Search of Nash Equilibrium in Quadratic Nonconvex Game with Weighted Potential*

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Abstract. We consider an n-player nonconvex continuous game with quadratic payoffs, multi-dimensional strategy spaces, and possibly shared constraints on strategies, and investigate conditions when this game admits a weighted potential. Since a potential is generally nonconvex in this case, we propose local and global search procedures for maximizing it over the set of admissible game profiles. The local search uses nonlinear support functions that are constructed through a d.c.-decomposition of the potential. The global search is based on reducing of a certain nonconvex quadratic programming problem to a mixed-integer linear programming problem.

Keywords: Nash equilibrium · Weighted potential · Nonconvex optimization · D.C decomposition

1 Introduction

One of the main features of potential games in the sense of computing Nash equilibria is a possibility of reducing the game to an optimization problem. More exactly, in a potential game the set of maxima of a certain function is a subset of the set of Nash equilibria. This function is called potential. Generally a potential is a nonconvex function. In this case the set of local non-global maxima may contain equilibrium points as well. For the games with differentiable payoffs, even a stationary point could be suspected to be an equilibrium. Thus two crucially important questions arise in practice: how to determine the existence of a potential, and how to specify this function. Both of these issues are solved for the differentiable case of exact potential games with scalar strategies [15]. Weighted potential games generalize exact potential games by introducing a positive weight for every player. The class of weighted potential games turns out to have properties, which are very similar to those of exact potential games.

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The class of exact potential games was introduced in [15], along with weighted, ordinal, and generalized ordinal potential games. Later even more general classes were investigated, such as best-response potential games [19], and pseudo-potential games [4]. A common property of all these classes stays the same: they admit an optimization problem statement that noticeably makes easier an analysis of the game. A survey on different classes of potential games, on relations between them, as well as examples of applications one can find in [12, 7, 13] (see also the references therein). In this study we are interested in continuous games, i.e. games with continuous sets of strategies. For games with twice continuously differentiable payoffs and interval strategy sets, a useful characteristic for identifying the existence of a potential was established [15]. Such a characteristic can be easily extended for weighted potential games with interval strategy sets. For ordinal and generalized ordinal potential games with the payoffs of the same class and with multi-dimensional strategies, necessary conditions were formulated in [5].

For the cases in which either a game is non-potential or an identification whether a potential exists is complicated, there are methods of variety of types that do not exploit potential optimization for computing Nash equilibria. For instance, the methods among them are gradient-type algorithms [17, 21, 1], relaxation algorithms [20, 8], Newton-type method [9, 3], nonlinear support function method [10, 14], KKT-conditions-based method [2], and many others. It is worth to note that convexity-type assumptions usually play significant role for the convergence properties of equilibrium computation algorithms. The standard assumption widely used is player-concavity of payoffs, i.e. every player's payoff function must be concave with respect to this player's strategic variable. Games with this assumption being violated we call nonconvex games, in contrast to convex games [16]. Also there exists a rather general approach to reducing an equilibrium problem to a global optimization problem [10] using the so-called Nikaido-Isoda function [16]. It is noticable that this approach in general does not employ any of convexity presumptions. The underlying method in [10] is based on the concept of nonlinear support function. Later such a technique was applied to quadratic nonconvex non-potential games [14]. As we see further, existence of a weighted potential can also be guaranteed without convexity-type conditions on payoffs.

The present paper focuses on n-player nonconvex games with quadratic payoffs and convex compact set of game profiles. A strategy of every player is assumed to be a multi-dimensional value. We admit that the strategy set of every player depends on chosen strategies of rival players through shared constraints. These constraints restrain the set of admissible profiles. Recall, the problem of finding Nash equilibrium in non-cooperative game with coupled strategy sets is called generalized Nash equilibrium problem (GNEP). In contrast, if the strategy sets are independent, the problem is referred to as Nash equilibrium problem (NEP); see [17] for details. We discuss a differentiable characterization of weighted potential games with vector strategies, and provide explicit expression of potential function. Then we formulate these results specifically for quadratic games. The potential turns out to be nonconvex in this statement. Hence we propose a local search and a global search methods for maximizing a potential function over the set of game profiles.

