding with a family of admissible guidance times s_t, we say that the idealized problem of package guidance corresponding to the original problem of guaranteed closed-loop control is solvable with the family of admissible guidance times s_t. These constructions suit for both Problems A and B.

Let G be the set of all homogeneous signals. We introduce the set $G_0(g(\cdot))$ of all homogeneous signals coinciding with $g(\cdot)$ in a right-sided neighborhood of the initial time t_0. The first splitting moment of the homogeneous signal $g(\cdot)$ is the time $\tau_1(g(\cdot)) = \max\Big\{\tau \in [t_0, \vartheta] \colon \max\limits_{g'(\cdot) \in G_0(g(\cdot))} \max\limits_{t \in [t_0,\tau]} \|g'(t) - g(t)\|_{\mathbb{R}^q} = 0\Big\}$. If $\tau_1(g(\cdot)) < \vartheta$, then, by analogy with $G_0(g(\cdot))$, we introduce the set $G_1(g(\cdot))$ of all homogeneous signals from $G_0(g(\cdot))$ coinciding with $g(\cdot)$ in a right-sided neighborhood of the splitting moment $\tau_1(g(\cdot))$. By analogy with $\tau_1(g(\cdot))$, we define the second splitting moment of the homogeneous signal $g(\cdot)$ and so on. Finally, we introduce the set of all the splitting moments of the homogeneous signal $g(\cdot)$: $T(g(\cdot)) = \{\tau_j(g(\cdot)) \colon j = 1, \dots, k_g\}$, $k_g \geq 1$, $\tau_{k_g}(g(\cdot)) = \vartheta$. Then, we consider the set (in ascending order) of all the splitting moments of all the homogeneous signals (possible switching moments for the "ideal" guiding open-loop control): $T = \bigcup_{g(\cdot) \in G} T(g(\cdot))$, $T = \{\tau_1, \dots, \tau_K\}$, $K \leq \sum_{g(\cdot) \in G} k_{g(\cdot)}$ is the number of elements of the set T. Obviously, the sets $T(g(\cdot))$ and T are finite due to the finiteness of the sets X_0 and G. For any $k = 1, \dots, K$, the set $\mathcal{X}_0(\tau_k) = \{X_0(\tau_k|g(\cdot)) \colon g(\cdot) \in G\}$ is called the cluster position at the time τ_k; each of its elements X_{0k} is called the cluster of initial states at this moment.

The constructions above were used for designing rather cumbersome criteria for the solvability of the original problems (A and B) mathematically based on solving finite-dimensional optimization problems; see [4] and [5] for details.

5 Properties of the statistical estimates

Lemma 1. *For a finite set of some specified times* $\tau_i \in T$, $i \in [1:l]$, *the standard estimates* m_i^N *of the mathematical expectation* $m(\tau_i)$ *and* D_i^N *of the covariance matrix* $D(\tau_i)$ *constructed through* N $(N>1)$ *realizations* $x^1(\tau_i), \ldots, x^N(\tau_i)$ *of the random variables* $x(\tau_i)$ *by the following rules* [9]:

$$m_i^N = \frac{1}{N}\sum_{r=1}^{N} x^r(\tau_i), \quad D_i^N = \frac{1}{N-1}\sum_{r=1}^{N}(x^r(\tau_i) - m_i^N)(x^r(\tau_i) - m_i^N)^*, \quad (12)$$

provide the validness of relation (4) (*consequently,* (7) *and* (11)).

The proof of the lemma is presented in [10]. Here, we restrict ourselves by the citation that it is possible to choose the same parameters in relations (4), (7), and (11), namely,

$$h(N) = C_h N^{\gamma - 1/2}, \quad g(N) = C_g N^{\max\{-1, -1/2 - 3\gamma\}}, \quad (13)$$

where $0 < \gamma < 1/2$, C_h and C_g are constants. For example, if $\gamma \to +0$, $h(N)$ and $g(N)$ have the power exponents of the value $1/N$ asymptotically equal to 1/2.

