The Dynamic Model of Advertising Costs with Continuously Distributed Lags*

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Abstract. The dynamic optimal control problem of advertising costs in case of a company's limited advertising budget is analyzed. Initial optimization problem is formulated as a system of nonlinear integral equations of Volterra type. The constructed model takes into account the accumulated advertising effect and the accumulated effect of previous sales on the consumer demand. The accumulation of these effects is supposed to be distributed on an interval of time from the beginning of the planning period to a given point in time, that is, the delay of consumer reaction on advertising does not take the specified value and tends to infinity in the long run. Also the equation of consumer demand is determined. The total profit maximization problem under restrictions is stated. The existence theorems for the solutions of the demand equation and the total profit maximization problem are proved. The mathematical model can be integrated in information systems of enterprises for obtaining an optimal decision of a problem of the best distribution of advertising costs. In addition, the mathematical model is tested with the real data. In this case the Pontryagin's maximum principle and numerical methods of solving the integral and differential equations are used.

Keywords: computer modeling, mathematical model of advertising, distributed lag, optimal control, Pontryagin's maximum principle, existence of solution

1 Introduction

The main purpose of any commercial company is to satisfy clients, encourage them to repeat purchases and to recommend company's products to other potential clients. In this case, it is important to increase brand's recognition in order to attract new customers. Also a company should keep the interest of old clients. One of the most effective and popular tools which can help to achieve those goals is advertising. Generally, the advertising costs should be included in the total costs as well as other company's costs. But advertising does not require any change in production technologies or quality of goods, and it makes sense to consider advertising separately from other costs.

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Advertisement can influence on client's decisions immediately, but most of them have a 'lagging' nature. In addition, there are other factors which have an impact on sales [10,12,14], e.g. quality of goods, company's reputation etc. These factors make clients repeat purchases and create the effect of previous sales.

Accordingly, the demand is changing under the impact of the accumulated effect of advertising costs and the accumulated effect of previous sales.

Obviously, an advertising budget is usually limited to some amount of money. Thus, the problem of the best distribution of advertising costs over the planning period can be considered as a dynamic optimal control problem. In this case, the total profit function can be chosen as an objective function which needs to be maximized.

The optimization advertising models were considered in [1, 3, 4, 6, 8, 11, 13]. Many modern advertising models are based on them. They are discussed by Jian Huang, Mingming Leng, Liping Liang in detail [5].

In our paper, in addition to advertising costs, other factors of influence on consumer demand are taken into account. The delayed consumer reaction is considered here, too. The paper also includes practical application. It is supposed that the consumer demand is non-linear with respect to the accumulated effect of advertising costs and the accumulated effect of previous sales. The practical application of the dynamic model includes the algorithm for solving the problem of the best distribution of advertising costs by using the Pontryagin's maximum principle [2] and some numerical methods for solving integral and differential equations.

2 State of the Problem

Let $y(t)$ is the revenue function at the moment t, $u(t)$ is the advertising costs, $v(t)$ is the function which determines the accumulated advertising effect, $w(t)$ determines accumulated effect of previous sales:

$$v(t) = \int_0^{t+\lambda} G_u(\tau) u(t-\tau)\, d\tau; \tag{1}$$

$$w(t) = \int_0^{t+\lambda} G_y(\tau) y(t-\tau)\, d\tau. \tag{2}$$

Here λ is the length of the time interval of influence of previous sales and previous advertising costs on $y(t)$. The revenue at the moment t depends on $v(t)$ and $w(t)$:

$$y(t) = f(v(t), w(t)). \tag{3}$$

The functions $G_u(\tau)$, $G_y(\tau)$ describe the effects of previous advertising costs and previous sales [10]. There are some assumptions about the functions and $f(v, w)$:

1. If v and w take small values (a market isn't saturated by company's products and advertising is positively received by consumers), the function $f(v, w)$

increases. But consumers' reaction may change after some time [1, 9], and the function $f(v, w)$ may become non-increasing with respect to v. Also, $f(v, w)$ may decrease because of market saturation and supply constraints of the company. Thus, the function $f(v, w)$ is concave with respect to w.

