Properties of Admissible Set of an Optimal Non-destructive System Exploitation Problem in Some General Formalization

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Abstract. We study the properties of admissible strategies for the problem of using of a renewable resource system that does not lead to its destruction, in some general formulation. It is assumed that the system evolution is described by some iterative process. The conditions for the stable existence of this system, originally formulated in terms of the asymptotic properties of this iterative process, reduce to the existence of admissible controls for some mathematical programming problem. We analyzed the properties of its admissible set in terms of the dominant eigenvalues of some positively homogeneous maps. We obtained the conditions for the existence of optimal controls under the influence of which the system can stably exist for an indefinitely long time. Introduced by author's generalization of the classical concept of the irreducibility of a non-linear map, the concept of local irreducibility, is essentially used. The problem under consideration has important applications in the tasks of rational exploitation of renewable natural resources, in particular, in the tasks of managing ecological populations.

Keywords: Concave programming · Irreducible map · Positive equilibrium · Rational exploitation of natural resources

1 Introduction

One of a vital problems that require the use of mathematical methods is a problem of a rational use of renewable resources. Particularly acute in practice is a problem of rational non-destructive ecosystem exploitation. This problem is characterized by an abundance of specific approaches for specific natural systems (see the review in [4]). In the matrix formulation, the approach to optimal exploitation of a population in a stationary state has prevailed for a long time. This approach uses as a structure of the population a stable distribution — the structure of an eigenvector corresponding to the dominant eigenvalue of the

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projection matrix (that must be greater than unity). If at some point in time the structure of the population coincides with a stable distribution, then it is possible to remove the surplus without damage to the population, returning the population to the previous state. This corresponds to the withdrawal of the same proportion of all its constituent groups.

One of the first theoretical works in this direction was [2], where the concept of sustainable yield was defined, and the solvability of the problem of finding the maximum level of permissible exploitation was proved. In work [1] some concepts natural for the exploitation of ecosystems were formalized and the corresponding statements were rigorously proved. The first successful attempts to generalize these results to the case of a nonlinear density dependence are related to papers [11, 3] that were considered the dependence on density only for the first age class. In [5] was considered the model with dependence on density for all age classes.

We are developing here a general approach to this problem that considers the ecosystem as a discrete system, with its representation in the form of an iterative process

$$x_{t+1} = F(x_t), \qquad t = 0, 1, 2, \dots \tag{1}$$

where x_t means the state of the system at time $t = 0, 1, 2, \dots$, the step operator F transforms the system from the previous state to the following. With this approach, one can naturally formalize the inadmissibility of excessive load on the dynamic system condition.

We will use the following notation: $\mathbb{R}^q_+$ denotes the non-negative orthant of $\mathbb{R}^q$; $x \le y$ means $y - x \in \mathbb{R}^q_+$; $x < y$ means $y - x \in \operatorname{int} \mathbb{R}^q_+$; $\operatorname{int} M$ — the interior of the set M; $x \lneqq y$ means $x \le y$ and $x \ne y$; $\mathbb{Z}$ — the set of integers; $\overline{m, n} = \{i \in \mathbb{Z} \mid m \le i \le n\}$. Vector $x = (x_1, x_2, \dots, x_q)$ can be written in the short-hand form as $x = (x_i)$. The iteratives of the map F are denoted by $F^t(x)$ $(t = 1, 2, \dots)$, $F^0(x) \equiv x$.

A controlled system is modelled by iterative process

$$x_{t+1} = F_u(x_t), \qquad t = 0, 1, 2, \dots, \tag{2}$$

where $x = (x_i)$, $u = (u_i)$, $F_u(x) = (F(x) - u))^+$, $a^+ = \max\{a, 0\}$, $x^+ = (x_i^+)$. We will denote by $\{x_t(x_0, u)\}_{t=0}^{+\infty}$, $x_t(x_0, u) = (x_t^i(x_0, u))$ the realization of this process starting from the initial vector x_0. In [7] we formulated an optimization problem for dynamical system (1): determine

$$\tilde{c} = \max\{c(u) \mid u \in \overline{U}\}, \tag{3}$$

where $c(u)$ — the total effect of using of system elements in the quantities $u_1, u_2, \dots, u_q$, the set $\overline{U}$ is the closure of the set

$$U = \{u \in \mathbb{R}^q_+ \mid X_0(u) \ne \varnothing\}$$

of controls. We will be more interested in the properties of the admissible set $\overline{U}$, so for simplicity we shall assume that $c(u)$ is linear nonnegative on $\mathbb{R}^q_+$ function.