2 Weighted Potential Games

Consider a non-cooperative n-person game. The set of players is $N = \{1, 2, \ldots, n\}$, a strategy of ith player represents m_i-dimensional vector, so as the set of admissible game profiles is $X \subset \mathbb{R}^m$, where $m = m_1 + \cdots + m_n$, and $\mathbb{R}$ denotes the set of real numbers. Payoff function of ith player is $f_i \colon X \to \mathbb{R}$. We suppose that X is a convex compact set, and f_i is twice continuously differentiable for every $i \in N$. The problem is to find generalized Nash equilibrium (or, shortly, Nash equilibrium) for the given game, i.e. to find a profile $x^* \in X$ meeting the following conditions:

$$f_i(x_i, x^*_{-i}) \leqslant f_i(x^*) \quad \forall x_i \colon \ (x_i, x^*_{-i}) \in X \ , \quad \forall i \in N \ , \tag{1}$$

where $(x_i, x^*_{-i}) = (x^*_1, \ldots, x^*_{i-1}, x_i, x^*_{i+1}, \ldots, x^*_n)$. Let us denote the game by $\Gamma = \langle N, X, \{f_i\}_{i \in N} \rangle$.

Recall that the game Γ is called a weighted potential game if positive weights $w_1, \ldots, w_n$ and weighted potential function $P \colon X \to \mathbb{R}$ exist such that for all $i \in N$ and for all y_i, z_i, x_{-i}: $(y_i, x_{-i}) \in X$, $(z_i, x_{-i}) \in X$ the following relation holds [15]:

$$f_i(y_i, x_{-i}) - f_i(z_i, x_{-i}) = w_i \left(P(y_i, x_{-i}) - P(z_i, x_{-i}) \right) \ . \tag{2}$$

If $w_1 = w_2 = \cdots = w_n$, then Γ is called an exact potential game and P is called an exact potential. In order to specify certain values of weights in (2) we will say that the game is w-potential game, and P is w-potential, where $w = (w_1, \ldots, w_n)$. The next useful results are well known for differentiable games with scalar strategies. Let $m_1 = m_2 = \cdots = m_n = 1$. If payoffs are continuously differentiable on X, then condition (2) can be replaced by

$$\frac{\partial f_i(x)}{\partial x_i} = w_i \frac{\partial P(x)}{\partial x_i} \ . \tag{3}$$

Moreover, if payoffs are twice continuously differentiable then Γ is a weighted potential game if and only if positive numbers $v_1, \ldots, v_n$ exist such that for all $i \in N$, $j \in N$ $(i \neq j)$, $x \in X$:

$$v_i \frac{\partial^2 f_i(x)}{\partial x_i \partial x_j} = v_j \frac{\partial^2 f_j(x)}{\partial x_j \partial x_i} \ , \tag{4}$$

If Γ is a weighted potential game, then a weighted potential function is expressed by

$$P(x) = \int_0^1 \sum_{i \in N} v_i \frac{\partial f_i(tx)}{\partial x_i} x_i \ dt \ . \tag{5}$$

More exactly, due to definition (2) the game Γ is $(1/v_1,\dots,1/v_n)$-potential in the latter case. Relation (3) at $w_1 = w_2 = \dots = w_n = 1$, and relations (4), (5) at $v_1 = v_2 = \dots = v_n = 1$ (for exact potential games) were provided in [15]. Extension of these results for weighted potential games with scalar strategies follows in a straightforward way.

In order to generalize the differentiable characterization (3–5) for multi-dimensional strategy spaces, let us introduce a game Γ' by replacing ith player in Γ by m_i players, each of them maximizing the same function f_i with respect to one of scalar variables from m_i-dimensional strategy of the ith player of Γ. Then the set of players in Γ' is $N' = \{1,\dots,m_1,m_1+1,\dots,m_1+m_2,\dots,m\}$. The first m_1 players try to maximize independently f_1 as their payoff, whereas the next m_2 players treat f_2 as a payoff, etc. The profile set in Γ' is X, and every player of Γ' operates one scalar variable.

