МОДЕЛИРОВАНИЕ ПРОГИБА МЕМБРАНЫ С ПОМОЩЬЮ МНОГОСЛОЙНЫХ ПОЛУЭМИПРИЧЕСКИХ МОДЕЛЕЙ НА ОСНОВЕ ЭКСПЕРИМЕНТАЛЬНЫХ ДАННЫХ

Аннотация
В данной статье мы излагаем решение задачи о построении адаптивной математической модели прогиба нагруженной круговой мембраны на основе уравнения Пуассона и экспериментальных данных. В результате мы получили полуэмпирическую функциональную модель, выражающую зависимость прогиба мембраны от расстояния до оси симметрии. Модель построена с помощью приближённого аналитического решения уравнения прогиба мембраны, коэффициенты которого определяются с использованием экспериментально полученных данных. Приближённое решение построено при помощи авторской модификации уточнённого метода Эйлера, основанной на применении указанного метода к интервалу с переменным верхним пределом. Данная модификация позволила построить математическую модель в виде функциональной зависимости, аргументами которой являются неизвестные параметры реальной мембраны. Данные параметры находятся по измерениям с помощью метода наименьших квадратов. В результате мы получили улучшенную модель, более достоверно выражающую зависимость прогиба от расстояния до центра, чем точное решение исходного уравнения прогиба мембраны. Разработанные методы мы рекомендуем применять к моделированию реальных объектов в ситуации, когда физические процессы в них описаны не очень точно, кроме того, имеются данные наблюдений за объектом моделирования, которые могут пополняться в процессе его функционирования.

Ключевые слова
Полуэмпирический метод; уточнённый метод Эйлера; круговая мембрана; оператор Лапласа; исследование зависимости прогиба от радиуса.

MODELING OF THE MEMBRANE BENDING WITH MULTILAYER SEMI-EMPIRICAL MODELS BASED ON EXPERIMENTAL DATA

Abstract
In this article we present the solution of the problem of constructing an adaptive mathematical model of deflection of a loaded circular membrane based on the Poisson equation and experimental data. As a result, we got a semi-empirical functional model that expresses the dependence of the deflection of the membrane from the distance from the axis of symmetry. The model is constructed using the approximate analytical solution of the equation of deflection of a membrane whose coefficients are determined using experimentally obtained data. The approximate solution is constructed using the author’s modifications to the revised Euler method based on the application of this method to the interval with a variable upper limit. This modification allowed us to construct a mathematical model in the form of functional dependence, the arguments of which are the unknown parameters of the real membrane. These parameters are measured by the method of least squares. As a result, we got an

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improved model that more accurately expresses the dependence of the deflection on the distance to the center than the exact solution of the original equation of the deflection of the membrane. Methods have been developed we recommend you to apply to the modeling of real objects in situations where physical processes are described not very accurately, in addition, there are monitoring data on the simulation object, which may be in the process of its functioning.

Keywords

Semiempirical method; qualified Euler method; circular membrane; Laplace operator; study of the dependence of deflection on the radius.

Introduction

This paper develops methods of work that allow us to construct approximate semiempirical models using differential equations and experimental data.

A circular membrane of radius $R$ is considered, alternating cargos of various masses are placed on it. The membrane is assumed to be weightless (the mass of the membrane is much less than the weight of the load), the cargo is placed in the center of the membrane, its radius $a \ll R$. It is assumed that the stretching is isotropic (the tension is the same in all directions).

In this paper, we compare the exact solution of a differential equation and its approximate solution obtained by our modification [1-4] of the two-step Euler method [5], according to their agreement with the experimental data. The modification consists in applying to the variable-length interval known recurrence formulas for numerical methods for solving differential equations.

The problem under consideration turned out to be one of several problems with real objects for which the approximate solution better reflects the experimental data than the exact one. The reason for this seemingly unexpected result is that the differential equation under consideration displays the simulated object inaccurately. The derivation of more precise equations is very time consuming and does not guarantee success, therefore, the approach of constructing semiempirical models under consideration has the right to exist.

