The Schouten Curvature for a Nonholonomic Distribution in Sub-Riemannian Geometry and Jacobi Fields

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Abstract. The paper shows that if the distribution is defined on a manifold with the special smooth structure and does not depend on the vertical coordinates, then the Schouten curvature tensor coincides with the Riemannian curvature tensor. The Schouten curvature tensor is used to write the Jacobi equation for the distribution. This leads to studies on second-order optimality conditions for the horizontal geodesics in sub-Riemannian geometry. Conjugate points are defined by the solutions of the Jacobi equation. If a geodesic passed a point conjugated with its beginning then this geodesic ceases to be optimal.

Keywords: Sub-Riemannian geometry · Nonholonomic distributions · Schouten curvature · Jacobi fields

1 Introduction

Distribution on a smooth manifold N is a family of subspaces $\mathcal{A}(x) \subset T_xN$, $x \in N$. General theory of the variational calculus with nonholonomic restrictions $\varphi(t, x, \dot{x}) = 0$ was published by G.A. Bliss [2]. If the restrictions are linear for velocities, $\omega_x(\dot{x}) = 0$, where ω is a 1-form, we get distributions [25, 26].

The Romanian mathematician Vranceanu first introduced the term of the nonholonomic structure on a Riemannian manifold in 1928 [27]. The Dutch mathematician Schouten defined the connection and appropriate curvature tensor for horizontal vector fields on a distribution in [23, 22]. The soviet geometrician Vagner extended this construction and built the general curvature tensor compatible with the Schouten tensor satisfying common requirements for the curvature in 1937, Kazan [24]. These results were published by Gorbatenko, Tomsk, in 1985 [6] with modern notations.

The Schouten curvature tensor is different from the Riemannian curvature. In this paper we prove that if the distribution is defined on a manifold with the

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special smooth structure and does not depend on the "vertical" coordinates then these two tensors are identical. This curvature tensor is necessary to write the Jacobi equation for a distribution.

We study also second-order optimality conditions for the horizontal geodesics in sub-Riemannian geometry. If a geodesic passed a point conjugated with its beginning then this geodesic ceases to be optimal. Conjugate points are defined by means of the solutions of the Jacobi equation. In this paper we discuss several examples of conjugate points in sub-Riemannian geometry.

2 Variations and Equations of Variations

Let $\gamma : [t_0, T] \to N$ be a C^1-smooth horizontal path, $\omega(\gamma') = 0$, where ω is a 1-form. *Horizontal variation* of γ is a 1-parametric family of maps $\sigma(\cdot, \tau) : [t_1, t_2] \to N$, $|\tau| < \varepsilon$, if there are continuous vector fields $X = \dfrac{\partial\sigma}{\partial t}$, $Y = \dfrac{\partial\sigma}{\partial\tau}$, the central line is just $\sigma(t, 0) = \gamma(t)$ and the field X is horizontal, $X(t, \tau) \in \mathcal{A}(\sigma(t, \tau))$ and there are continuous second derivatives $\dfrac{\partial^2\sigma}{\partial t\partial\tau}$, $\dfrac{\partial^2\sigma}{\partial\tau\partial t}$ for all allowed t and τ.

Since the horizontality condition $\omega^\alpha(\dfrac{\partial\sigma}{\partial t}) = 0$ is fulfilled as an identity we can differentiate it for τ. We get $\sum\limits_{j,k=1}^{n} \dfrac{\partial\omega_k^\alpha}{\partial x^j}\dfrac{\partial\sigma^j}{\partial\tau}\dfrac{\partial\sigma^k}{\partial t} + \sum\limits_{k=1}^{n} \omega_k^\alpha \dfrac{\partial^2\sigma^k}{\partial t\partial\tau} = 0$. At $\tau = 0$ we obtain *the equations of variations along* γ*:*

$$\sum_{k=1}^{n} \omega_k^\alpha \frac{dY^k}{dt} + \sum_{j,k=1}^{n} \frac{\partial\omega_k^\alpha}{\partial x^j}\gamma'^k Y^j = 0, \quad \alpha = m{+}1, \ldots, n. \tag{1}$$

The vector field $Y(\cdot, 0)$ along γ is denoted as Y. These equations are of the form $\Phi^\alpha(Y', Y) = 0$, $\alpha = m{+}1, \ldots, n$. This is a system of $n - m$ differential equations. Since the rank of the matrix (ω_k^α) is $n - m$, the horizontal projection of the field Y is arbitrary and the vertical components Y^α are defined by the initial conditions.

3 The Accessory Problem of G.A. Bliss (Example)

3.1 The Lagrange Problem

Let us consider the 2-dimensional distribution $\mathcal{A}$ on $\mathbb{R}^3$ defined by the 1-form $\omega = x^2 dx^1 + dx^3$. The energy functional is $J(x(\cdot)) = \frac{1}{2}\int_0^T \big((\dot{x}^1)^2 + (\dot{x}^2)^2\big)dt$, the metric tensor for this distribution is the identity matrix. The Lagrangian of the variational problem is $L = \frac{1}{2}\big((\dot{x}^1)^2 + (\dot{x}^2)^2\big) + l\omega(\gamma')$. The horizontal geodesics starting from the origin in $\mathbb{R}^3$ for $l \neq 0$ are

$$\begin{cases} x^1(t) = \frac{1}{l}\big(-v_2 + v_2\cos lt + v_1 \sin lt\big), \\ x^2(t) = \frac{1}{l}\big(v_1 - v_1\cos lt + v_2 \sin lt\big), \\ x^3(t) = \frac{1}{4l^2}\Big(2lt(v_1^2 + v_2^2) + 2v_1v_2 - 4v_1v_2\cos lt - 4v_1^2 \sin lt + \\ \qquad +2v_1v_2\cos 2lt + (v_1^2 - v_2^2)\sin 2lt\Big). \end{cases} \tag{2}$$

Note that $\dot{x}^1(0) = v_1$ and $\dot{x}^2(0) = v_2$. These geodesics are shown at Fig. 1 for