Recall a definition from potential game theory useful for the present study. A path in X [15] with respect to Γ is a sequence of game profiles $(p^1,p^2,\dots)$ such that $p^k \in X$, $k = 1,2,\dots$, and for every $k \geqslant 1$ there exists a unique deviating player of Γ, say ith player, such that $p^k = (x_i, p^k_{-i})$ for some $x_i \neq p^{k-1}_i$, $(x_i, p^k_{-i}) \in X$. In the further discussion $x^{(i)}_k$ denotes the kth component of vector strategy x_i. Besides, $w' \in \mathbb{R}^m$ is a vector of weights, where $w_1 = w'_1 = w'_2 = \dots = w'_{m_1}$, $w_2 = w'_{m_1+1} = w'_{m_1+2} = \dots = w'_{m_1+m_2}$, etc.

The next result holds without continuity assumption on payoffs.

Theorem 1. *Let there exist a path $(p^1,p^2,\dots,p^K)$ in X with respect to Γ' connecting the profiles $p^1 = (y_i,x_{-i})$ and $p^K = (z_i,x_{-i})$ for every y_i, z_i, x_{-i} such that $p^1 \in X$, $p^K \in X$, and for every $i \in N$. Then the game Γ is a w-potential game if and only if the game Γ' is a w'-potential. If Γ and Γ' are a w-potential and a w'-potential games respectively, their weighted potentials coincide up to constant.*

Proof. If Γ is a w-potential game then (2) holds for some function $P\colon X \to \mathbb{R}$. For every $i \in N$ and for every $k \in \{1,2,\dots,m_i\}$ fix arbitrarily $y^{(i)}_{-k} = z^{(i)}_{-k} = x^{(i)}_{-k}$ such that there exists a scalar value $a^{(i)}_k\colon (a^{(i)}_k, x^{(i)}_{-k}, x_{-i}) \in X$. Then the condition (2) yields

$$
\begin{gathered}
f_i\big(y^{(i)}_k, x^{(i)}_{-k}, x_{-i}\big) - f_i\big(z^{(i)}_k, x^{(i)}_{-k}, x_{-i}\big) = \\
w_i\left(P\big(y^{(i)}_k, x^{(i)}_{-k}, x_{-i}\big) - P\big(z^{(i)}_k, x^{(i)}_{-k}, x_{-i}\big)\right) .
\end{gathered}
$$

The latter relation provides that Γ' is a w'-potential game with potential P.

Conversely, let Γ' be a w'-potential game. For every $i \in N$ choose arbitrary vectors y_i, z_i, x_{-i} such that $(y_i,x_{-i}) \in X$ and $(z_i,x_{-i}) \in X$. Define a finite path $(p^1,p^2,\dots,p^K)$ in X with respect to Γ' connecting the profiles $p^1 = (y_i,x_{-i})$ and $p^K = (z_i,x_{-i})$, where the unique deviators are the m_i players of Γ' with f_i as their payoff. In other words, $p^k = (p^k_i, x_{-i})$, $k = 2,3,\dots,K-1$. Then due to (2) the following relations take place:

$$\begin{aligned} f_i(y_i, x_{-i}) - f(p^2) &= w_i \left(P(y_i, x_{-i}) - P(p^2) \right) \ , \\ f_i(p^2) - f(p^3) &= w_i \left(P(p^2) - P(p^3) \right) \ , \\ &\dots \\ f_i(p^{K-2}) - f(p^{K-1}) &= w_i \left(P(p^{K-2}) - P(p^{K-1}) \right) \ , \\ f_i(p^{K-1}) - f(z_i, x_{-i}) &= w_i \left(P(p^{K-1}) - P(z_i, x_{-i}) \right) \ . \end{aligned}$$

Summing up these equalities we at once obtain that Γ is a w-potential game with P as a w-potential function. □

In particular, the assumption in Theorem 1 holds for games with independent strategy sets (for NEP). If this assumption is violated let us define an open convex neighborhood Ω of the set X, and suppose that payoffs and a potential are defined on Ω. Replace in Γ and Γ' the profile set X by Ω. Denote the derived games by Γ_Ω and Γ'_Ω respectively. It is obvious that if Γ_Ω is a weighted potential game, then Γ is also a weighted potential game with the same potential function, since $X \subset \Omega$. Moreover, the assumption of Theorem 1 holds for a game with the profile set Ω, as for every $x \in \Omega$ there exists a neighborhood $B(x)$ such that $B(x) \subset \Omega$.

