УДК 51-72

Бортковская М.Р.[1], Васильев П.И.[1], Зулькарнай И.У.[2], Семенова Д.А[1]., Тархов Д.А.[1], Удалов П.П.[1], Шишкина И.А.[1]

[1] Санкт-Петербургский политехнический университет имени Петра Великого, г. Санкт-Петербург, Россия
[2] Башкирский государственный университет, г. Уфа, Россия

МОДЕЛИРОВАНИЕ ПРОГИБА МЕМБРАНЫ С ПОМОЩЬЮ МНОГОСЛОЙНЫХ ПОЛУЭМПИРИЧЕСКИХ МОДЕЛЕЙ НА ОСНОВЕ ЭКСПЕРИМЕНТАЛЬНЫХ ДАННЫХ*

Аннотация

В данной статье мы излагаем решение задачи о построении адаптивной математической модели прогиба нагруженной круговой мембраны на основе уравнения Пуассона и экспериментальных данных. В результате мы получили полуэмпирическую функциональную модель, выражающую зависимость прогиба мембраны от расстояния до оси симметрии. Модель построена с помощью приближённого аналитического решения уравнения прогиба мембраны, коэффициенты которого определяются с использованием экспериментально полученных данных. Приближённое решение построено при помощи авторской модификации уточнённого метода Эйлера, основанной на применении указанного метода к интервалу с переменным верхним пределом. Данная модификация позволила построить математическую модель в виде функциональной зависимости, аргументами которой являются неизвестные параметры реальной мембраны. Данные параметры находятся по измерениям с помощью метода наименьших квадратов. В результате мы получили улучшенную модель, более достоверно выражающее зависимость прогиба от расстояния до центра, чем точное решение исходного уравнения прогиба мембраны. Разработанные методы мы рекомендуем применять к моделированию реальных объектов в ситуации, когда физические процессы в них описаны не очень точно, кроме того, имеются данные наблюдений за объектом моделирования, которые могут пополняться в процессе его функционирования.

Ключевые слова

Полуэмпирический метод; уточнённый метод Эйлера; круговая мембрана; оператор Лапласа; исследование зависимости прогиба от радиуса.

Bortkovskaya M.R.[1], Vasilyev P.I.[1], Zulkarnay I.U.[2], Semenova D.A.[1], Tarkhov D.A.[1], Udalov P.P.[1], Shishkina I.A.[1]

[1] Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russia
[2] Bashkir State University, Ufa, Russia

MODELING OF THE MEMBRANE BENDING WITH MULTILAYER SEMI-EMPIRICAL MODELS BASED ON EXPERIMENTAL DATA

Abstract

In this article we present the solution of the problem of constructing an adaptive mathematical model of deflection of a loaded circular membrane based on the Poisson equation and experimental data. As a result, we got a semi-empirical functional model that expresses the dependence of the deflection of the membrane from the distance from the axis of symmetry. The model is constructed using the approximate analytical solution of the equation of deflection of a membrane whose coefficients are determined using experimentally obtained data. The approximate solution is constructed using the author's modifications to the revised Euler method based on the application of this method to the interval with a variable upper limit. This modification allowed us to construct a mathematical model in the form of functional dependence, the arguments of which are the unknown parameters of the real membrane. These parameters are measured by the method of least squares. As a result, we got an

* **Труды II Международной научной конференции «Конвергентные когнитивно-информационные технологии» (Convergent'2017), Москва, 24-26 ноября, 2017**

Proceedings of the II International scientific conference "Convergent cognitive information technologies" (Convergent'2017), Moscow, Russia, November 24-26, 2017

improved model that more accurately expresses the dependence of the deflection on the distance to the center than the exact solution of the original equation of the deflection of the membrane. Methods have been developed we recommend you to apply to the modeling of real objects in situations where physical processes are described not very accurately, in addition, there are monitoring data on the simulation object, which may be in the process of its functioning.

Keywords

Semiempirical method; qualified Euler method; circular membrane; Laplace operator; study of the dependence of deflection on the radius.