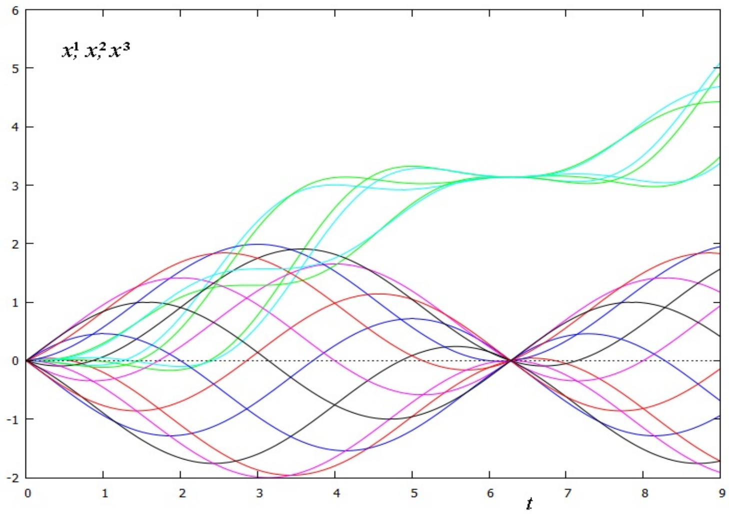

Fig. 1. Horizontal geodesics.

$l = -1$. These geodesics are optimal for the interval $[0, 2\pi/|l|)$. If $|lt| > 2\pi$, then the geodesic ceased to be optimal.

3.2 The Accessory Problem

Let us consider the minimization problem for the functional $\int_{t_0}^{T} f(t, x, \dot{x})\, dt$ with the nonholonomic constrains $\varphi(t, x, \dot{x}) = 0$. Assume that the matrix $(\frac{\partial \varphi^\alpha}{\partial \dot{x}^k})_{\substack{\alpha=m+1,..,n, \\ k=1,..,n}}$ has the rank $n-m$ for all $(t, x, \dot{x})$. Let $L(t, x, \dot{x}, l) = f(t, x, \dot{x}) + \sum\limits_{\alpha=m+1}^{n} l_\alpha \varphi^\alpha(t, x, \dot{x})$. The second variation of the functional

$$J(\tau) = \int_{t_0(\tau)}^{T(\tau)} L(t, \sigma(t, \tau), \sigma'_t(t, \tau), l)\, dt \tag{3}$$

is as following:

$$\delta^2 J = \left.\frac{d^2 J}{d\tau^2}\right|_{\tau=0} = L(t, x, \dot{x})\xi'_\tau \Big|_{t_0}^{T} + 2\left(\sum_{k=1}^{n} \frac{\partial L}{\partial x^k}\eta^k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{x}^k}\dot{\eta}^k\right)\xi \Big|_{t_0}^{T} + \frac{dL}{dt}\xi^2 \Big|_{t_0}^{T} +$$
$$+ \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{x}^k}\frac{\partial \eta^k}{\partial \tau}\Big|_{t_0}^{T} + \int_{t_0}^{T} \sum_{i,j=1}^{n} \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\dot{\eta}^i\dot{\eta}^j + 2\frac{\partial^2 L}{\partial x^i \partial \dot{x}^j}\eta^i\dot{\eta}^j + \frac{\partial^2 L}{\partial x^i \partial x^j}\eta^i\eta^j\right) dt, \tag{4}$$

where $\xi(t_0) = \frac{dt_0}{d\tau}$, $\xi(T) = \frac{dT}{d\tau}$, $\xi'_\tau(t_0) = \frac{d^2 t_0}{d\tau^2}$, $\xi'_\tau(T) = \frac{d^2 T}{d\tau^2}$, $\eta = \dfrac{\partial \sigma}{\partial \tau}$ at $\tau = 0$. This functional should be minimized for variables η satisfying the equations of variations along γ [2]. Since the distribution $\mathcal{A}$ is defined by the 1-form $\omega = x^2 dx^1 + dx^3$ in our example, the constrains (1) in the accessory problem are

$$\varPhi \equiv \dot\eta^3 + x^2 \dot\eta^1 + \eta^2 \dot x^1 = 0. \tag{5}$$

This is the equation of variations along γ. Since the Lagrangian in our example is $L = \frac{1}{2}\big((\dot x^1)^2 + (\dot x^2)^2\big) + l(x^2 \dot x^1 + \dot x^3)$, the Lagrangian for the accessory problem is $\varOmega = \frac{1}{2}\big((\dot\eta^1)^2 + (\dot\eta^2)^2\big) + l\eta^2\dot\eta^1 + \lambda(\dot\eta^3 + x^2\dot\eta^1 + \eta^2 \dot x^1)$. The generalized momenta $p_1 = \dot\eta^1 + l\eta^2 + \lambda x^2$, $p_2 = \dot\eta^2$ and $p_3 = \lambda$. The generalized forces $f_1 = 0$, $f_2 = l\dot\eta^1 + \lambda \dot x^1$ and $f_3 = 0$. The Euler—Lagrange equations for the accessory problem are $\ddot\eta^1 + l\dot\eta^2 + \lambda \dot x^2 = 0$, $\ddot\eta^2 = l\dot\eta^1 + \lambda \dot x^1$ and $\dot\lambda = 0$. Applying the geodesics (2) we get

$$\begin{cases} \ddot\eta^1 + l\dot\eta^2 + \lambda\big(v_1 \sin lt + v_2 \cos lt\big) = 0 \\ \ddot\eta^2 - l\dot\eta^1 - \lambda\big(v_1 \cos lt - v_2 \sin lt\big) = 0 \\ \dot\eta^3 + \frac{1}{l}\big(v_1 - v_1 \cos lt + v_2 \sin lt\big)\dot\eta^1 + \big(v_1 \cos lt - v_2 \sin lt\big)\eta^2 = 0 \\ \dot\lambda = 0. \end{cases} \tag{6}$$