The control vector u is considered as admissible if there is at least one initial state x_0, for which all units of the system stably exist for an indefinitely long time:

$$X_0(u) = \{x_0 \in \mathbb{R}^q_+ \mid \inf_t x^i_t(x_0, u) > 0 \ (\forall i \in \overline{1, q})\}$$

Thus, the size of each unit of the system should not decrease with time to zero. As can be seen from this definition, the set U formalizes the requirement of the indestructability of the managed object.

A necessary condition for $U \neq \varnothing$ is the existence of a positive fixed point of the map F, i. e., condition $N^+_F \neq \varnothing$. In this case the set U contains at least the trivial control: $0 \in U$.

Unfortunately, the set U is not always closed; thus, in problem (3), its closure is considered as the admissible set.

Denote by $\tilde{u}$ and $\widetilde{U}$ the optimal vector and optimal set of problem (3); and by N_u and N^+_u the sets of nonzero fixed points and positive fixed points of F_u, respectively. It is assumed that the original model without control (1) has the trivial equilibrium state: $F(0) = 0$.

We are interested in the positive realizations of the iterative process (2) only, when $x_t(x_0, u) > 0$ $(\forall t = 1, 2, \ldots)$. In this case $F_u(x) = F(x) - u > 0$ for all $x = x_t(x_0, u)$. Note that if $x \in N^+_u$, then $F_u(x) > 0$ and $F(x) = x + u$ also must be fulfilled.

The main requirement imposed on the class of maps under consideration is a concavity or its weakening, a subhomogeneity [6]. The definitions of subhomogeneous and superhomogeneous maps, as well as other standard concepts used below, are given, for example, in [8]. The subhomogeneity of a map make it possible to depict adequately a so-called saturation effect observed in many natural systems with limited resources.

In what follows, the concept of irreducibility (indecomposability) of a map is essentially used. Along with the classical concept of irreducibility [10], we will also use its local analogues [12], such as a notion of irreducibility of a map at zero. We give a classical definition of the irreducibility of a map in a somewhat different way, using the following sets:

$$I^+(x, y) = \{j \in \overline{1, q} \mid x_j > y_j\}, \quad I^0(x, y) = \{j \in \overline{1, q} \mid x_j = y_j\},$$

$$I^+(x) = \{i \in \overline{1, q} \mid x_i > 0\}, \qquad I^0(x) = \{i \in \overline{1, q} \mid x_i = 0\}.$$

A map F is said to be *reducible at a point* $y \in \mathbb{R}^q_+$, if

$$\exists x \in \mathbb{R}^q_+ : \quad x \gneqq y, \quad I^0(x, y) \neq \varnothing, \quad I^0(x, y) \subseteq I^0(F(x), F(y)). \tag{4}$$

A map F is said to be *irreducible at point* y, if it is not reducible at point y.

A map reducible (or irreducible) at every point of set M is said to be *reducible* (or *irreducible*, respectively) *on the set* M.

We consider separately the case of irreducibility at the point $y = 0$. A map $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ is said to be *reducible at point* $y = 0$ (*reducible at zero*), if

$$\exists x \in \mathbb{R}^q_+ : \quad x \gneqq 0, \quad I^0(x) \neq \varnothing, \quad I^0(x) \subseteq I^0(F(x)). \tag{5}$$

We note that the concept of reducibility at zero makes sense only for maps satisfying condition $F(0) \not> 0$.

The global irreducibility of a map means its irreducibility at all points $\mathbb{R}^q_+$ and, in particular, irreducibility at zero.

In a research of asymptotic properties of iterative processes a primitivity property of a map [10] is usually used. Although irreducibility at zero is a weaker property in comparison with primitivity at zero, it also guarantees the positivity of nonzero points of a map. Indeed, suppose, by way of contradiction, that $I^0(\bar{x}) \neq \varnothing$. Then from $f_i(\bar{x}) = \bar{x}_i$ $(\forall i \in I^0(\bar{x}))$ we get $I^0(\bar{x}) \subseteq I^0(F(\bar{x}))$, which conradicts, by (5), the irreducibility of F at zero. Therefore $I^0(\bar{x}) = \varnothing$ and $\bar{x} > 0$.

The local irreducibility properties for monotone increasing subhomogeneous maps are considered in [8, 9].