6 Criteria for the solvability of the problems

Let us define additional notions for ODEs (5) and (9). Let $G^1 = \{g^1(\cdot)\}$ and $G^2 = \{g^2(\cdot)\}$ be the sets of all homogeneous signals for (5) and (9), respectively. The sets of all splitting moments of all homogeneous signals for (5) and (9) are denoted by $T^1 = \{\tau_1^1, \ldots, \tau_{K_1}^1\}$ and $T^2 = \{\tau_1^2, \ldots, \tau_{K_2}^2\}$; the cluster positions and clusters of initial states at the times τ_k^1 and τ_k^2, by $\mathcal{M}_0(\tau_k^1)$ and M_{0k}, $\mathcal{D}_0(\tau_k^2)$ and D_{0k}. Recall that $\tau_{K_1}^1 = \tau_{K_2}^2 = \vartheta$ and assume that $\tau_0^1 = \tau_0^2 = t_0$. Let us introduce the sets of pairs of homogeneous signals splitted at the moments τ_k^1, $k \in [0 : K_1 - 1]$ and τ_k^2, $k \in [0 : K_2 - 1]$: $G_k^{1*} = \{(g_i^1(\tau_k^1), g_j^1(\tau_k^1))\}$, $G_k^{2*} = \{(g_i^2(\tau_k^2), g_j^2(\tau_k^2))\}$, $i \neq j$. A moment from the interval $(\tau_k^1, \tau_k^1 + C\varepsilon]$ $(\tau_k^1 + C\varepsilon < \tau_{k+1}^1$, C is a constant), at which all the pairs from G_k^{1*} are distinguishable, is denoted by τ_k^{1*} and is called a distinguishing moment for all the signals splitted at the time τ_k^1. Similarly, we define a distinguishing moment τ_k^{2*}. For all $\tau_k^1 \in T^1$, $k \in [0 : K_1 - 1]$ and $\tau_k^2 \in T^2$, $k \in [0 : K_2 - 1]$, the corresponding moments τ_k^{1*} and τ_k^{2*} are defined uniquely; at these moments, the signal's values differ in all the pairs from G_k^{1*} and G_k^{2*} [10]. The set of all such moments for (5) and (9)

$$T^* = T^{1*} \cup T^{2*}, \quad T^{1*} = \{\tau_0^{1*}, \ldots, \tau_{K_1-1}^{1*}\}, \quad T^{2*} = \{\tau_0^{2*}, \ldots, \tau_{K_2-1}^{2*}\}, \quad (14)$$

determines both the aforesaid set of l $(l < K_1 + K_2)$ times, at which the N trajectories of the original process are measured, and the set of times, which are possible for switching the closed-loop control. As an example, we formulate the solvability conditions for problem B2 [5]:

$$\sup_{(l_{m_0})_{m_0 \in \mathcal{M}_0} \in \mathcal{S}_1} \gamma_1((l_{m_0})_{m_0 \in \mathcal{M}_0}) \leq 0, \quad \gamma_1((l_{m_0})_{m_0 \in \mathcal{M}_0}) = \sum_{m_0 \in \mathcal{M}_0} \langle l_{m_0}, F_1(\vartheta, t_0) m_0 \rangle$$

$$+\sum_{k=1}^{K_1}\int\limits_{\tau_{k-1}^1}^{\tau_k^1}\sum_{M_{0k}\in\mathcal{M}_0(\tau_k^1)}\rho^-\Big(\sum_{m_0\in M_{0k}}B_1^*(s)F_1^*(\vartheta,s)l_{m_0}\big|S_{u1}\Big)ds$$

$$+\int\limits_{t_0}^{\vartheta}\Big\langle\sum_{m_0\in\mathcal{M}_0}l_{m_0},\,F_1(\vartheta,s)f(s)\Big\rangle\,ds-\sum_{m_0\in\mathcal{M}_0}\rho^+(l_{m_0}|\mathcal{M}(\vartheta)),\tag{15}$$

$$\sup_{(l_{d_0})_{d_0\in\mathcal{D}_0}\in\mathcal{S}_2}\gamma_2((l_{d_0})_{d_0\in\mathcal{D}_0})\le 0,\quad \gamma_2((l_{d_0})_{d_0\in\mathcal{D}_0})=\sum_{d_0\in\mathcal{D}_0}\langle l_{d_0},F_2(\vartheta,t_0)d_0\rangle$$

$$+\sum_{k=1}^{K_2}\int\limits_{\tau_{k-1}^2}^{\tau_k^2}\sum_{D_{0k}\in\mathcal{D}_0(\tau_k^2)}\rho^-\Big(\sum_{d_0\in D_{0k}}\bar{B}^*(s)F_2^*(\vartheta,s)l_{d_0}\big|S_v\Big)\,ds-\sum_{d_0\in\mathcal{D}_0}\rho^+(l_{d_0}|\mathcal{D}(\vartheta)).$$

Here, $(l_{m_0})_{m_0\in\mathcal{M}_0}$ and $(l_{d_0})_{d_0\in\mathcal{D}_0}$ are families of vectors parameterized by the corresponding initial states; $\mathcal{S}_1$ and $\mathcal{S}_2$ are the sets of families $(l_{m_0})_{m_0\in\mathcal{M}_0}$, $\sum\|l_{m_0}\|^2_{\mathbb{R}^n}=1$ and $(l_{d_0})_{d_0\in\mathcal{D}_0}$, $\sum\|l_{d_0}\|^2_{\mathbb{R}^{n_d}}=1$; $F_1(\cdot,\cdot)$ and $F_2(\cdot,\cdot)$ are the fundamental matrices of systems (5) and (9). We present the key result of [10].