This is the system of linear homogeneous differential equations for the variables η, λ. To find points conjugate with the starting point $t_0 = 0$ one should find the solutions of these equations which are zero $\eta(0) = 0$ at the initial point but $\eta'(0) \neq 0$. The solutions have the form

$$(\eta^1, \eta^2, \eta^3)^T = P(a_1, a_2, \lambda)^T, \tag{7}$$

where a_1, a_2, λ are constants and P is the following part of the fundamental matrix of the system (6):

$$P = \begin{pmatrix} \dfrac{\sin lt}{l} & \dfrac{\cos lt - 1}{l} & \dfrac{(v_1 lt - v_2)\cos lt - (v_1 + v_2 lt)\sin lt + v_2}{l^2} \\ \dfrac{1 - \cos lt}{l} & \dfrac{\sin lt}{l} & \dfrac{(v_1 lt - v_2)\sin lt + (v_1 + v_2 lt)\cos lt - v_1}{l^2} \\ \dfrac{\begin{array}{c}\big(2v_1 lt - 4v_1 \sin lt + \\ + v_1 \sin 2lt - 2v_2 \cos lt + \\ + v_2 \cos 2lt + v_2\big)\end{array}}{2l^2} & \dfrac{\begin{array}{c}\big(2v_2 lt - v_2 \sin 2lt + \\ + v_1 \cos 2lt + v_1 - \\ - 2v_1 \cos lt\big)\end{array}}{2l^2} & \dfrac{\begin{array}{c} -\big((v_1^2+v_2^2)lt - 4v_1^2 \sin lt + (v_1^2 - v_2^2 + 2v_1v_2lt)\sin 2lt + \\ + 2(v_1 lt - 2v_2)v_1 \cos lt - 2v_1v_2 lt \sin lt + 2v_1v_2 + \\ + (2v_1v_2 + (v_2^2 - v_1^2)lt)\cos 2lt\big)\end{array}}{2l^3} \end{pmatrix}.$$

If t_1 is a point conjugated with the initial point then there is the solution η of (6) satisfying $\eta(t_1) = 0$. To find such a solution we should calculate the determinant

$$\det P = -2t\big(lt \cos\frac{lt}{2} - 2\sin\frac{lt}{2}\big)\sin(\frac{lt}{2})(v_1^2 + v_2^2)/l^4. \tag{8}$$

Since $\det P(t_1) = 0$ we get two series of conjugate points. For the conjugate points of first series $\sin\frac{lt}{2} = 0$ and $t_k = 2\pi k/l$, $k \in \mathbb{Z}$. The first conjugate point of this series is $t_1 = 2\pi/l$ (Fig. 1). The appropriate Jacobi field is defined by (7) with the parameters

$$a_1 = v_2, \quad a_2 = -v_1, \quad \lambda = 0. \tag{9}$$

For the conjugate points of second series $lt\cos\frac{lt}{2} - 2\sin\frac{lt}{2} = 0$. Approximately $t_n \approx \pm(\pi + 2\pi n)/l$, $n \in \mathbb{N}$. The first conjugate point of this series is $t_1 \approx 8.99/l$ (Fig. 2). The appropriate Jacobi field is defined by (7) with the parameters

$$a_1 \approx 4.49 v_2/l, \quad a_2 \approx -4.49 v_1/l, \quad \lambda = 1. \tag{10}$$

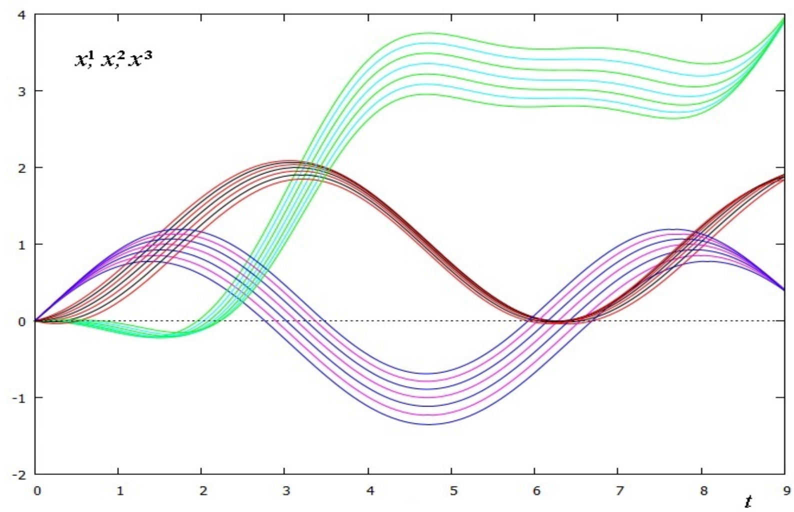

Fig. 2. Conjugate point of second series $t_1 \approx 8.99$.

The equations (2) define the exponential map for the geodesics with the initial velocity vector $\mathbf{v} = (v_1, v_2)$ and Lagrange multiplier l: $x(t) = \exp_0^l(t\mathbf{v})$. This exponential map was studied in [9] and its differential was found. Let us consider the family of paths

$$y(t,\tau) = \exp_0^{l+\tau\mu}(t(\mathbf{v} + \tau\mathbf{b})), \tag{11}$$

where $\mathbf{b} = (b_1, b_2)$. This is the variation of the geodesic $x(\cdot)$. Let us find the derivatives

$$\left.\frac{\partial y_1}{\partial \tau}\right|_{\tau=0} = -\big((l\mu t\sin lt + \mu\cos lt - \mu)v_2 + (\mu\sin lt - l\mu t\cos lt)v_1 - \\ b_1 l\sin lt - b_2 l\cos lt + b_2 l\big)/l^2,$$

$$\left.\frac{\partial y_2}{\partial \tau}\right|_{\tau=0} = -\big((\mu\sin lt - l\mu t\cos lt)v_2 + (-l\mu t\sin lt - \mu\cos lt + \mu)v_1 - \\ b_2 l\sin lt + b_1 l\cos lt - b_1 l\big)/l^2,$$

$$\left.\frac{\partial y_3}{\partial \tau}\right|_{\tau=0} = \big((\mu \sin 2lt - l\mu t \cos 2lt - l\mu t)v_2^2 +$$
$$((-2l\mu t \sin 2lt - 2\mu \cos 2lt + 2l\mu t \sin lt + 4\mu \cos lt - 2\mu)v_1 -$$
$$b_2 l \sin 2lt + b_1 l \cos 2lt - 2b_1 l \cos lt + 2b_2 l^2 t + b_1 l)v_2 +$$
$$(-\mu \sin 2lt + l\mu t \cos 2lt + 4\mu \sin lt - 2l\mu t \cos lt - l\mu t)v_1^2 +$$
$$(b_1 l \sin 2lt + b_2 l \cos 2lt - 4b_1 l \sin lt - 2b_2 l \cos lt + 2b_1 l^2 t + b_2 l)v_1\big)/(2l^3).$$