We refine our assumptions about properties of dynamical system (1). The step operator F is assumed to be nonnegative and concave on $\mathbb{R}^q_+$ (and hence it is monotone increasing on $\mathbb{R}^q_+$). It is also natural to assume that the condition for the absence of identically zero coordinates of F is satisfied:

$$\forall\, i \in \overline{1,q}\ \exists\, x \in \mathbb{R}^q_+ :\ f_i(x) > 0. \tag{6}$$

It is important that this property for monotone increasing subhomogeneous map F implies that $F(\operatorname{int}\mathbb{R}^q_+) \subseteq \operatorname{int}\mathbb{R}^q_+$. A sufficient condition for (6) is the irreducibility of F at zero. Since irreducibility at zero also guarantees the positiveness of non-zero fixed points, it is convenient to assume from now on that the map F is irreducible at zero.

2 Some Preliminary Results

For the subhomogeneous map F we can define the following positively homogeneous maps:

$$F_0(x) = \lim_{\alpha \to +0} \alpha^{-1} F(\alpha x), \qquad F_\infty(x) = \lim_{\alpha \to +\infty} \alpha^{-1} F(\alpha x), \tag{7}$$

It is easy to see that

$$F_\infty(x) \leq F(x) \leq F_0(x)\ (\forall x \in \mathbb{R}^q_+). \tag{8}$$

Define the following sets:

$$P_F^+ = \{x > 0 \mid F(x) > x\}, \quad Q_F^+ = \{x > 0 \mid F(x) < x\}. \tag{9}$$

It is shown in [13] that for a subhomogeneous, monotone increasing and irreducible at zero map $F \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$ the following properties are valid:

$$\lambda(F_0) > 1 \Leftrightarrow P_F^+ \neq \varnothing; \qquad \lambda(F_\infty) < 1 \Leftrightarrow Q_F^+ \neq \varnothing. \tag{10}$$

Likewise, for the superhomogeneous, monotone increasing and irreducible at zero map F the following properties are valid:

$$\lambda(F_0) < 1 \Leftrightarrow Q_F^+ \neq \varnothing; \qquad \lambda(F_\infty) > 1 \Leftrightarrow P_F^+ \neq \varnothing. \tag{11}$$

We introduce the set

$$M_F(x) = \{\alpha > 0 \mid F(\alpha x = \alpha F(x)\}, \tag{12}$$

and quantities

$$\alpha_F(x) = \inf M_F(x),\ \beta_F(x) = \sup M_F(x), \tag{13}$$

where x is an arbitrary nonzero vector. Note that the set $M_F(x)$ always contains unity as an element.

Existence conditions of positive fixed points for subhomogeneous maps on cone $\mathbb{R}_+^q$ were studied in [13]. It follows from (8) due to [10, Theorem 10.3] that the condition $\lambda(F_\infty) \leq 1 \leq \lambda(F_0)$ is necessary for existence of a positive fixed point for monotone increasing, subhomogeneous and irreducible at zero map. A similar condition with strict inequalities:

$$\lambda(F_\infty) < 1 < \lambda(F_0) \tag{14}$$

is sufficient for existence of a positive fixed point. Sufficient conditions with $\lambda(F_\infty) = 1$ or $\lambda(F_0) = 1$ require the use of the global irreducibility property.

With some work it can be shown that if F is monotone increasing, subhomogeneous, and condition (14) is satisfied, then N_F^+ is bounded, convex and contains a largest element $\bar{x}_F$. Moreover, a concave globally irreducible map satisfying condition (14) has a unique positive fixed point [13, Corollary 2.2.1].

If $N_u^+ = \varnothing$ for all $u > 0$, then it follows from condition $N_F^+ \neq \varnothing$ that $U = \{0\}$. This case is uninteresting to us, therefore, in addition to condition $N_F^+ \neq \varnothing$, it is necessary to assume that $N_u^+ \neq \varnothing$ for some $u > 0$. In this case $F(x) = x + u > x$ ($\forall x \in N_u^+$), and by (10), we have $\lambda(F_0) > 1$. For this reason, the condition (14) is assumed everywhere below. Then the set U is also convex, contains some positive vector and with each positive vector u also contains the segment $[0, u]$; those U is so-called downward set. Besides the following equalities hold:

$$U = \{u \in \mathbb{R}_+^q \mid N_u^+ \neq \varnothing\}, \quad \overline{U} = \{u \in \mathbb{R}_+^q \mid N_u \neq \varnothing\}, \tag{15}$$

Finally, it was shown in [7] that under above assumptions the problem of mathematical programming

$$\max\{c(u) \mid x = F(x) - u,\ x \geq 0,\ u \geq 0\} \tag{16}$$

is solvable in the sense that $\tilde{c} < +\infty$ and this value is attained on certain admissible vectors $\tilde{u}$, $\tilde{x}$. Moreover, $\tilde{c}$, $\tilde{u}$ is a solution of problem (3) if and only if $\tilde{c}$, $\tilde{u}$, $\tilde{x}$ is a solution of problem (16).