Material and methods

Let $u(r)$ is the deflection of the membrane from the equilibrium position. For its description we use the equation:

$$u''_r + \frac{1}{r} u'_r = \begin{cases} B, & \text{если } r \in [0, a], \\ 0, & \text{если } r \in (a, R], \end{cases}$$

(1)

which is the Poisson equation in polar coordinates, where $u(r, \varphi) = u(r)$, that is the desired function does not depend on the direction, but depends only on the distance $r$ of the point from the center of the membrane. Here $B = \frac{A}{T} A \cdot$ the weight of the load, $T \cdot$ the absolute value of the tensile force applied to the edge of the membrane.

Since the weight of the membrane is small in comparison with the weight of the load, its effect is neglected. The above approach to the physical situation studied is taken from the book [6], where additionally the term due to the membrane weight is taken into account on the right-hand side of the equation.

The equation under consideration is an ordinary differential equation of the second order. Let us write down its exact solution $u(r)$, taking into account continuity for $r = a$ and boundedness with $r = 0$:

$$u(r) = \begin{cases} \frac{1}{2} B a^2 \ln \frac{a}{r} + u_0 + \frac{1}{4} B(r^2 - a^2) npu \text{ } r \in [0, a], \\ \frac{1}{2} B a^2 \ln \frac{r}{R} + u_0 npu \text{ } r \in (a, R]. \end{cases}$$

(2)

Here $u_0 = u(R)$ is taken from measurements. We choose the parameter $B$ by using the least squares method so as to minimize the value $\sum_{i=1}^{m} (u(r_i) - u'_i)^2$. Here $r_i$ are the values of $r$ for which the deflection measurements were made, $u'_i$ - the results of the corresponding measurements, $u(r_i)$ - the values of the function found by the formula (2). Obviously, finding the value $B$, we will know the corresponding value $z_0 = u'(R)$. Taking into account the above formulas, knowing the weight of the cargo from the experiment, and determining the value $B$, we determine the value of the tensile force $T$.  

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To obtain an approximate solution, we reduce equation (1) to the normal system of differential equations:

\[
\begin{aligned}
&u' = z, \\
&z' = - \frac{z}{r} + f(r).
\end{aligned}
\]  

(3)

Here \( f(r) \) is the right-hand side of equation (1).

We seek an approximate solution of (1) in the form of a piecewise given function. To do this, we apply Euler's method, which for the equation \( \dot{y} = g(x, y) \) has the form of a recurrence formula

\[
y_{k+1} = y_k + h_k g(x_k, y_k).
\]

In accordance with [1-4], we apply this method to a gap of variable length. In this problem we will construct solutions from two sides of the interval \([0, R]\), joining them by the continuity of the function and the derivative at the point \( r = a \).

For the interval \((a, R]\) we construct a solution starting at its right end. After changing the variable \( x = R - r \), solve the system (3) by the two-step Euler method with \( h_k = x / 2 \), we obtain:

\[
\begin{aligned}
&u(x) = u_0 - \frac{x^2 z_0}{4R}, \\
&z(x) = (z_0 + \frac{x z_0}{2R}) \cdot \frac{2R}{2R - x}.
\end{aligned}
\]  

(4)

The value of \( u_0 \), as before, is taken from the experiment. The value \( z_0 \) is not defined yet. An approximate solution of (4) is considered for \( x \in [0, R - a] \), that is, for \( r \in (a, R] \).

For the interval \( r \in [0, a] \), we solve the system (3) by the same method, assuming the value of the deflection \( \tilde{U}_0 \) unknown for the \( r = 0 \), and the value of the derivative \( u'_0 \) zero for \( r = 0 \). Then we get:

\[
\begin{aligned}
&u(r) = \tilde{u}_0 + \frac{r^2 B}{4}, \\
&z(r) = \frac{r B}{2}.
\end{aligned}
\]  

(5)

Demanding the continuity of the solution \( u \) and its derivative \( z \) at a point \( r = a \), we obtain the following conditions:

\[
\begin{aligned}
&\tilde{u}_0 + \frac{a^2 B}{4} = u_0 - (R - a) z_0 - \frac{(R - a)^2 z_0}{4R}, \\
&\frac{1}{2} a B = \frac{3R - a}{R + a}.
\end{aligned}
\]  

(6)

From the continuity conditions (6), we find the expressions for the parameters \( \tilde{u}_0 \) and \( B \) in terms of the value \( z_0 \), and the last one is determined using the least squares method so as to minimize the value \( \sum_{i=1}^{10} u_i (r_i) - u_i \rangle^2 \), where we calculate \( u(r_i) \) by formula (5) for \( r_i \leq a \) and from (4) for \( r_i \geq a, x_i = R - r_i \).