The rank of exponential geodesic map is not its maximum rank at a conjugate point. If t_1 is a point conjugated with the initial point then $\left.\frac{\partial y}{\partial \tau}\right|_{\tau=0}(t_1) = 0$ for some μ and $\mathbf{b}$. For the conjugate points of first series this is

$$\begin{cases} (2\pi\mu v_1)/l^2 = 0 \\ (2\pi\mu v_2)/l^2 = 0 \\ (2\pi(\mu(v_1^2 + v_2^2) - l(b_2 v_2 + b_1 v_1)))/l^3 = 0. \end{cases} \tag{12}$$

Therefore $\mu = 0$ and $b_2 = -b_1 v_1/v_2$. This means that the Lagrange multiplier l is fixed and $\mathbf{b} \perp \mathbf{v}$ matching (9).

For the conjugate points of second series the equation $\left.\frac{\partial y}{\partial \tau}\right|_{\tau=0}(t) = 0$ should be solved numerically. For the point $t_1 \approx 8.99/l$ the solution is $b_1 \approx 4.49 v_2 \mu/l$, $b_2 \approx -4.49 v_1 \mu/l$, $\mu \neq 0$ matching (10). With this solution in mind we can draw the geodesic variation with these parameters. This variation is shown on Fig. 2 for $l = 1$. The endpoints of the geodesics are close to each other at $t_1 \approx 8.99/l$ as should be expected for a conjugate point.

4 The Schouten Curvature and the Schouten–Vranceanu Connection

Let us consider the distribution $\mathcal{A}$ of dimension m on a smooth manifold N of dimension n. The coordinates x^k, $k = 1, \ldots, n$, on an open and sufficiently small domain $U \subset N$ can be chosen so as to maximize the projection of the distribution $\mathcal{A}$ on the first m coordinates. Then the basis of the distribution $\mathcal{A}$ can be chosen as

$$e_k = \partial_k - \sum_{\alpha=m+1}^{n} A_k^\alpha \partial_\alpha, \quad k = 1, \ldots, m, \tag{13}$$

where $\partial_k = \frac{\partial}{\partial x^k}$ are the coordinate vector fields. The functions A_k^α will be called *the potentials of the distribution*. We assume that they are C^1-smooth. The distribution $\mathcal{A}$ can be defined also by differential 1-forms $\omega^\alpha = \sum_{s=1}^{m} A_s^\alpha dx^s + dx^\alpha$, $\alpha = m+1, \ldots, n$. Here and further Latin indexes are in the range $1, \ldots, m$ and Greek indexes are in the range $m+1, \ldots, n$. The Lie brackets $[e_i, e_j] = \sum_{k=1}^{m} c_{ij}^k e_k +$

$\sum\limits_{\alpha=m+1}^{n} c_{ij}^{\alpha}\partial_{\alpha}$. According to the choice (13) the only non-zero components are $F_{ij}^{\alpha} = c_{ij}^{\alpha}$. The tensor F_{ij}^{α} is the tensions tensor.

For any point $x \in N$ the quadratic form $\langle \cdot, \cdot \rangle_x$ is defined on $\mathcal{A}(x)$. The metric tensor in the basis (13) is

$$g_{ij}(x) = \langle e_i, e_j \rangle_x. \tag{14}$$

To define covariant differentiation ∇ on the distribution, we must introduce a symmetric Riemannian connection. The property of being Riemannian is defined in a standard way:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{15}$$

while the symmetry condition must be modified as

$$\nabla_X Y - \nabla_Y X = \mathrm{pr}([X, Y]), \tag{16}$$

where $\mathrm{pr} = \sum\limits_{k=1}^{m} e_k \otimes dx^k$ is the (horizontal) projection on the distribution. To make this projection invariant with respect to transformations of coordinates, we must impose the following constraints on the smooth structure of the manifold: $\dfrac{\partial x^k}{\partial y^\alpha} = 0$, $k = 1, \ldots, m$, $\alpha = m+1, \ldots, n$, and $\big(\dfrac{\partial x^\beta}{\partial y^\alpha}\big)_{\alpha,\beta=m+1,\ldots,n}$ is the identity matrix [13, 14, 16]. The coordinates x^α will be called *verticals.* The differentials of the transfer maps $h_U \circ h_V^{-1}$ for all charts $h_U : U \to \mathbb{R}^n$, $U, V \subset N$, admissible in this smooth structure are the block matrices

$$\begin{pmatrix} \left(\dfrac{\partial x^i}{\partial y^j}\right) & \left(\dfrac{\partial x^\alpha}{\partial y^j}\right) \\ \left(\dfrac{\partial x^i}{\partial y^\beta}\right) & \left(\dfrac{\partial x^\alpha}{\partial y^\beta}\right) \end{pmatrix} = \begin{pmatrix} & * \\ \star & \vdots \\ & * \\ 0 \ldots 0 & I_{n-m} \end{pmatrix} \tag{17}$$

Since these matrices form a group our definition is correct. The basis vectors (13) change just as the coordinate basis of a manifold of dimension m: $e_i = \sum_{s=1}^{m} \dfrac{\partial y^s}{\partial x^i}\tilde{e}_s$. The potentials of the distribution are also subject to the gauge transformation: $\tilde{A}_j^\alpha = \sum_{s=1}^{m} A_s^\alpha \dfrac{\partial x^s}{\partial y^j} + \dfrac{\partial x^\alpha}{\partial y^j}$.