We now turn to properties of the admissible set for this problem. Proceeding from its basic interpretation (10), we will be interested first of all in properties and conditions for existence of the positive equilibria $\tilde{x}$ for the control $\tilde{u}$.

3 Some Properties of the Admissible Set of Problem (3)

Note that the set N_u for $u \in \overline{U}$ contains the largest element $\bar{x}_u$ and the map $\bar{x}(u)\colon u \to \bar{x}_u$ is monotone decreasing on $\overline{U}$. It can also be shown that $\bar{x}(u)$ inherits the concavity of the map F.

We introduce the set

$$D = \{u \in \mathbb{R}^q_+ \mid N_u \neq \varnothing,\ N_v = \varnothing\ (\forall v > u)\} \tag{17}$$

that forms part of a boundary of U. The set D is of interest in connection with the fact that it contains all potentially optimal vectors of the problem (3): $\widetilde{U} \subset D$. We divide this set into two disjoint parts by the criterion of the presence or absence of common points with U:

$$D' = D \setminus D'', \quad D'' = \overline{U} \setminus U. \tag{18}$$

From equalities (15) we obtain the following representation of D'':

$$D'' = \{u \in \mathbb{R}^q_+ \mid N_u \neq \varnothing,\ N_u^+ = \varnothing\}. \tag{19}$$

We note that, by definition of D'', for elements of N_u with $u \in D''$ the following property is true:

$$\bar{x}_u \notin \operatorname{int} \mathbb{R}^q_+\ (\forall u \in D''). \tag{20}$$

Directly from (17)–(19) the following representation follows:

$$D' = \{u \mid N_u^+ \neq \varnothing,\ N_v = \varnothing\ (\forall v > u)\}. \tag{21}$$

The representations (19), (21) can be clarified. Indeed, it follows from (21), in view of (19), that $D' \subset U$. Since $D' \subseteq D$, we get $D' \subseteq D \cap U$. On the other hand, $D \cap U \subseteq D'$, because $u \in U$ together with the second equality in (18) means $u \notin D''$. Hence, taking into account $u \in D$ and the first equality in (18), we obtain $u \in D'$. Therefore, $D' = D \cap U$.

Further, from the second equality in (18) it follows that $D \setminus U \subseteq D''$. On the other hand, $D'' \subseteq D \setminus U$, since $D'' \subseteq D$ and $D'' \cap U = \varnothing$. Therefore $D'' = D \setminus U$. Thus, we have the following equalities:

$$D' = D \cap U, \quad D'' = D \setminus U. \tag{22}$$

These equalities show that the positive optimal vector $\tilde{x}$ for the optimal control $\tilde{u}$ of the problem (3) exist if and only if $\tilde{u} \in D'$. Using (22), we can refine that $U \setminus D = U \setminus D'$.

The set $U \setminus D$ does not a fortiori contain the optimal vectors of the problem (3). We give without proof the following assertion characterizing this set.

Lemma 1. *If F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (14) *is satisfied, then $u \in U \setminus D$ if and only if there exists $v > u$ belonging to the same set: $v \in U \setminus D$; or equivalently, if and only if there exists $x > 0$ such that $F_u(x) > x$.*

Let us define the maps $\Phi_y \in \{\mathbb{R}^q_+ \mapsto \mathbb{R}^q_+\}$, $\Psi_y \in \{[0,y] \mapsto \mathbb{R}^q_+\}$, where $y \in N_u$, $\Phi_y(x) = (\phi^i_y(x))$, $\Psi_y(x) = (\psi^i_y(x))$, with the following equalities:

$$\Phi_y(x) = F(x+y) - F(x), \quad \Psi_y(x) = F(y) - F(y-x). \tag{23}$$

Obviously, $\bar{x}_u \leq \bar{x}_F$; therefore we can assume that y, which plays the role of a parameter here, varies in $[0, \bar{x}_F]$.

It is easy to see that the maps Φ_y, Ψ_y inherit the monotonicity of the map F, as well as the presence of a trivial fixed point $x = 0$. Furthermore, the map Φ_y is also concave on $\mathbb{R}^q_+$, and the map Ψ_y is convex on $[0, y]$. Therefore, for these maps there exist positively homogeneous (of the first degree) maps $(\Phi_y)_0, (\Psi_y)_0$ defined for F in (7). For their components the following equalities hold:

$$(\phi^i_y)_0(x) = f'_i(y,x), \quad (\psi^i_y)_0(y,x) = -f'_i(y,-x) \quad (\forall i \in \overline{1,q}),$$

where $f'_i(y,x)$ is one-sided directional derivative of the function $f_i(x)$ at the point y in the direction x.