2. Until the certain moment τ_u^* the advertising influence on the demand increases. After that, the advertising effect begins to decrease until it disappears. The possibility of negative advertising effect is excluded, i.e. advertising costs of the company do not reduce product demand. Thus, the function $G_u(\tau)$ is non-negative and it has unique maximum $\tau = \tau_u^*$ which is the moment of the highest advertising effect. If this function is differentiable, then assumption is equivalent to the following conditions:

$$G_u(\tau) \geq 0, \forall \tau \in [0; +\infty); \quad \lim_{\tau \to +\infty} G_u(\tau) = 0;$$

$$G'_u(\tau) \geq 0, \tau \in [0; \tau_u^*); \quad G'_u(\tau) \leq 0, \tau \in (\tau_u^*; +\infty).$$

3. Also we can note that consumers repeat purchases because of their positive experience. In this case, the consumer demand becomes higher and the function $G_y(\tau)$ increases. However, the experience of the first purchases is usually forgotten. It affects the current purchase weakly, giving place to the recent experience. Therefore, the function $G_y(\tau)$ has the properties:

$$G_y(\tau) \geq 0, \forall \tau \in [0; +\infty); \quad \lim_{\tau \to +\infty} G_y(\tau) = 0;$$

$$G'_y(\tau) \geq 0, \tau \in [0; \tau_y^*); \quad G'_y(\tau) \leq 0, \quad \tau \in (\tau_y^*; +\infty).$$

The financial result of the company is the profit or loss which is determined by the condition $\pi(y(t), u(t)) = y(t) - c(y(t), t)$. Here $c(y(t), t)$ is the function of total costs which include fixed and variable costs. Thus, we can write the objective function:

$$\Pi(T) = \int_0^T \pi(y(t), u(t))\, dt = \int_0^T (f(v(t), w(t)) - u(t) - c(y(t), t))\, dt$$

where $c(y(t), t) = c_1(y(t), t) + c_2$, c denotes total costs of the company which are connected with the production of goods excepting the advertising costs, c_1 - variable costs, c_2 - fixed costs.

It is logical that variable costs are in direct ratio to the volume of output with the rate μ. We have following:

$$\Pi(T) = \int_0^T \pi(x(t), u(t))\, dt = \int_0^T ((1 - \mu) f(v(t), w(t)) - u(t))\, dt. \tag{4}$$

Let the advertising budget be limited in the following way:

$$0 \leq u(t) \leq b, \quad t \in [0; T]. \tag{5}$$

The main task of the company is to maximize profit taking into account the fixed restrictions. So, we can formulate the following optimization problem: to maximize the functional (4) under the conditions (1), (2), (3), (5).

Rewrite (1) and (2):

$$v(t) = \int_{-\lambda}^{t} G_u(t-s)u(s)\,ds, \quad w(t) = \int_{-\lambda}^{t} G_y(t-s)y(s)\,ds,$$

where $s = t - \tau$.

Denote by $\phi_u(t)$, $\phi_y(t)$ the following functions:

$$\phi_u(t) = \int_{-\lambda}^{0} G_u(t-s)u(s)\,ds, \quad \phi_y(t) = \int_{-\lambda}^{0} G_y(t-s)y(s)\,ds.$$

The functions $v(t)$, $w(t)$, $\Pi(t)$ can be represented:

$$v(t) = \phi_u(t) + \int_{0}^{t} G_u(t-s)u(s)\,ds, \tag{6}$$

$$w(t) = \phi_y(t) + \int_{0}^{t} G_y(t-s)f(v(s), w(s))\,ds, \tag{7}$$

$$\Pi(t) = \int_{0}^{t} ((1-\mu)f(v(s), w(s)) - u(s))\,ds. \tag{8}$$

Thus, the maximization of $\Pi(T)$ under conditions (5), (6), (7), (8) is equivalent to the maximization of (4) under conditions (5), (3), (1), (2), and it presents the optimal control problem with integral equations of Volterra type.

3 Existence of the Solution

Suppose that advertising costs function $u(t)$ is piecewise-continuous on the right in $[0, T]$.

Consider the question of the solution existence of the equations (6), (7), (8).

If $G_u(\tau)$ is continuous, then $v(t)$ is continuous in $[0; T]$. Considering (5), (6) we can state the existence of the value $b_1 > 0$:

$$0 \le \phi_u(t) + \int_{0}^{t} G_u(t-s)u(s)\,ds \le \max_{0\le t\le T}\left(\phi_u(t) + \int_{0}^{t} G_u(t-s)b\,ds\right) \le b_1.$$

Thus, the function of the accumulated advertising effect satisfies the condition $v(t) : 0 \le v(t) \le b_1$ for any advertising strategy $u(t)$ that satisfies the condition (5). The problem of the solution existence of the equation (7) is solved by the theorem 1 [10].