Since the connection ∇ is Riemannian and pr-symmetric (15), (16) we get

$$2\langle \nabla_X Y, Z\rangle = (X\langle Y, Z\rangle - \langle Y, \mathrm{pr}[X, Z]\rangle) + (Y\langle Z, X\rangle - \langle X, \mathrm{pr}[Y, Z]\rangle) - \\ - (Z\langle X, Y\rangle - \langle Z, \mathrm{pr}[X, Y]\rangle). \tag{18}$$

Due to (13) $\mathrm{pr}[e_i, e_j] = 0$. Let us assume that the metric tensor of the distribution does not depend on the vertical coordinates x^α. Then the equation (18) matches the Levi-Civita connection on a manifold: $\nabla_{e_i} e_j = \sum_{k=1}^{m} \Gamma_{ij}^k e_k$ and

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{s=1}^{m} g^{sk}\big(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}\big). \tag{19}$$

Theorem 1. *For a distribution of dimension m on a manifold with the smooth structure* (17) *the vectors of the basis* (13) *change over the coordinate transformations just as the coordinate basis of a manifold of the same dimension. If the distribution and its metric tensor do not depend on the vertical coordinates then the Riemannian and pr-symmetric connection ∇ matches the Riemannian and symmetric connection of a manifold with metric* (14)*.*

For any point $x \in N$ and any three vectors $u, v, w \in \mathcal{A}(x)$ *the curvature map of distribution* $\mathcal{A}$ at point x is defined by the Schouten tensor [23, 22, 6]

$$R(u,v)w = \nabla_{\tilde{u}}\nabla_{\tilde{v}}\tilde{w} - \nabla_{\tilde{v}}\nabla_{\tilde{u}}\tilde{w} - \nabla_{\mathrm{pr}[\tilde{u},\tilde{v}]}\tilde{w} - \mathrm{pr}\big[(1-\mathrm{pr})[\tilde{u},\tilde{v}],\tilde{w}\big], \qquad (20)$$

where $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are horizontal vector fields on a neighbourhood of x such that $\tilde{u}(x) = u$, $\tilde{v}(x) = v$, $\tilde{w}(x) = w$. The curvature does not depend on the way of expansion of u, v, w to vector fields. Assume that $\tilde{w}$ does not depend on vertical coordinates x^α and consider $R(e_i, e_j)e_k$ for the basis (13). The horizontal projection of the Lie derivative on the vertical vector field is zero: $\mathrm{pr}\big[(1-\mathrm{pr})[e_i, e_j], e_k\big] = 0$, because e_k does not depend on x^α and the result of differentiation of the Lie bracket $[e_i, e_j]$ on e_k has the zero horizontal projection. Therefore the equation (20) can be simplified: $R(e_i, e_j)e_k = \nabla_{e_i}\nabla_{e_j}e_k - \nabla_{e_j}\nabla_{e_i}e_k - \nabla_{\mathrm{pr}[e_i,e_j]}e_k$. The term $\nabla_{\mathrm{pr}[e_i,e_j]}e_k = 0$ since $\mathrm{pr}[e_i, e_j] = 0$. The corresponding term in Riemannian geometry is also zero for any coordinate vector fields since $[\partial_i, \partial_j] = 0$. Final equation for the curvature of distribution in the basis (13) matches the equation for the curvature of a manifold: $R_{ijkl} = \langle R(e_k, e_l)e_j, e_i\rangle$ and

$$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \sum_{s=1}^{m}\big(\Gamma^s_{lj}\Gamma^i_{ks} - \Gamma^s_{kj}\Gamma^i_{ls}\big). \qquad (21)$$

Theorem 2. *Let the distribution be defined on a manifold with the smooth structure* (17)*. If the distribution and its metric tensor do not depend on the vertical coordinates then its Schouten tensor matches the Riemannian curvature tensor of a manifold with metric* (14)*.*

"Look inside": we cannot distinguish a distribution from a manifold locally. We know that a distribution is involved in our physics because the equations of motion of a charged particle are just the equations of horizontal geodesics [8, 15]. The potentials A_j for a 4-dimensional distribution on a 5-manifold are just the 4-potentials of the electromagnetic field. The existence of these potentials is confirmed by the Aharonov—Bohm effect [1]. The Maxwell equations and the Dirac equation use this 4-potential and there is a problem how to find soliton-like solution for the 4-potential satisfying these equations [4, 5]. If the distribution and its metric tensor do not depend on the vertical coordinates we say that the distribution satisfies the *cyclicity condition.*

5 The Equations of Geodesics

Let the distribution $\mathcal{A}$ on a manifold N^n be defined by the differential 1-forms $\omega^\alpha = \sum\limits_{s=1}^{m} A^\alpha_s dx^s + dx^\alpha$, $\alpha = m{+}1, \ldots, n$. The geodesics equations for a distribution with the cyclicity condition are [10]

$$a_0 \frac{D\gamma'}{dt} + \sum_{\alpha=m+1}^{n} l_\alpha \hat{F}^\alpha \gamma' = 0, \tag{22}$$

where $\frac{D}{dt}$ – covariant derivative, (a_0, l) – the Lagrange multipliers, which cannot be altogether zero. A geodesic γ is called *normal (or regular)*, iff there is the only one set of multipliers (a_0, l) with $a_0 = 1$ for γ. Operator $\hat{F}^\alpha$ is the tensions tensor $F^\alpha_{ij} = \partial_j A^\alpha_i - \partial_i A^\alpha_j$ with the second index raised by the inverse metric tensor of the distribution.