Introduction

This paper develops methods of work that allow us to construct approximate semiempirical models using differential equations and experimental data.

A circular membrane of radius R is considered, alternating cargos of various masses are placed on it. The membrane is assumed to be weightless (the mass of the membrane is much less than the weight of the load), the cargo is placed in the center of the membrane, its radius $a << R$. It is assumed that the stretching is isotropic (the tension is the same in all directions).

In this paper, we compare the exact solution of a differential equation and its approximate solution obtained by our modification [1-4] of the two-step Euler method [5], according to their agreement with the experimental data. The modification consists in applying to the variable-length interval known recurrence formulas for numerical methods for solving differential equations.

The problem under consideration turned out to be one of several problems with real objects for which the approximate solution better reflects the experimental data than the exact one. The reason for this seemingly unexpected result is that the differential equation under consideration displays the simulated object inaccurately. The derivation of more precise equations is very time consuming and does not guarantee success, therefore, the approach of constructing semiempirical models under consideration has the right to exist.

Material and methods

Let $u(r)$ is the deflection of the membrane from the equilibrium position. For its description we use the equation:

$$u''_{rr} + \frac{1}{r}u'_r = \begin{cases} B, \text{если } r \in [0,a], \\ 0, \text{если } r \in (a,R], \end{cases} \tag{1}$$

which is the Poisson equation in polar coordinates, where $u(r,\varphi) = u(r)$, that is the desired function does not depend on the direction, but depends only on the distance r of the point from the center of the membrane. Here $B = \frac{A}{T}$, A - the weight of the load, T - the absolute value of the tensile force applied to the edge of the membrane. Since the weight of the membrane is small in comparison with the weight of the load, its effect is neglected. The above approach to the physical situation studied is taken from the book [6], where additionally the term due to the membrane weight is taken into account on the right-hand side of the equation.

The equation under consideration is an ordinary differential equation of the second order. Let us write down its exact solution $u(r)$, taking into account continuity for $r = a$ and boundedness with $r = 0$:

$$u(r) = \begin{cases} \frac{1}{2}Ba^2 \ln\frac{a}{R} + u_0 + \frac{1}{4}B(r^2 - a^2) \text{ при } r \in [0,a], \\ \frac{1}{2}Ba^2 \ln\frac{r}{R} + u_0 \text{ при } r \in (a,R]. \end{cases} \tag{2}$$

Here $u_0 = u(R)$ is taken from measurements. We choose the parameter B by using the least squares method so as to minimize the value $\sum_{i=1}^{10}(u(r_i) - u_i)^2$. Here r_i are the values of r for which the deflection measurements were made, u_i - the results of the corresponding measurements, $u(r_i)$ - the values of the function found by the formula (2). Obviously, finding the value B, we will know the corresponding value $z_0 = u'(R)$. Taking into account the above formulas, knowing the weight of the cargo from the experiment, and determining the value B, we determine the value of the tensile force T.

To obtain an approximate solution, we reduce equation (1) to the normal system of differential equations:

$$\begin{cases} u' = z, \\ z' = -\dfrac{z}{r} + f(r). \end{cases} \tag{3}$$

Here $f(r)$ is the right-hand side of equation (1).

We seek an approximate solution of (1) in the form of a piecewise given function. To do this, we apply Euler's method, which for the equation $\dot{y} = g(x, y)$ has the form of a recurrence formula $y_{k+1} = y_k + h_k g(x_k, y_k)$. In accordance with [1-4], we apply this method to a gap of variable length. In this problem we will construct solutions from two sides of the interval $[0, R]$, joining them by the continuity of the function and the derivative at the point $r = a$.

For the interval $(a, R]$ we construct a solution starting at its right end. After changing the variable $x = R - r$, solve the system (3) by the two-step Euler method with $h_k = x/2$, we obtain:

$$u(x) = u_0 - xz_0 - \frac{x^2 z_0}{4R},$$
$$z(x) = (z_0 + \frac{xz_0}{2R}) \cdot \frac{2R}{2R - x}. \tag{4}$$

The value of u_0, as before, is taken from the experiment. The value z_0 is not defined yet. An approximate solution of (4) is considered for $x \in [0, R-a)$, that is, for $r \in (a, R]$.