Theorem 2. *Let conditions* (15) *be fulfilled, let the information on N trajectories of* SDE (1) *be received at the times composing the set T^** (14)*, let the constants C_h, C_g, and γ be taken from Lemma 1* (*see* (13)) *and*

$$N>(2C_h/\rho(\varepsilon))^{2/(1-2\gamma)}\,,\tag{16}$$

$$\rho(\varepsilon)=\min\Big\{\min_{\tau_k^{1*}\in T^{1*},\,(g_i^1,g_j^1)\in G_k^{1*}}\|g_i^1(\tau_k^{1*})-g_j^1(\tau_k^{1*})\|_{\mathbb{R}^q},$$

$$\min_{\tau_k^{2*}\in T^{2*},\,(g_i^2,g_j^2)\in G_k^{2*}}\|g_i^2(\tau_k^{2*})-g_j^2(\tau_k^{2*})\|_{\mathbb{R}^{n_q}}\Big\}.$$

Then, problem $B1$ is solvable with the probability $1-C_gN^{\max\{-1,-1/2-3\gamma\}}$ and there exists an ε-guiding control in equation (1) based on open-loop control packages for systems (5) and (9).

7 Illustrative example

Consider the linear SDE of the first order:

$$dx(t)=-x(t)dt+u_1(t)dt+u_2(t)d\xi(t),\quad t\in T=[0,2],\quad u_1,u_2\in[0,1],\tag{17}$$

with the unknown initial state $x_0\in X_0$, X_0 consists of four normally distributed random variables with numerical parameters (m_0,d_0), where the mathematical expectation $m_0\in\mathcal{M}_0=\{m_0^1,m_0^2\}$, $m_0^1=(3-e)e$, $m_0^2=e^2$, and the dispersion $d_0\in\mathcal{D}_0=\{d_0^1,d_0^2\}$, $d_0^1=e^2/2$, $d_0^2=e^4$. We use the signal

$$y(t)=Q(t)x(t),\quad Q(t)=\begin{cases}0, & t\in[0,1]\\ t-1, & t\in(1,2]\end{cases}.\tag{18}$$

Let us write ODEs for the mathematical expectation and dispersion, as well as the observed signals, using the formulas from Section 3:

$$\dot{m}(t) = -m(t) + u_1(t), \quad m(0) = m_0 \in \{m_0^1, m_0^2\}, \quad y_m(t) = Q(t)m(t); \quad (19)$$

$$\dot{d}(t) = -2d(t) + u_2^2(t), \quad d(0) = d_0 \in \{d_0^1, d_0^2\}, \quad y_d(t) = Q^2(t)d(t). \quad (20)$$

Let the set of admissible guidance times $S_t = \{3/2, 2\}$ and the target sets $\mathcal{M}(3/2) = [1 + 2/\sqrt{e} - \sqrt{e}, 1]$, $\mathcal{M}(2) = \{1\}$, $\mathcal{D}(3/2) = [1/2, 1]$, $\mathcal{D}(2) = \{1\}$ be given. The control aim consists in forming, for an arbitrary small $\varepsilon > 0$, an open-loop control (u_1, u_2) guaranteeing, whatever the initial states $m_0 \in \mathcal{M}_0$ and $d_0 \in \mathcal{D}_0$, by the information on N trajectories of equation (17), the attainability (with a probability close to 1) of ε-neighborhoods of the target sets $\mathcal{M}(t)$ and $\mathcal{D}(t)$ by the mathematical expectation m and the dispersion d, respectively, at the moment $t = 2$ (Problem B2) and by the moment $t = 2$, i.e., at one of the moments of the set S_t (Problem A2). The example reflects the natural fact that Problem B2 is not solved, whereas Problem A2 is solvable.

The splitting moments of the homogeneous signals for equations (19) and (20) coincide, $K = 2$, $\tau_1 = 1$ (then it is possible to distinguish different homogeneous signals), $\tau_2 = 2$. In the case when the controls u_1 and u_2^2 are piecewise constant functions ($u_{[0,1]}$, $u_{(1,2]}$ and $v_{[0,1]}$, $v_{(1,2]}$, respectively), we obtain $m(2) = e^{-2}m_0 + (1 - e^{-1})(e^{-1}u_{[0,1]} + u_{(1,2]})$, $d(2) = e^{-4}d_0 + (1 - e^{-2})(e^{-2}v_{[0,1]} + v_{(1,2]})/2$. It follows from the form of $m(2)$ that the solution of equation (19), starting from the greater initial state $m_0^2 = e^2$, reaches (at $t = 2$) the set $\mathcal{M}(2)$ only under the action of zero control u_1 on the whole interval $[0, 2]$, i.e., $u_{[0,1]} = u_{(1,2]} = 0$. Note that this boundary cannot be reached before the time $t = 2$; so the set $\mathcal{M}(3/2)$ is not attainable in case of the action of zero control u_1. At the same time, if the real initial state coincides with the smaller possible value $m_0^1 = (3 - e)e$, then, after the necessary action of zero control till the splitting moment $t = 1$, the choice of $u_{(1,2]} = 1$ cannot already force the trajectory to reach the set $\mathcal{M}(2)$. However, it is easily seen that the lower boundary of the set $\mathcal{M}(3/2)$ can be reached at the time $t = 3/2$. A similar argument is applicable to equation (20). The open-loop controls for equations (19), (20) solving Problem B2 are unique.