Corollary 1. *If Γ'_Ω is a w'-potential game then Γ is a w-potential game with the same potential function as in Γ'_Ω.*

In what follows, we suppose that the assumption of Theorem 1 holds.

With Theorem 1 relations (3) and (4) are easily extended for games with vector strategies. Since payoffs in Γ are continuously differentiable, due to Theorem 1 the condition (3) for multi-dimensional strategies can be replaced by the following one:

$$\nabla_{x_i} f_i(x) = w_i \nabla_{x_i} P(x) \ . \tag{6}$$

Using the assumption that payoffs are twice continuously differentiable, Theorem 1 and (4) imply that Γ is a weighted potential game if and only if positive numbers $v_1, \dots, v_n$ exist such that for all $i \in N$, $j \in N$ $(i \neq j)$, $k \in \{1, \dots, m_i\}$, and $l \in \{1, \dots, m_j\}$, and for every feasible x:

$$v_i \frac{\partial^2 f_i(x)}{\partial x_k^{(i)} \partial x_l^{(j)}} = v_j \frac{\partial^2 f_j(x)}{\partial x_l^{(j)} \partial x_k^{(i)}} \ . \tag{7}$$

Condition (7) is equivalent to

$$v_i \nabla^2_{x_i, x_j} f_i(x) = v_j [\nabla^2_{x_i, x_j} f_j(x)]^\top \ .$$

If we define a multi-valued function

$$g_v(x) = (v_1 \nabla_{x_1} f_1(x), \dots, v_n \nabla_{x_n} f_n(x)) \ ,$$

then relation (7) is also equivalent to that the Jacobian $G_v(x)$ of $g_v(x)$ is symmetric for every feasible x.

Proposition 1. *If payoffs are twice continuously differentiable then the game Γ is a weighted potential game if and only if positive numbers $v_1, \ldots, v_n$ exists such that the Jacobian $G_v(x)$ of the function $g_v(x)$ is a symmetric matrix for every $x \in X$.*

Formula (5) of a potential function is also adopted for multi-dimensional strategies through Theorem 1.

Proposition 2. *If Γ is a $(1/v_1, \ldots, 1/v_n)$-potential game then a weighted potential for Γ is given by*

$$P(x) = \int_0^1 \sum_{i \in N} \sum_{k=1}^{m_i} v_i \frac{\partial f_i(tx)}{\partial x_k^{(i)}} x_k^{(i)} \; dt \; . \tag{8}$$

Note that existence of a continuous potential function implies existence of a Nash equilibrium in the game if the profile set X is compact.

3 Quadratic Games

Let ith player's payoff function in the game Γ be defined as

$$f_i(x) = x_i^\top \left(\frac{1}{2} B_i x_i + d_i \right) + \sum_{j \in N}^{j \neq i} x_i^\top C_{ij} x_j \; . \tag{9}$$

Without loss of generality matrices B_i are supposed to be symmetric. For payoffs (9) necessary and sufficient condition (7) for Γ to be a weighted potential game is presented by the following one:

$$v_i C_{ij} = v_j C_{ji}^\top \tag{10}$$

for some positive $v_1, \ldots, v_n$ and for every $i \in N$ and $j \in N$ $(i \neq j)$. As player-concavity of (9) is not used in the condition (10), weighted potential quadratic games include nonconvex games. Moreover, even the term in payoff, which does not depend on rival players variables, has no impact on the existence of a potential function. Indeed, the equality (10) plays the role of necessary and sufficient condition also for the game with the following payoffs:

$$f_i(x) = \widehat{f}_i(x_i) + \sum_{j \in N}^{j \neq i} x_i^\top C_{ij} x_j \; , \tag{11}$$

where $\widehat{f}_i$ generally is non-quadratic.