Theorem 1. *Let the functions $G_u(\tau) \in C([0; T])$, $G_y(\tau) \in C([0; T])$, the function $f(v, w)$ is continuous and it satisfies a Lipschitz condition with respect to w for all w. Then, for any piecewise-continuous function $u(t)$, $t \in [0; T]$ that satisfies the restriction (5), there exists continuous and unique function $w(t)$, $t \in [0; T]$ that satisfies the condition (7).*

Proof. The proof of the solution existence of the linear equation of Volterra type is described in [7]. Based on this statement we prove a solution existence of the integral equation (7).

Initially, the function $f(v,w)$ satisfies the Lipschitz condition with respect to w. That is, there exists the constant $L : |f(v,w_1) - f(v,w_2)| \le L|w_1 - w_2|$, $\forall w_1, w_2$. Define the operator A in the following form:

$$Aw(t) \equiv \phi_y(t) + \int_0^t G_y(t-s) f(v(s), w(s))\, ds.$$

Evidently, there is a finite number $M = \max\limits_{0\le\tau\le t} G_y(\tau)$. Due to properties of the function $G_y(\tau)$, there is $\max\limits_{0\le\tau\le t} G_y(\tau)$ which is equal M. We can write the following inequality:

$$\begin{aligned} &|Aw_1(t) - Aw_2(t)| = \\ &\quad = \left| \int_0^t G_y(t-s)(f(v(s), w_1(s)) - f(v(s), w_2(s)))\, ds \right| \le \\ &\qquad \le MLt \max_{0\le s\le T} |w_1(s) - w_2(s)|. \end{aligned}$$

Introduce A^k, k is the multiple successive applying of the operator A. In this case $A^2 w \equiv A(Aw)$, $A^k w \equiv A(A^{k-1}w)$. We have the inequality:

$$\begin{aligned} &\left|A^2 w_1(t) - A^2 w_2(t)\right| = \\ &\quad = \left| \int_0^t G_y(t-s)(f(v(s), Aw_1(s)) - f(v(s), Aw_2(s)))\, ds \right| \le \\ &\quad \le ML \int_0^t |Aw_1(s) - Aw_2(s)|\, ds \le \frac{(MLt)^2}{2} \max_{0\le s\le T} |w_1(s) - w_2(s)|. \end{aligned}$$

Similarly,

$$\left|A^k w_1(t) - A^k w_2(t)\right| = \frac{(MLt)^k}{k!} \max_{0\le s\le T} |w_1(s) - w_2(s)|.$$

Using the metric in the space of continuous functions

$$\rho(w_1, w_2) = \max_{0\le s\le T} |w_1(s) - w_2(s)|$$

it is possible to write:

$$\rho\left(A^k w_1, A^k w_2\right) \le \frac{(MLt)^k}{k!} \rho(w_1, w_2).$$

Evidently, there exists $k : \dfrac{(MLt)^k}{k!} < 1$. It means, that operator A^k is contracting. Thus, the solution $w(t)$ of the equation (7) exists, it is unique and continuous in $[0;T]$. □

Theorem 1 gives the conditions of the existence of the global solution of the equation (7).

Remark 1. Let us suppose that the $f(v,w)$ is non-negative, concave and non-decreasing monotonously with respect to w. If the finite partial derivative f'_w exists with respect to $w=0$, and the function $f(v,w)$ satisfies the Lipschitz condition with respect to w for any w, then the solution of the equation (7) exists and it is non-negative in $[0;T]$.

Consider the optimization problem: to maximize $\Pi(T)$ under the conditions (5), (6), (7), (8).

A finite value of the accumulated profit $\Pi(T)$ is a functional of control $u(\cdot)$. Introduce $J(u(\cdot)) \equiv \Pi(T)$. Formulate the theorem of the solution existence of the optimization dynamic problem.

Theorem 2. *Assume that the conditions of the theorem 1 are fulfilled, $f(v,w)$ does not decrease monotonously with respect to v; then there are two alternatives:*

1. There exists $\{u^*(t), 0 \le t \le T\} : (5)$, a solution to the equations (6), (7), (8) $\{v^*(t), w^*(t), \Pi^*(t), 0 \le t \le T\} : J(u^*(\cdot)) \ge J(u(\cdot))$ for any $u(\cdot) : (5)$.