We pass to constructing the closed-loop control using the open-loop control package. Note that, since the splitting moments for equations (19) and (20) coincide, it is sufficient to perform measurements of N ($N > 1$) trajectories $x^1(\tau_*), \ldots, x^N(\tau_*)$ of the original SDE at the unique distinguishing moment $\tau_* = 1 + \varepsilon$; the zero controls are fed onto equations (19) and (20) till this time. Then, we construct by (12) the estimates $y_m^N(\tau_*)$ and $y_d^N(\tau_*)$ of the signals $y_m(\tau_*)$ and $y_d(\tau_*)$ satisfying the relations like (7), (11), and (13):

$$P\left(\max\left\{\left|y_m^N(\tau_*) - y_m(\tau_*)\right|, \left|y_d^N(\tau_*) - y_d(\tau_*)\right|\right\} \le h(N)\right) = 1 - g(N).$$

Let us derive the condition providing the detection at the time τ_* of the real initial states of equations (19) and (20) (m_0^1 or m_0^2 and d_0^1 or d_0^2) and, consequently, of equation (17). Actually, taking into account that $u_1(t) = 0$, $u_2^2(t) = 0$, $t \in [0, 1 + \varepsilon]$, and the form of Q, we should distinguish the values $y_m^1(\tau_*) =$

$\varepsilon e^{-(1+\varepsilon)} m_0^1$ and $y_m^2(\tau_*) = \varepsilon e^{-(1+\varepsilon)} m_0^2$, as well as $y_d^1(\tau_*) = \varepsilon^2 e^{-2(1+\varepsilon)} d_0^1$ and $y_d^2(\tau_*) = \varepsilon^2 e^{-2(1+\varepsilon)} d_0^2$. Therefore, N must be such that

$$h(N) < \min\left\{ (\varepsilon e^{-(1+\varepsilon)} |m_0^1 - m_0^2|)/2,\ (\varepsilon^2 e^{-2(1+\varepsilon)} |d_0^1 - d_0^2|)/2 \right\}. \qquad (21)$$

Then, only one of the inequalities $\left| y_m^N(\tau_*) - y_m^i(\tau_*) \right| \le h(N)$, $i = 1, 2$, holds with the probability $1-g(N)$; the same is valid for the inequalities $\left| y_d^N(\tau_*) - y_d^i(\tau_*) \right| \le h(N)$, $i = 1, 2$. In case equation (19) starts from the initial state m_0^1, we decide to apply the control $u_1(t) = 1$ on the interval $(1+\varepsilon, 3/2)$; otherwise (from the state m_0^2), the control $u_1(t) = 0$, $t \in (1+\varepsilon, 2)$. In the first variant, in view of the time delay in switching the control to optimal, $m(3/2)$ takes a value at the ε-neighborhood of the set $\mathcal{M}(3/2)$. In the second variant, as a result, we have exactly $m(2) = 1$. By analogy, we proceed with equation (20): if the real initial state is d_0^1, then we apply the control $u_2^2(t) = 1$ on the interval $(1+\varepsilon, 3/2)$; if d_0^2, then $u_2^2(t) = 0$, $t \in (1+\varepsilon, 2)$. In the first case, $d(3/2)$ takes a value at the ε-neighborhood of the set $\mathcal{D}(3/2)$. In the second case, we have exactly $d(2) = 1$.

Thus, the closed-loop control method described above solves the original ε-guidance problem: it guarantees the attaintment of the solution of equation (19) to the ε-neighborhood of the target set $\mathcal{M}(t)$, $t \in S_t$, and the attaintment of the solution of equation (20) to the ε-neighborhood of the target set $\mathcal{D}(t)$, $t \in S_t$, with a probability close to 1. The computations by formulas (13) and (21) showed that $N = 10^3$ guarantees the guidance accuracy $\varepsilon = 0.1$ with a probability $P \ge 0.95$, whereas $N = 10^5$ guarantees $\varepsilon = 0.01$ with $P \ge 0.995$.

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