Proposition 3. *The game Γ with payoffs (11) is a weighted potential game if and only if positive numbers $v_1, \ldots, v_n$ exist such that for every $i \in N$ and $j \in N$ $(i \neq j)$ condition (10) holds.*

For instance, consider a 2-player game with payoffs (11) and with scalar strategies:

$$f_1(x) = \widehat{f}_1(x_1) + c_1 x_1 x_2 \ ,$$
$$f_2(x) = \widehat{f}_2(x_2) + c_2 x_1 x_2 \ .$$

Generally $c_1 \neq c_2$, therefore existence of an exact potential is not guaranteed. However we can always choose positive numbers v_1, v_2 such that $v_1 c_1 = v_2 c_2$ being true, if c_1 and c_2 are nonzero and have the same sign:

$$c_1 c_2 > 0 \ .$$

In particular, $v_1 = 1$, $v_2 = c_1/c_2$ would be appropriate. For an n-player game with multi-dimensional strategy sets, in order to ensure existence of a weighted potential in accordance with Proposition 1 we should provide $v > 0$ such that the Jacobian

$$G_v(x) = \begin{pmatrix} v_1 \nabla^2 \widehat{f}_1(x_1) & v_1 C_{1,2} & \dots & v_1 C_{1,n} \\ v_2 C_{2,1} & v_2 \nabla^2 \widehat{f}_2(x_2) & \dots & v_2 C_{2,n} \\ \dots & \dots & \dots & \dots \\ v_n C_{n,1} & v_n C_{n,2} & \dots & v_n \nabla^2 \widehat{f}_n(x_n) \end{pmatrix}$$

be symmetric.

If (10) holds, by Proposition 2 a $(1/v_1, \dots, 1/v_n)$-potential for Γ with payoffs (9) is given by

$$P(x) = \sum_{i \in N} v_i \left[x_i^\top \left(\frac{1}{2} B_i x_i + d_i \right) + \frac{1}{2} \sum_{j \in N}^{j \neq i} x_i^\top C_{ij} x_j \right] . \tag{12}$$

The validity of (12) one can examine directly through the verifying the equality (6).

Example 1 (quadratic payoffs [14]). Consider a 2-player nonconvex game with quadratic payoffs and scalar strategies:

$$f_1(x) = x_1^2 + x_1 x_2 \ , \quad f_2(x) = -x_2^2 + \frac{1}{2} x_1 x_2 \ , \quad X = [-1, 1] \times [-1, 1] \ .$$

Obviously the game does not admit exact potential. However with $v_1 = 1$, $v_2 = 2$ in (12) $(1, 1/2)$-potential is

$$P(x) = x_1^2 - 2x_2^2 + x_1 x_2 \ .$$

There are two Nash equilibria in the game: $x' = (1, 1/4)$ and $x'' = (-1, -1/4)$. Both of them provide maximal value to P on X. The vector field $(\partial f_1(x)/\partial x_1,$ $\partial f_2(x)/\partial x_2)$ and the plot of the weighted potential are presented at Fig. 1. Due to (6) the vector field coincides with $(\partial P(x)/\partial x_1, \partial P(x)/\partial x_2)$.

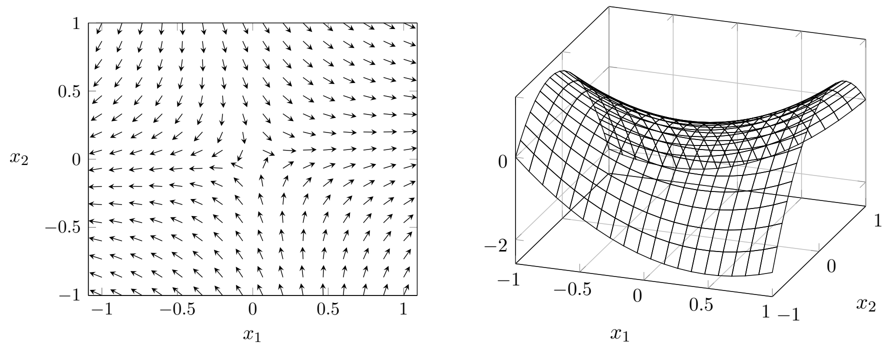

Fig. 1. Vector field $(\partial f_1(x)/\partial x_1, \partial f_2(x)/\partial x_2)$ (left) and plot of the weighted potential (right) in Example 1

To illustrate Proposition 3 consider an example with non-quadratic payoffs.