For the interval $r \in [0, a]$, we solve the system (3) by the same method, assuming the value of the deflection $\tilde{u}_0$ unknown for the $r = 0$, and the value of the derivative u_r' zero for $r = 0$. Then we get:

$$u(r) = \tilde{u}_0 + \frac{r^2 B}{4},$$
$$z(r) = \frac{rB}{2}. \tag{5}$$

Demanding the continuity of the solution u and its derivative z at a point $r = a$, we obtain the following conditions:

$$\tilde{u}_0 + \frac{a^2 B}{4} = u_0 - (R-a)z_0 - \frac{(R-a)^2 z_0}{4R},$$
$$\frac{1}{2} aB = z_0 \frac{3R - a}{R + a}. \tag{6}$$

From the continuity conditions (6), we find the expressions for the parameters $\tilde{u}_0$ and B in terms of the value z_0, and the last one is determined using the least squares method so as to minimize the value $\sum_{i=1}^{10} (u(r_i) - u_i)^2$, where we calculate $u(r_i)$ by formula (5) for $r_i \le a$ and from (4) for $r_i \ge a$, $x_i = R - r_i$.

Now, in the approximate solution u expressed by formulas (4) and (5), all the parameters will be found, and we can compare it with the exact solution.

Calculation

In the first experiment (Fig. 1, 2) with the mass of 100 grams the following results were obtained: z_0 = 0.0455 for the exact solution, z_0 = 0.070 for an approximate solution; B = 40.5 for an exact solution, B = 23.2 for an approximate solution; T = 0.00247 for the exact solution, T = 0.00431 for the approximate solution.

In the second experiment (Fig. 3, 4) with the mass of 228 grams the following results were obtained: z_0 = 0.904 for the exact solution, z_0 = 0,140 for an approximate solution; B = 80.3 for an exact solution, B = 46.4 for an approximate solution; T = 0.00124 for the exact solution, T = 0.00215 for the approximate solution.

In the third experiment (Fig. 5, 6) with the mass of 456 grams the following results were obtained: = 0.0817 for the exact solution, = 0.127 for an approximate solution; = 72.6 for an exact solution, = 42.0 for an approximate solution; T = 0.00138 for the exact solution, T = T = 0.00238 for the approximate solution.

Results and Discussion

For all three values of the mass of the cargo, the exact solution deviates more strongly from the results of the experiment than the approximate one. At the same time, the same parameter (the deflection derivative at the edge of the membrane) is selected for approximate and exact solutions from the experimental data, from which the value of the tensile force is determined. The insufficient accuracy of the solution (2) indicates that the model (1) needs to be refined. This refinement can be done by refining the physical model of the membrane. The most obvious way is to take into account the weight of the membrane itself, but it is doubtful that this weight could explain the large deviation of the experimental results from formula (2), since the weight of the membrane is small compared to the weight of the goods.

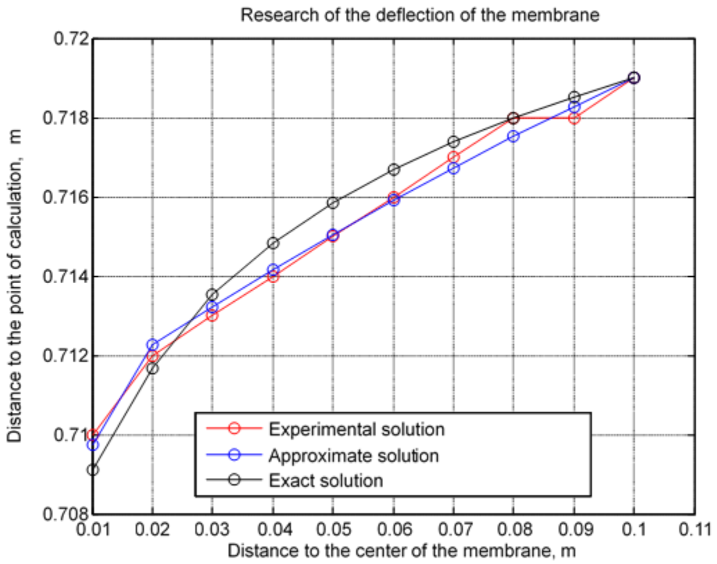

Fig. 1. Diagrams of the deflection of the membrane for the mass of the cargo 100 grams