2. There exists a sequence of control functions $\{u^s(t), 0 \le t \le T\} : (5)$ and a value $\bar{J} : J(u^s(\cdot) \to \bar{J}$, $s \to \infty$, $J(u(\cdot)) \le \bar{J}$ for any $u(\cdot) : (5)$.

Proof. Let us estimate a solution of the equation (7)

$$w(t) = \phi_y(t) + \int_0^t G_y(t-s) f(v(s), w(s))\, ds \le \phi_y(t) + \int_0^t G_y(t-s) f(b_1, w(s))\, ds = w_{b_1}(t).$$

Here $w_{b_1}(t)$ is a solution of the equation (7): $v(s) \equiv b_1$.

Thus, for the advertising strategy (5) the accumulated influence of previous sales $w(s)$ is limited by a constant value $K : w(t) \le K = \max\limits_{0 \le t \le T} w_{b_1}(t)$.

It is possible to demonstrate that the total profit is limited:

$$\Pi(T) = \int_0^T ((1-\mu) f(v(s), w(s)) - u(s))\, ds \le \int_0^T f(v(s), w(s))\, ds \le T \max_{(v,w) \in D} f(v,w),$$

where $D = \{(v,w) : 0 \le w \le K, 0 \le v \le b_1\}$.

Thus, the range of the functional $J(u(\cdot))$ of the optimal control problem (5), (6), (7), (8) is limited. Denote this range as L.

Let us suppose that $\bar{J} = \sup L$. Evidently, $\bar{J}$ exists and it is limited.

If $\bar{J} \in L$, then the first alternative is realized else the second alternative is realized [8]. $\square$

Remark 2. If the second alternative of the Theorem 2 is realized, then there exists an approximated solution of the optimal control problem (5), (6), (7), (8) so that for any $\varepsilon > 0$ there exists such a control function $u_\varepsilon(\cdot)$: (5) and appropriate solutions of (6), (7), (8) that $\bar{J} - J(u_\varepsilon(\cdot)) < \varepsilon$.

4 The maximum principle for optimization problem of advertising costs with a Volterra integral equation

Consider the maximum principle for the problem.

Denote by $x_1(t)$, $x_2(t)$, $x_3(t)$ the functions of Volterra type:

$$x_1(t) = v(s) - \phi_u(s),\ x_2(t) = w(s) - \phi_y(s),\ x_3(t) = \Pi(t).$$

And $x(t)$:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \int_0^t \begin{pmatrix} G_u(t-s)u(s) \\ G_y(t-s)y(s) \\ (1-\mu)(y(s)-u(s)) \end{pmatrix} ds, \tag{9}$$

where $y(s) = f(\phi_u(s) + x_1(s), \phi_y(s) + x_2(s))$. Here the objective function is $x_3(T)$.

Introduce the modified Hamilton-Pontryagin function:

$$H(s,x,u,\psi) = (\psi(s), F(s,s,x,u)) + \int_s^T (\psi(t), F'_t(t,s,x,u))\,dt, \tag{10}$$

where

$$\psi(s) = (\psi_1(s), \psi_2(s), \psi_3(s)),$$

$$F(t,s,x,u) = \begin{pmatrix} G_u(t-s)u \\ G_y(t-s)f(\phi_u(s)+x_1, \phi_y(s)+x_2) \\ (1-\mu)f(\phi_u(s)+x_1, \phi_y(s)+x_2) - u \end{pmatrix}.$$

Define the adjoint variables $\psi(s)$:

$$\frac{d\psi_i(s)}{ds} = -\frac{\partial H(s,x,u,\psi)}{\partial x_i}, i = 1,2,3, \psi_1(T) = \psi_2(T) = 0, \psi_3(T) = 1. \tag{11}$$

Note that the Hamiltonian function (10) is linear with respect to u. That is, the maximum can be obtained, if only u takes the extreme values b or 0. The partial derivative of the Hamiltonian function with respect to u is determined:

$$\frac{\partial H(s,x,u,\psi)}{\partial u} = \psi_1(s)G_u(0) - \psi_3(s) + \int_s^T \psi_1(t)\frac{\partial G_u(t-s)}{\partial t}\,dt.$$

Obviously, that

$$u(s) = \begin{cases} clb, & \dfrac{\partial H(s,x,u,\psi)}{\partial u} > 0, \\ 0, & \dfrac{\partial H(s,x,u,\psi)}{\partial u} < 0, \\ \bar{u}, & \dfrac{\partial H(s,x,u,\psi)}{\partial u} = 0, \quad 0 \le \bar{u} \le b. \end{cases} \tag{12}$$

It is required to solve the boundary problem (9), (11), under the condition (12) in order to obtain a solution of the optimal control problem. Generally, the boundary problem (9), (11) under (12) cannot be solved by analytical methods, so we need to apply numerical schemes.