Example 2 (qubic payoffs). Define a 2-player game with payoffs (11), where $\widehat{f}_1(x_1)$ and $\widehat{f}_2(x_2)$ are qubic polynoms:

$$\begin{aligned} f_1(x) &= x_1^3 + 5x_1^2 - 10x_1 + 10x_1x_2 \ , \\ f_2(x) &= x_2^3 - 7x_2^2 + 13x_2 + 15x_1x_2 \ , \\ X &= [-10, 5] \times [-10, 5] \ . \end{aligned}$$

As in the previous example, the given game is not an exact potential. Setting $v_1 = 1$, $v_2 = 2/3$ at (8) we obtain $(1, 3/2)$-potential function:

$$P(x) = x_1^3 + 5x_1^2 - 10x_1 + \frac{2}{3}x_2^3 - \frac{14}{3}x_2^2 + \frac{26}{3}x_2 + 10x_1x_2 \ .$$

Potential P attains its maximum on X at $x' = (5, 5)$, $P(x') \approx 460$. Besides local non-global maximum exists: $x'' \approx (-5.73, -3.12)$, $P(x'') \approx 119.39$. One can easily verify by (1) that x'' is also a Nash equilibrium in the game. The vector field $(\partial f_1(x)/\partial x_1, \partial f_2(x)/\partial x_2) = (\partial P(x)/\partial x_1, \partial P(x)/\partial x_2)$ and the plot of the weighted potential are depicted at Fig. 2.

The next example shows that locally non-optimal stationary point of a potential may be a Nash equilibrium.

Example 3. Consider an exact potential game with $X \subseteq \mathbb{R}^2$ such that $(0, 0)$ is an interior point of X, and payoffs are defined as follows:

$$\begin{aligned} f_1(x) &= x_2x_1^3 - x_1^2 + x_2^3x_1 + 3x_1x_2 \ , \\ f_2(x) &= x_1x_2^3 - x_2^2 + x_1^3x_2 + 3x_1x_2 \ . \end{aligned}$$

A potential is given by

$$P(x) = x_1^3x_2 - x_1^2 + 3x_1x_2 + x_1x_2^3 - x_2^2 \ .$$

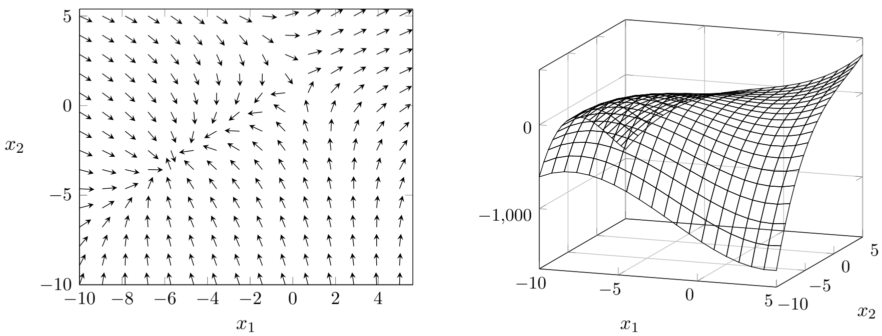

Fig. 2. Vector field $(\partial f_1(x)/\partial x_1, \partial f_2(x)/\partial x_2)$ (left) and plot of the weighted potential (right) in Example 2

The profile $x' = (0, 0)$ is an equilibrium since $f_1(x_1, 0) = -x_1^2$ and $f_2(0, x_2) = -x_2^2$. Besides $\nabla P(x') = 0$. However P increases along directions $d^1 = (-1, -1)$ and $d^2 = (1, 1)$ from x'. Indeed,

$$P(x' + td^1) = P(x' + td^2) = 2t^4 + t^2 > P(x') \qquad \text{for every} \quad t > 0 \, .$$

Hence x' is not locally optimal.