Example 1. Let the 2-dimensional distribution in $\mathbb{R}^3$ be defined by $\omega = -y^2 dx + dz$. The tensions tensor $F = \begin{pmatrix} 0 & -2y \\ 2y & 0 \end{pmatrix}$. Non-trivial abnormal geodesics appear at $y = 0$. It can be any absolutely continuous function $x = x(t)$ with $z = \text{const}$ due to $\omega(\gamma') = 0$. Since any path with backward movements cannot be optimal, we can choose parametrization so that

$$\begin{cases} x = x_0 + ct \\ y = 0 \\ z = \text{const} \end{cases}. \tag{23}$$

N.N. Petrov [19–21] and R. Montgomery [18] proved independently that if this path is short enough it is a solution to the minimization problem for some length functional.

Example 2. Let the 2-dimensional distribution in $\mathbb{R}^3$ be defined by $\omega = f(x)g(y)dx - xdy + dz$, where f, g are some sufficiently smooth functions. The tensions tensor is

$$F = \begin{pmatrix} 0 & f(x)g'(y) + 1 \\ -f(x)g'(y) - 1 & 0 \end{pmatrix}. \tag{24}$$

If the equation $f(x)g'(y) + 1 = 0$ has a continuous solution for all $y \in [y_0, y_1]$ with some y_0, y_1, then this is an abnormal geodesic. Since $x = f^{-1}(-1/g'(y))$, we assume that $g' : [y_0, y_1] \to \mathbb{R}$ is a diffeomorphism to its image and there is the inverse function f^{-1} continuous in the domain of $-1/g'$. Let us choose $g(y) = y^2/2$ and $f(x) = x$, then $x = -1/y$. If $y = y_0 + ct$, then $x = -1/(y_0 + ct)$ where $c = \text{const} \neq 0$ and $t \neq -y_0/c$. The coordinate $z = z(t)$ is defined by the horizontality condition $dz = xdy - \frac{1}{2}xy^2 dx = -\frac{1}{y}dy + \frac{y}{2}d(-\frac{1}{y}) = -\frac{1}{y}dy + \frac{1}{2y}dy = -\frac{1}{2y}dy$. Hence, $z(t) = z_0 - \frac{1}{2}\ln|y_0 + ct|$. We proved that this distribution admits the abnormal geodesics

$$\begin{cases} x = -\dfrac{1}{y_0 + t} \\ y = y_0 + t \\ z = z_0 - \frac{1}{2}\ln|y_0 + t| \end{cases}, \quad t \neq -y_0, \quad y_0, z_0 \in \mathbb{R}. \tag{25}$$

It is an open question whether some of these abnormal geodesics are solutions of the minimization problem for some length functional on the distribution.

Further we consider regular geodesics only.

6 The Jacobi Equation

Let us assume that both the distribution and the metric tensor of the distribution are independent of vertical coordinates (the cyclicity condition). Hence the Lagrange multipliers are time-independent (constant). The Jacobi equation for this class of distributions can be written in geometric covariant form. The distribution is assumed to be totally nonholonomic. The Hilbert condition [2] for geodesics is always fulfilled in this theory (all geodesics are non-singular).

Consider the minimization problem for the energy functional $J(\gamma) = \frac{1}{2}\int_{t_0}^{T}\langle\gamma',\gamma'\rangle\,dt$ for horizontal paths with fixed endpoints and time. This is the Lagrange problem. We shall omit the subscript α and the summation sign in sums involving Lagrange multipliers such as $\sum\limits_{\alpha=m+1}^{n} l_\alpha\omega^\alpha$ and $\sum\limits_{\alpha=m+1}^{n} l_\alpha F^\alpha$.

By definition, a two-parameter variation σ of a geodesic γ is a two-parameter family of curves $\sigma(t,\mu,\tau)$ such that $\sigma(t,0,0)\equiv\gamma(t)$ [2]. A geodesic γ and its variation σ are assumed to be C^2-smooth. In addition, we assume that the third derivatives $\dfrac{\partial^3\sigma}{\partial\mu\partial t\partial\tau}$ and $\dfrac{\partial^3\sigma}{\partial t\partial\mu\partial\tau}$ do exist for all admissible t, μ, τ and are continuous along γ. In the paper [11], it was shown that the second variation of the energy functional for a distribution is as follows: $\dfrac{\partial^2 J}{\partial\mu\partial\tau}\Big|_{\substack{\tau=0\\ \mu=0}} = \big\langle\gamma',\dfrac{DZ}{\partial\mu}\big\rangle\Big|_{t_0}^{T} + l\dfrac{\partial(\omega(Z))}{\partial\mu}\Big|_{t_0}^{T} + I(Y,Z)$, where $Y=\dfrac{\partial\sigma}{\partial\mu}$ and $Z=\dfrac{\partial\sigma}{\partial\tau}$. The functional

$$I(Y,Z)=\int_{t_0}^{T}\Big(\Big\langle\frac{DY}{dt},\frac{DZ}{dt}\Big\rangle-\big\langle R(Y,\gamma')\gamma',Z\big\rangle\Big)dt-\\ -\int_{t_0}^{T} l\Big((\nabla_Y F)(\gamma',Z)+F\big(\frac{DY}{dt},Z\big)\Big)dt \quad (26)$$

is called the *index form of a geodesic γ with Lagrange multipliers l*. The vector fields $Y(\cdot,0,0)$ and $Z(\cdot,0,0)$ along γ will be denoted by the same letters Y, Z. If one of the fields Y or Z is vertical, then $I(Y,Z)=0$.

The metric tensor of a distribution is positively definite in sub-Riemannian geometry. Therefore the functional $I(Y,Y)>0$ for variations of geodesics which are sufficiently short. It is one of the necessary conditions of optimality. The optimality may be lost if $I(Y,Y)=0$ and it will be lost if $I(Y,Y)<0$. To find critical variations we should consider the minimization problem for the functional

If the distribution is integrable in some sence, *i.e.* the sequence of commutators (the flag of the distribution) does not span the whole tangent bundle of the manifold, then some vertical coordinates of Jacobi fields are dependend and the fundamental matrix of the Jacobi equation is degenerated for all points.