5 Practical application

In order to test the dynamic model of advertising costs, the company's monthly data set of advertising costs and revenue (in Russian rubles) from January 2009 to July 2014 is analyzed. The company is a producer of ready-made clothes located in Ulyanovsk, Russia.

Consider the case when the function $f(v, w)$ is nonlinear with respect to v and w : $f(v, w) = \alpha v^{\beta_1} w^{\beta_2}$. That is, the equation (3) can be written:

$$y(t) = \alpha \left(\phi_u(t) + \int_0^t G_u(t-s)u(s)\,ds \right)^{\beta_1} \left(\phi_y(t) + \int_0^t G_y(t-s)y(s)\,ds \right)^{\beta_2}.$$

Based on the properties of the functions $G_u(\tau)$, $G_y(\tau)$, it is possible to define them:

$$G_u(\tau) = \exp\left(a_u\tau^2 + b_u\tau\right), \quad G_y(\tau) = \exp\left(a_y\tau^2 + b_y\tau\right).$$

For example, the parameters a_u, b_u, a_y, b_y, β_1, β_2, α can be assessed by the method of least squares.

Results of the experiment

Denote by T the planning period. The limited number of advertising budget b is equal to 72,525.50 Russian rubles (maximal advertising costs in according to the data of the last year). The assessments of parameters of the model are obtained by the method of least squares: $\hat{a}_u = -0.35$, $\hat{b}_u = 0.97$, $\hat{a}_y = 0$, $\hat{b}_y = -1.37$, $\hat{\beta}_1 = 0.16$, $\hat{\beta}_2 = 0.93$, $\hat{\alpha} = 1.38$. The functions $\phi_u(t)$ and $\phi_y(t)$ are defined as approximation functions for the statistical data.

The present model was analyzed by numerical scheme:

- a combination of the method of local variations and the method of successive approximations (Krylov I.A., Chernousko F.L.) was used for solving the optimization problem;
- the trapezoidal rule was used for calculating the integrals (6), (7), (8) (integration step $h = 0.001$).

The solutions are found for different planning periods: $T = 1, 2$ and 3 months. In each case the solutions have the same structure:

$$u(t) = \begin{cases} b, & 0 \le t < t^*, \\ 0, & t^* \le t \le T, \end{cases}$$

where t^* is a switch point of control. The necessary conditions (9), (11), (12) are fulfilled for the solutions.

Table 1. The results of the experiment for the different planning periods T

T, months	$\Pi(T)$, Russian rubles	t^*, months
1	2.90843×10^7	0.989
2	1.52341×10^8	1.994
3	6.43752×10^8	2.998

Table 1 demonstrates the obtained values of the total profit and the switch point for the different planning periods.

As shown in Table 1, the maximum of the total profit can be achieved if the company has the highest possible advertising costs at the beginning of the planning period. It is a logical strategy. First, companies draw attention of customers to the goods. But later there is no need to spend money on advertising because the previous costs continue to bring high profits. In this case, it is a waste of money to invest in advertising at the end of the planning period.

6 Conclusion

In the practical application of the model (1), (2), (3), (4), (5), some difficulties can appear. They are related to properties of the function $f(v, w)$ which ensure existence of the solution of the optimal control problem. Particularly, the multiplicative function $f(v, w) = \alpha v^{\beta_1} w^{\beta_2}$ is continuous for any $\beta_1 > 0$, $\beta_2 > 0$ in $\{(v, w) : v \geq 0, w \geq 0\}$, but in the case when $0 < \beta_2 < 1$, the Lipschitz condition is not fulfilled for $w = 0$.

There is a special problem for a researcher: to identify the function type $f(v, w)$ which accords to empirical assumptions and requirements of Theorems 1 and 2. Specifically, in the practical application we have the nonlinear function $f(v(t), w(t)) = \alpha \left(\phi_u(t) + v(t)\right)^{\beta_1} \left(\phi_y(t) + w(t)\right)^{\beta_2}$ which is continuous and satisfies Lipschitz condition with respect to w. Thus, the optimal control problem has a solution in this case. Based on the numerical scheme, we got solutions which satisfy necessary conditions of Pontryagin's maximum principle, and it is in accordance with the understanding of the optimal advertising strategy.

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