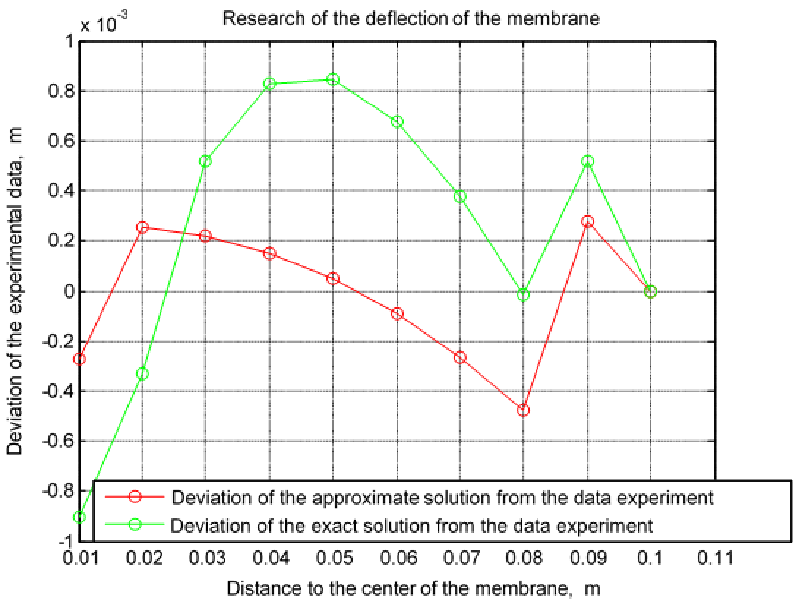

Fig. 2. Graphs of deviation of solutions from experimental values for the mass of the cargo 100 grams

We can take into account the thickness of the membrane: in the experiments carried out, we are dealing with a large deflection $u(R)-u(0)$, whose values are only several times smaller than the radius of the membrane, whereas the thickness of the membrane is small compared to its radius. In problems of material resistance associated with the calculation of the deflection of membranes, a change in thickness is considered when deformation of the membrane is in the case of a large deflection (see, for example, the book [9]). But on the other hand, when writing the Poisson equation for a membrane, the thickness is neglected. Probably, the question of introducing into the model the thickness of the membrane should be related to the refinement of its physical properties in further research. It is possible to build a deflection model based on the filamentary structure of the tissue, but such a model will contain many difficult-to-identify parameters.

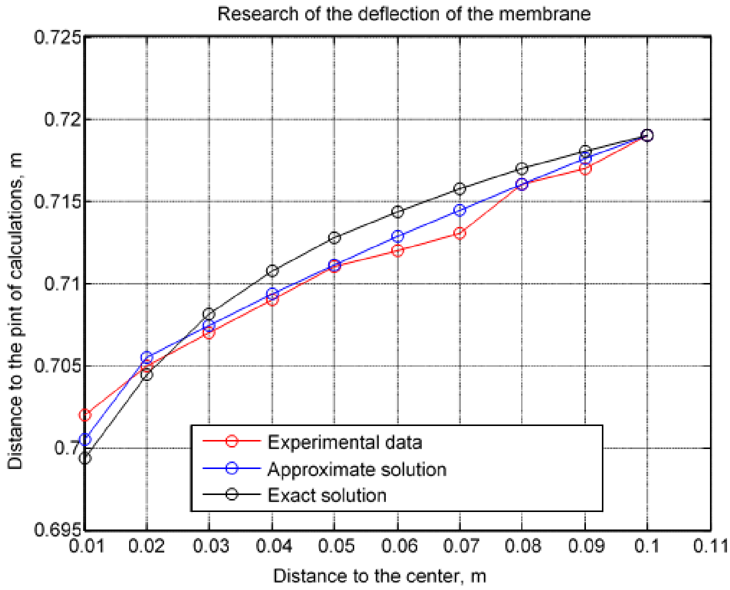

Fig. 3. Diagrams of the deflection of the membrane for the mass of the cargo 228 grams