$I(Y,Y)$ with the constrains $\Phi^\alpha(Y',Y)=0$ (1). Hence we should minimize

$$I_\lambda(Y)=\frac{1}{2}\int_{t_0}^{T}\Big(\Big\langle\frac{DY}{dt},\frac{DY}{dt}\Big\rangle-\big\langle R(Y,\gamma')\gamma',Y\big\rangle\Big)dt-$$
$$-\frac{1}{2}\int_{t_0}^{T}l\Big((\nabla_Y F)(\gamma',Y)+F\big(\frac{DY}{dt},Y\big)\Big)dt+\int_{t_0}^{T}\lambda\Big(\sum_{k=1}^{n}\omega_k\frac{dY^k}{dt}+\sum_{j,k=1}^{n}\frac{\partial\omega_k}{\partial x^j}\gamma'^kY^j\Big)dt.$$

Consider the variation $Y\mapsto Y+\delta Y$ and collect linear for δY terms. We get

$$\delta I_\lambda(Y)=\int_{t_0}^{T}\Big(\Big\langle\frac{DY}{dt},\frac{D\delta Y}{dt}\Big\rangle-\big\langle R(Y,\gamma')\gamma',\delta Y\big\rangle\Big)dt-$$
$$-\frac{1}{2}\int_{t_0}^{T}l\Big((\nabla_{\delta Y}F)(\gamma',Y)+(\nabla_Y F)(\gamma',\delta Y)+F\big(\frac{D\delta Y}{dt},Y\big)+F\big(\frac{DY}{dt},\delta Y\big)\Big)dt+$$
$$+\int_{t_0}^{T}\lambda\Big(\sum_{k=1}^{n}\omega_k\frac{d\delta Y^k}{dt}+\sum_{j,k=1}^{n}\frac{\partial\omega_k}{\partial x^j}\gamma'^k\delta Y^j\Big)dt. \quad (27)$$

Assuming that the vector field Y is C^1-smooth and that the derivative $\frac{DY}{dt}$ is absolutely continuous, we have

$$\delta I_\lambda(Y)=\big\langle\frac{DY}{dt},\delta Y\big\rangle\Big|_{t_0}^{T}-\int_{t_0}^{T}\Big(\Big\langle\frac{D}{dt}\frac{DY}{dt},\delta Y\Big\rangle+\big\langle R(Y,\gamma')\gamma',\delta Y\big\rangle\Big)dt-$$
$$-\frac{l}{2}F\big(\delta Y,Y\big)\Big|_{t_0}^{T}-\frac{1}{2}\int_{t_0}^{T}l\Big((\nabla_{\delta Y}F)(\gamma',Y)+(\nabla_Y F)(\gamma',\delta Y)-(\nabla_{\gamma'}F)\big(\delta Y,Y\big)\Big)dt-$$
$$-\frac{1}{2}\int_{t_0}^{T}l\Big(F\big(\delta Y,\frac{DY}{dt}\big)-F\big(\frac{DY}{dt},\delta Y\big)\Big)dt+\lambda\omega(\delta Y)\Big|_{t_0}^{T}+\int_{t_0}^{T}\lambda F\big(\delta Y,\gamma'\big)dt. \quad (28)$$

Since $F=d\omega$, we get $(\nabla_{\delta Y}F)(\gamma',Y)+(\nabla_Y F)(\gamma',\delta Y)-(\nabla_{\gamma'}F)\big(\delta Y,Y\big)=(\nabla_{\delta Y}F)(\gamma',Y)+(\nabla_{\gamma'}F)\big(Y,\delta Y\big)-(\nabla_Y F)(\delta Y,\gamma')=-2(\nabla_Y F)(\delta Y,\gamma')$.

Definition 1. *A pair $(\tilde Y,\lambda)$, where $\tilde Y$ is a vector field along a geodesic γ with Lagrange multipliers l, will be called a Jacobi field if $\tilde Y$ satisfies the variations equations* (1) *and its horizontal projection $Y=\operatorname{pr}\tilde Y$ satisfies the nonholonomic Jacobi equation*

$$\frac{D}{dt}\frac{DY}{dt}+R(Y,\gamma')\gamma'+l\hat F\big(\frac{DY}{dt}\big)+l(\nabla_Y\hat F)(\gamma')+\lambda\hat F(\gamma')=0. \quad (29)$$

The $\hat F$ operator is the tensions tensor F with the second index raised by the inverse metric tensor of the distribution. We assume that both the distribution and the metric tensor of the distribution are independent of vertical coordinates (the cyclicity condition). Hence the Lagrange multipliers (both l_α and λ_α) are time-independent. The equations (1), (29) together with $\lambda'=0$ are a system of linear homogeneous differential equations with the variables $(\tilde Y,\lambda)$. The set of solutions of this system is a linear space. Therefore there are two types of Jacobi fields. For the first type $\lambda \equiv 0$ (zero vector). For the second type of Jacobi fields $\lambda \neq 0$.

A horizontal vector field Y along a geodesic γ with Lagrange multipliers l will be called a *horizontal Jacobi field* iff it satisfies (29) with some multipliers λ.

Definition 2. *Points $t_1, t_2 \in [t_0, T]$, $t_1 \neq t_2$, are said to be conjugated along a horizontal geodesic γ if there exists a nontrivial Jacobi field Y (with some λ) along γ which vanishes at these points: $Y(t_1) = 0$ and $Y(t_2) = 0$.*

A smooth variation $\sigma(\cdot,\cdot) : [t_0, T] \times (-\epsilon, \epsilon) \to N$ is a *geodesic variation* iff all its longitudinal lines are geodesics.