Therefore we are interested in maximizing $P(x)$ subject to $x \in X$, possibly locally, or at least finding a stationary point of the potential over X. To this end, a local search and a global search procedures will be examined further.

4 Local Search

As we discussed before, equilibria of a potential game should be searched among stationary points of a potential function. Hence it is reasonable to use a local ascending method in addition to a global search technique due to much less computational costs of the former one.

For a weighted potential quadratic game with weights $(1/v_1, \ldots, 1/v_n)$ define a vector $d_v = (v_1 d_1, \ldots, v_n d_n)$ and the following matrices:

$$B_w = \begin{pmatrix} v_1 B_1 & 0 & \ldots & 0 \\ 0 & v_2 B_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & v_n B_n \end{pmatrix}, \quad C_w = \begin{pmatrix} 0 & v_1 C_{1,2} & \ldots & v_1 C_{1,n} \\ v_2 C_{2,1} & 0 & \ldots & v_2 C_{2,n} \\ \ldots & \ldots & \ldots & \ldots \\ v_n C_{n,1} & v_n C_{n,2} & \ldots & 0 \end{pmatrix},$$

and $Q_v = B_v + C_v$, where Q_v is symmetric since the game is potential. Then represent the potential (12) as follows:

$$P(x) = \frac{1}{2} x^\top Q_v x + x^\top d_v \, .$$

Consider the general case when the matrix Q_v is indefinite, and represent Q_v as follows:

$$Q_v = D_1 + D_2 \ , \qquad D_1 = Q_v - \alpha I \ , \qquad D_2 = \alpha I \ , \tag{13}$$

where $\alpha > 0$ is the largest eigenvalue of Q_v. The matrix D_1 is negative semidefinite, whereas the matrix D_2 is positive definite. Using (13) we represent P as the sum of concave and convex functions φ and ψ respectively:

$$P(x) = \varphi(x) + \psi(x) \ , \quad \varphi(x) = \frac{1}{2}x^\top D_1 x + x^\top d_v \ , \quad \psi(x) = \frac{1}{2}x^\top D_2 x \ .$$

In other words, one can say that P is presented as a difference of two convex functions. Such a representation is referred to as a d.c. decomposition. Define an iterative local search process for the problem

$$P(x) \to \max, \qquad x \in X \ , \tag{14}$$

as follows. Let an initial profile $x^0 \in X$ and the current iteration point $x^k \in X$ be given. Then the next point x^{k+1} is obtained as a solution of a maximization problem with concave objective:

$$x^{k+1} \in \operatorname{Arg\,max}\left\{\rho(x, x^k) \mid x \in X\right\} \ , \qquad k = 0, 1, \dots \ , \tag{15}$$

$$\rho(x, x^k) = \varphi(x) + \nabla\psi(x^k)^\top (x - x^k) + \psi(x^k) \ . \tag{16}$$

The function $\rho(x, x^k)$ may be considered as a nonlinear support minorant function for P, since

$$\rho(x, x^k) \leqslant P(x) \qquad \forall (x, x^k) \in X \times X \ ,$$
$$\rho(x^k, x^k) = P(x^k) \qquad \forall x^k \in X \ .$$

An investigation on the algorithm (15), (16), including its convergence properties, one can find in [18]. It had been shown that the process (15), (16) converges to a stationary point of the nonconcave optimization problem (14).

5 Global Search

It is possible to point out two cases when a using of a global search is essential. The first one is a verification of a stationary point obtained by a local search whether this point is an equilibrium. Such a verification consists of n nonconvex programming problems that we need to solve in global sense (see (1)). In order to obtain various stationary points it is supposed to start local ascending from different, say randomly chosen, initial points (multistart). Secondly, we need to use a global search method if after the verification of considerable number of stationary points we failed to find an equilibrium. In this case we have to solve globally initial problem of maximizing a potential. In both cases we face a quadratic nonconvex programming problem with a convex feasible set. One of existing methods for solving such a problem consists in its reduction to a mixed-integer linear programming problem [11]. In what follows we recall an approach from [11].