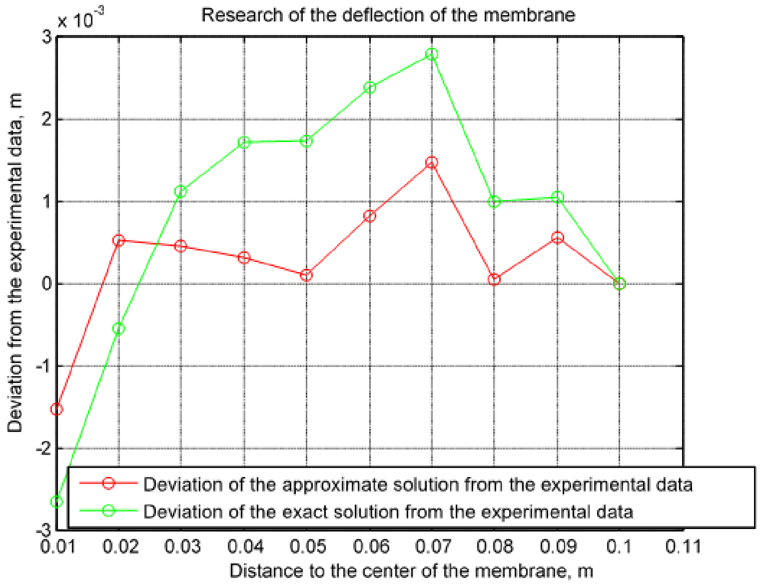

Fig. 4. Graphs of the deviation of solutions from the experimental values for the mass of the cargo 228 grams

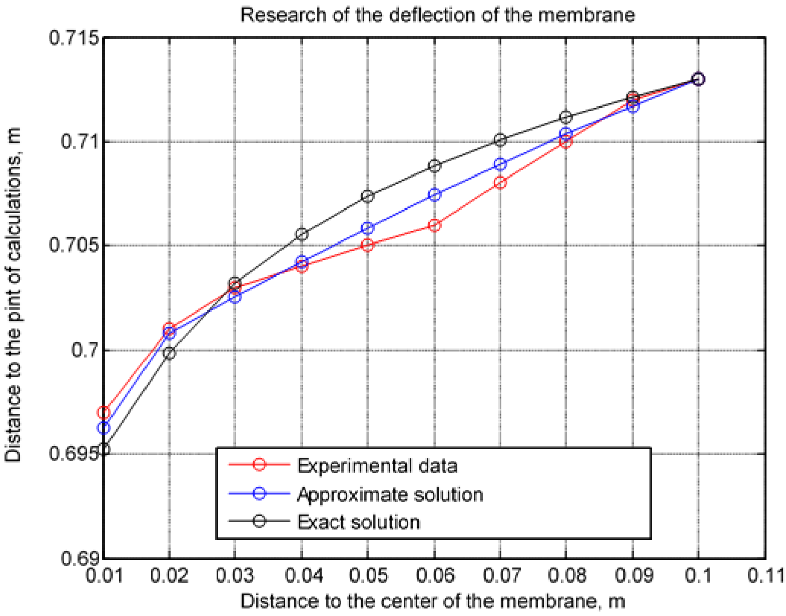

Fig. 5. Diagrams of the deflection of the membrane for the mass of the cargo 456 grams