Now, in the approximate solution \( u \) expressed by formulas (4) and (5), all the parameters will be found, and we can compare it with the exact solution.

**Calculation**

In the first experiment (Fig. 1, 2) with the mass of 100 grams the following results were obtained: \( z_0 = 0.0455 \) for the exact solution, \( z_0 = 0.070 \) for an approximate solution; \( B = 40.5 \) for an exact solution, \( B = 23.2 \) for an approximate solution; \( T = 0.00247 \) for the exact solution, \( T = 0.00431 \) for the approximate solution.

In the second experiment (Fig. 3, 4) with the mass of 228 grams the following results were obtained: \( z_0 = 0.904 \) for the exact solution, \( z_0 = 0.140 \) for an approximate solution; \( B = 80.3 \) for an exact solution, \( B = 46.4 \) for an approximate solution; \( T = 0.00124 \) for the exact solution, \( T = 0.0025 \) for the approximate solution.

In the third experiment (Fig. 5, 6) with the mass of 456 grams the following results were obtained: \( z_0 = 0.0817 \) for the exact solution, \( z_0 = 0.127 \) for an approximate solution; \( B = 72.6 \) for an exact solution, \( B = 42.0 \) for an approximate solution; \( T = 0.00138 \) for the exact solution, \( T = T = 0.00238 \) for the approximate solution.

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Results and Discussion

For all three values of the mass of the cargo, the exact solution deviates more strongly from the results of the experiment than the approximate one. At the same time, the same parameter (the deflection derivative at the edge of the membrane) is selected for approximate and exact solutions from the experimental data, from which the value of the tensile force is determined. The insufficient accuracy of the solution (2) indicates that the model (1) needs to be refined. This refinement can be done by refining the physical model of the membrane. The most obvious way is to take into account the weight of the membrane itself, but it is doubtful that this weight could explain the large deviation of the experimental results from formula (2), since the weight of the membrane is small compared to the weight of the goods.

![Figure 1. Diagrams of the deflection of the membrane for the mass of the cargo 100 grams](image1)

![Figure 2. Graphs of deviation of solutions from experimental values for the mass of the cargo 100 grams](image2)

We can take into account the thickness of the membrane: in the experiments carried out, we are dealing with a large deflection $u(R) - u(0)$, whose values are only several times smaller than the radius of the membrane, whereas the thickness of the membrane is small compared to its radius. In problems of material resistance associated with the calculation of the deflection of membranes, a change in thickness is considered when deformation of the membrane is in the case of a large deflection (see, for example, the book [9]). But on the other hand, when writing the Poisson equation for a membrane, the thickness is neglected. Probably, the question of introducing into the model the thickness of the membrane should be related to the refinement of its physical
properties in further research. It is possible to build a deflection model based on the filamentary structure of the tissue, but such a model will contain many difficult-to-identify parameters.

**Fig. 3.** Diagrams of the deflection of the membrane for the mass of the cargo 228 grams

**Fig. 4.** Graphs of the deviation of solutions from the experimental values for the mass of the cargo 228 grams
Fig. 5. Diagrams of the deflection of the membrane for the mass of the cargo 456 grams

Fig. 6. Graphs of deviation of solutions from the experimental values for the mass of the cargo 456 grams

Conclusions

In the future, it is proposed to study the dependence of the tension force $T$ on the mass and shape of the load. This requires more experiments. The approach proposed in this paper can be useful for the rapid construction of semi-empirical models in situations when the theoretical model in the form of a differential equation does not accurately describe the available experimental data, and the ways of its refinement are not obvious or unnecessarily time-consuming.

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