To this moment, we have not yet specified the profile set X. From now we assume that X is defined by linear constraints:

$$X = \{x \in \mathbb{R}^m \mid Ax \leqslant b\} \ , \tag{17}$$

where $A \in \mathbb{R}^{s\times m}$ and $b \in \mathbb{R}^s$. Write out the KKT conditions for (14), (17):

$$\nabla P(x) - \sum_{i=1}^{s} \lambda_i a^i = 0 \ , \tag{18}$$

$$\lambda_i (x^\top a^i - b_i) = 0 \ , \quad 1 \leqslant i \leqslant s \ , \tag{19}$$

$$Ax \leqslant b \ , \quad \lambda \geqslant 0 \ , \tag{20}$$

where

$$\nabla P(x) = Q_v x + d_v \ , \tag{21}$$

and a^i denotes the ith row of the matrix A. The relation (21) implies

$$x^\top \nabla P(x) = x^\top Q_v x + x^\top d_v = 2P(x) - x^\top d_v \ . \tag{22}$$

From (18), (19), and (22) we have

$$x^\top \nabla P(x) - \sum_{i=1}^{s} \lambda_i x^\top a^i = 2P(x) - x^\top d_v - \sum_{i=1}^{s} \lambda_i b_i = 0 \ .$$

Hence every solution of the system (18)–(20) satisfies the equality

$$P(x) = \frac{1}{2}\left(x^\top d_v + \lambda^\top b\right) \ .$$

For convenience define auxiliary variables $y_i = b_i - x^\top a^i$ for $1 \leqslant i \leqslant s$. The problem (14), (17) is equivalent to the following one:

$$x^\top d_v + \lambda^\top b \to \max_{(x,\lambda,y)} \ , \tag{23}$$

$$Q_v x + d_v - \lambda^\top A = 0 \ , \tag{24}$$

$$\lambda_i y_i = 0 \ , \quad 1 \leqslant i \leqslant s \ , \tag{25}$$

$$Ax + y = b \ , \tag{26}$$

$$\lambda \geqslant 0 \ , \quad y \geqslant 0 \ . \tag{27}$$

With respect to the sensible assumptions [11] we can suppose that a constant M exists such that $\lambda < M$ and $y < M$. Let us introduce binary variables z_i, $1 \leqslant i \leqslant s$. Then complementarity slackness (25) can be replaced by the following equivalent constraints:

$$\lambda_i - M z_i \leqslant 0 \ , \quad y_i - M(1 - z_i) \leqslant 0 \ , \quad 1 \leqslant i \leqslant s \ . \tag{28}$$

Thus we derive the mixed-integer linear programming (MILP) problem (23), (24), (26)–(28) with binary variables. Although the original quadratic nonconvex problem (14), (17) is rather difficult, we believe that the derived MILP one would be more tractable due to highly developed MILP solvers (such as CPLEX, for example).

If we want to verify a stationary point x of the problem (14), (17) whether x is an equilibrium, we can maximize the payoffs (9) in accordance with the definition (1) by the same way as it was just described. Note that in this case we need to solve n nonconvex quadratic problems of dimensions $m_1, m_2, \ldots, m_n$, in contrast to one large problem (14), (17) of the dimension m. Hence it may be reasonable first to search equilibria through the local ascending, since the large dimension plays crucial role in global search procedures.

6 Conclusion

In this study we generalize the differentiable characterization for weighted potential games with multi-dimensional strategy spaces, and specify the corresponding conditions for an n-player quadratic nonconvex game with continuous set of game profiles. The set of profiles is defined by shared constraints. It is shown that the fulfilment of these conditions does not depend on convexity-type assumptions on payoffs. A weighted potential for the given game turns out to be a nonconcave function. Since every global maximum of a potential is a Nash equilibrium, and moreover even a stationary point may be an equilibrium, we propose a local and a global search procedures for maximizing a potential function of quadratic weighted potential game. The local search process represents the sequence of convex optimization problems, and it converges to a stationary point of a potential. The global search is based on the reduction to an MILP problem.

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