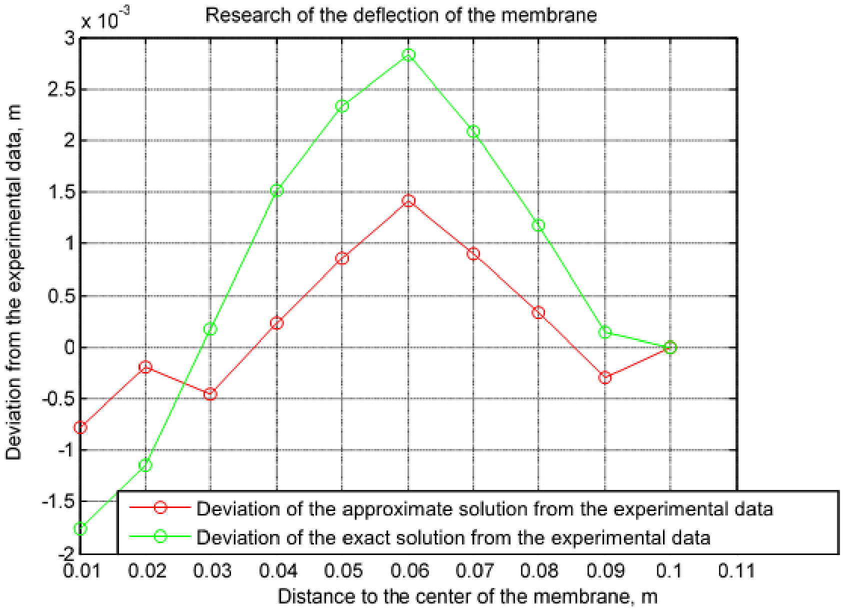

Fig. 6. Graphs of deviation of solutions from the experimental values for the mass of the cargo 456 grams

Conclusions

In the future, it is proposed to study the dependence of the tension force T on the mass and shape of the load. This requires more experiments. The approach proposed in this paper can be useful for the rapid construction of semi-empirical models in situations when the theoretical model in the form of a differential equation does not accurately describe the available experimental data, and the ways of its refinement are not obvious or unnecessarily time-consuming.

Литература

1. Lazovskaya T., Tarkhov D. Multilayer neural network models based on grid methods, IOP Conf. Series: Materials Science and Engineering 158 (2016) http://iopscience.iop.org/article/10.1088/1757-899X/158/1/01206
2. Vasilyev A., Tarkhov D., Bolgov I., Kaverzneva T., Kolesova S., Lazovskaya T., Lukinskiy E., Petrov A., Filkin V. Multilayer neural network models based on experimental data for processes of sample deformation and destruction // Selected Papers of the First International Scientific Conference Convergent Cognitive Information Technologies (Convergent 2016) Moscow, Russia, November 25-26, 2016 p.6-14 http://ceur-ws.org/Vol-1763/paper01.pdf
3. Vasilyev A., Tarkhov D., Shemyakina T. Approximate analytical solutions of ordinary differential equations // Selected Papers of the

XI International Scientific-Practical Conference Modern Information Technologies and IT-Education (SITITO 2016) Moscow, Russia, November 25-26, 2016 p.393-400 http://ceur-ws.org/Vol-1761/paper50.pdf

4. Tarkhov D., Shershneva E. Approximate analytical solutions of Mathieu's equations based on classical numerical methods // Selected Papers of the XI International Scientific-Practical Conference Modern Information Technologies and IT-Education (SITITO 2016) Moscow, Russia, November 25-26, 2016 p.356-362 http://ceur-ws.org/Vol-1761/paper46.pdf
5. Вербжицкий В.М. Численные методы. Математический анализ и обыкновенные дифференциальные уравнения // М.: Высшая школа. — 2001. — С. 218-220.
6. Ландау Л. Д., Лифшиц Е.М. Курс теоретической физики. Т.7. Теория упругости. М.: Физматлит, 2007. — 259 с.
7. Работнов Ю.Н. Сопротивление материалов. М., «Физматгиз», 1962, 456 с.