Lemma 1. *If $\sigma(\cdot,\cdot) : [t_0, T] \times (-\epsilon, \epsilon) \to N$ is a geodesic variation then the horizontal projection of the field $\dfrac{\partial\sigma}{\partial\tau}$ is a horizontal Jacobi field along each longitudinal line of this variation.*

Proof. Let $X = \dfrac{\partial\sigma}{\partial t}$, then due to the equations of geodesics $\frac{DX}{\partial t} + l(\tau)\hat{F}X = 0$. If a field Y is horizontal then $R(Y,X)X = \frac{D}{\partial\tau}\frac{DX}{\partial t} - \frac{D}{\partial t}\frac{DX}{\partial\tau}$ [11]. Let $Y = \mathrm{pr}\big(\dfrac{\partial\sigma}{\partial\tau}\big)$, therefore we can continue this equation $R(Y,X)X = \frac{D}{\partial\tau}(-l(\tau)\hat{F}X) - \frac{D^2Y}{\partial^2 t} = -l'_\tau\hat{F}X - l(\tau)(\nabla_Y\hat{F})X - l(\tau)\hat{F}\frac{DY}{\partial t} - \frac{D^2Y}{\partial^2 t}$. Assigning $\lambda = l'_\tau$ we get (29). □

Theorem 3. *Let γ be a geodesic with the origin $x_0 = \gamma(t_0)$ and endpoint $x_1 = \gamma(T)$. The point $x_1 = \exp^l_{x_0}(u)$ is conjugated with the point x_0 along γ iff the rank of the differential $d_{(u,l)}\exp_{x_0}$ is not its maximum, i.e. iff (u,l) is a critical point of the mapping $(u,l) \mapsto \exp^l_{x_0}(u)$.*

Proof. The Jacobi equation which appears in the Bliss accessory problem is the result of the linearisation of geodesics equations

$$\begin{cases} \gamma' = X \\ \frac{DX}{dt} + l\hat{F}X = 0 \\ \omega(X) = 0 \end{cases} . \tag{30}$$

The fundamental matrix in the Bliss problem matches the matrix of the differential $d_{(u,l)}\exp_{x_0}$, because this differential satisfies the linearised geodesics equations as the result of differentiation of a solution of a differential equation by initial conditions [17, p. 289], [7]. Since we consider distributions with the cyclicity condition the index form $I(Y,Z)$ depends on the horizontal projections of fields Y, Z only. Hence the horizontal projection of a Jacobi field in the Bliss problem satisfies (29). Therefore we can consider a (horizontal) solution of the equation (29) and find its vertical components by means of the variations equation (1). The fundamental matrix of the obtained solution matches the fundamental matrix of the Bliss problem and therefore the differential matrix $d_{(u,l)}\exp_{x_0}$. □

In the next part of this paper we consider Jacobi fields of the first type.

The Jacobi fields of first type ($\lambda \equiv 0$).

Lemma 2. *If Y, Z are horizontal Jacobi fields along a geodesic γ with Lagrange multipliers l, then $f = \langle Y, Z'\rangle - \langle Y', Z\rangle - lF(Y,Z)$ is a constant function.*

Proof. $f' = \langle Y', Z'\rangle + \langle Y, Z''\rangle - \langle Y'', Z\rangle - \langle Y', Z'\rangle - l(\nabla_{\gamma'}F)(Y,Z) - lF(Y',Z) - lF(Y,Z') = -\langle Y, R(Z,\gamma')\gamma' + l(\nabla_Z \hat{F})\gamma' + l\hat{F}Z'\rangle + \langle R(Y,\gamma')\gamma' + l(\nabla_Y \hat{F})\gamma' + l\hat{F}Y', Z\rangle - l(\nabla_{\gamma'}F)(Y,Z) - lF(Y',Z) - lF(Y,Z') = -l(\nabla_Z F)(\gamma', Y) - l(\nabla_Y F)(Z,\gamma') - l(\nabla_{\gamma'}F)(Y,Z) = -ldF(\gamma', Y, Z) = 0.$ □

Hence the Jacobi fields preserve the definite antisymmetric form. Note that the vector field $X = \gamma'$ is a horizontal Jacobi field. Indeed due to the geodesics equations $X' = -l\hat{F}X$, therefore $X'' = -l(\nabla_X \hat{F})X - l\hat{F}X'$, and this is the equation (29) (the curvature contribution $R(X,X)X = 0$).

Applying Lemma 2 to an arbitrary horizontal Jacobi field Y and γ' we obtain $\langle Y, \gamma''\rangle - \langle Y', \gamma'\rangle - lF(Y,\gamma') = C$. Since γ is a horizontal geodesic, $\gamma'' = -l\hat{F}\gamma'$ and $\langle Y', \gamma'\rangle = \text{const}$.

Lemma 3. *A geodesic $\gamma : [t_0, T] \to N$ may contain only a finite number of points which are conjugated to t_0.*

The proof is similar to that given in [3, p. 149].

If a (regular) geodesic $\gamma : [t_0, T] \to N$ is a solution to the minimization problem for the functional J, then the functional $I_\lambda(Y)$ is non-negative for any vector field Y along γ satisfying (1).

Theorem 4. *Let $\gamma : [t_0, T] \to N$ be a geodesic and the semi-interval $(t_0, T]$ does not contain points conjugated with t_0. Then the functional $I_\lambda(Y)$ is positively definite for all vector fields Y along γ satisfying* (1) *wich are zero at the endpoints of γ.*

Theorem 5. *Let the cyclicity condition be satisfied for a distribution. Suppose that the metric tensor of a distribution is positive definite, a (regular) geodesic γ connects two given points x_0 and x_1, and there are no points on the semi-interval $(t_0, T]$ that are conjugated to t_0. Then, on the path γ, the energy functional attains its weak local minimum in the problem with fixed endpoints.*

The proof is given in [12].

7 Conclusion

Hence we propose the Jacobi equation for horizontal geodesics on a distribution in sub-Riemannian geometry which involves the curvature tensor of a distribution and its tensions tensor. The classical variational theory treats this equation as a sum of derivatives of the given functional. We established the geometric sense of these sums in terms of well-defined geometric objects. The Jacobi equation (29) is applicable to regular geodesics. Since there are also abnormal geodesics in the sub-Riemannian geometry, we discussed two examples of such geodesics. There is one more way where geodesics may loss its optimality: it is the case when two given points can be connected by more than one geodesic. We could not discuss this case here yet it should be always noted.

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