References

1. Lazovskaya T., Tarkhov D. Multilayer neural network models based on grid methods, IOP Conf. Series: Materials Science and Engineering 158 (2016) http://iopscience.iop.org/article/10.1088/1757-899X/158/1/01206
2. Vasilyev A., Tarkhov D., Bolgov I., Kaverzneva T., Kolesova S., Lazovskaya T., Lukinskiy E., Petrov A., Filkin V. Multilayer neural network models based on experimental data for processes of sample deformation and destruction // Selected Papers of the First International Scientific Conference Convergent Cognitive Information Technologies (Convergent 2016) Moscow, Russia, November 25-26, 2016 p.6-14 http://ceur-ws.org/Vol-1763/paper01.pdf
3. Vasilyev A., Tarkhov D., Shemyakina T. Approximate analytical solutions of ordinary differential equations // Selected Papers of the XI International Scientific-Practical Conference Modern Information Technologies and IT-Education (SITITO 2016) Moscow, Russia, November 25-26, 2016 p.393-400 http://ceur-ws.org/Vol-1761/paper50.pdf
4. Tarkhov D., Shershneva E. Approximate analytical solutions of Mathieu's equations based on classical numerical methods // Selected Papers of the XI International Scientific-Practical Conference Modern Information Technologies and IT-Education (SITITO 2016) Moscow, Russia, November 25-26, 2016 p.356-362 http://ceur-ws.org/Vol-1761/paper46.pdf
5. Verbigitsky V.M. Chislennyye metody. Matematicheskiy analiz i obyknovennyye differentsial'nyye uravneniya [Numerical methods. Mathematical analysis and ordinary differential equations] // M.: Vysshaya shkola. — 2001. — P. 218-220.
6. Landau L.D., Lifshits E.M. Course of theoretical physics. T.7. Theory of elasticity. Moscow: Fizmatlit, 2007. – 259 p.
7. Rabotnov Yu.N. Strength of materials. M., "Fizmatgiz", 1962, 456 p.

Об авторах:

Бортковская Мария Романовна, кандидат физико-математических наук, доцент, Санкт-Петербургский политехнический университет Петра Великого, mbort@mail.ru

Васильев Пётр Игоревич, студент, Санкт-Петербургский политехнический университет Петра Великого, p.vasilyev@inbox.ru

Зулькарнай Ильдар Узбекович, доктор экономических наук, заведующий лабораторией исследований проблем социально-экономического развития регионов, Башкирский государственный университет, zulkar@mail.ru

Семенова Дарья Алексеевна, студентка, Санкт-Петербургский политехнический университет Петра Великого, dasha49fax.ru@gmail.com

Тархов Дмитрий Альбертович, доктор технических наук, профессор, Санкт-Петербургский политехнический университет Петра Великого, dtarkhov@gmail.com

Удалов Павел Павлович, студент, Санкт-Петербургский политехнический университет Петра Великого, forsteam1231@mail.ru

Шишкина Ирина Александровна, студентка, Санкт-Петербургский политехнический университет Петра Великого, irina-shishkina-1998@mail.ru

Note on the authors:

Bortkovskaya Mariya R., Candidate of mathematics, Peter the Great St. Petersburg Polytechnic University, mbort@mail.ru

Vasilyev Petr I., student, Peter the Great St. Petersburg Polytechnic University, p.vasilyev@inbox.ru

Zulkarnay Ildar U., doctor of economic sciences, head of the laboratory of research of problems of social and economic development of the regions, Bashkir State University, zulkar@mail.ru

Semenova Darya A., student, Peter the Great St. Petersburg Polytechnic University, dasha49fax.ru@gmail.com

Tarkhov Dmitry A., Doctor of Engineering Sciences, Full Professor of the department "Higher Mathematics", Peter the Great St. Petersburg Polytechnic University, dtarkhov@gmail.com

Udalov Pavel P., student, Peter the Great St. Petersburg Polytechnic University, forsteam1231@mail.ru

Shishkina Irina A., student, Peter the Great St. Petersburg Polytechnic University, irina-shishkina-1998@mail.ru