The maps Φ_y, Ψ_y (and, hence, their dominant eigenvalues) are related by the followings inequalities:

$$\Phi_y(x) \leq \Psi_y(x) \quad (\forall x \in [0,y]), \tag{24}$$

$$\lambda((\Phi_y)_0) \leq \lambda((\Psi_y)_0). \tag{25}$$

If F is differentiable at point y and F' is its derivative, then the following equations are valid:

$$(\Phi_y)_0(x) = (\Psi_y)_0(x) = F'(y)x,$$

$$\lambda((\Phi_y)_0) = \lambda((\Psi_y)_0) = \lambda(F'(y)).$$

The following relations between the sets of fixed points for the maps F_u, Φ_y, Ψ_y and the sets (9) for these maps follow directly from their definitions.

Lemma 2. *Suppose that F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (14) *is satisfied. Then the following properties are valid:*

(1) $x \in P^+_{\Phi_y}$, $y \in N_u \Rightarrow x + y \in P^+_{F_u}$;
(2) $x \in Q^+_{\Psi_y}$, $y \in N_u, x < y \Rightarrow y - x \in P^+_{F_u}$;
(3) $x \in N_{\Phi_y}$, $y \in N_u \Rightarrow x + y \in N_u$;
(4) $x \in N_{\Psi_y}$, $y \in N_u, x \leq y \Rightarrow y - x \in N_u$;
(5) $x \in N_u$, $y \in N_u$, $y \leq x \Rightarrow x - y \in N_{\Phi_y}$.
(6) $x \in N_u$, $y \in N_u$, $y \leq x \Rightarrow x - y \in N^+_{\Psi_x}$.

We show that Φ_y and Ψ_y inherit also the global irreducibility of the map F.

Lemma 3. *Suppose that F is concave on $\mathbb{R}^q_+$ and the condition* (14) *is satisfied. If the map F is globally irreducible, then the map Φ_y $(y \in \mathbb{R}^q_+)$ is globally irreducible, and the map Ψ_y $(y \in \operatorname{int} \mathbb{R}^q_+)$ is irreducible on $[0, y)$.*

Proof. Let F be globally irreducible and suppose for contradiction that Φ_y is not globally irreducible for some $y \in \mathbb{R}^q_+$. By (4) this means that there exist x, x' such that $x' \gneqq x$, $I^0(x, x') \neq \varnothing$ and $\phi_i(x, y) = \phi_i(x', y)$ for $i \in I^0(x, x')$. Then for $z = x + y$, $z' = x' + y$ we have: $z' \gneqq z$, $I^0(z, z') = I^0(x, x') \neq \varnothing$ and $f_i(z) = f_i(z')$ for all $i \in I^0(z, z')$. Hence F is not globally irreducible, which contradicts our assumptions. Thus, Φ_y is globally irreducible. The proof of the irreducibility for Ψ_y is carried out analogously. The proof is complete.

We proceed to describe the set D by means of the maps (23).

Theorem 1. *Suppose that F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (14) *is satisfied. If $u \in U$, then $u \notin D$ if and only if $\lambda((\Psi_{\bar{x}_u})_0) < 1$.*

Proof. Necessity. Using (15) and Lemma 1 for $u \in U \setminus D$, we obtain that $N_v^+ \neq \varnothing$ for some $v > u$. Then, denoting $\bar{x}_u - \bar{x}_v > 0$ by $\bar{x}$, we get: $\Psi_{\bar{x}_u}(\bar{x}) = F(\bar{x}_u) - F(\bar{x}_u - \bar{x}) = F(\bar{x}_u) - F(\bar{x}_v) = (\bar{x}_u + u) - (\bar{x}_v + v) = \bar{x} + (u - v) < \bar{x}$, i. e. $\Psi_{\bar{x}_u}(\bar{x}) < \bar{x}$. It then follows, by (11) due to convexity (and, hence, superhomogenity) of the map $\Psi_{\bar{x}_u}$, that $\lambda((\Psi_{\bar{x}_u})_0) < 1$.

Sufficiency. If $\lambda((\Psi_y)_0) < 1$, then, by (11), there exists $\bar{x}$, such that $0 < \bar{x} < y$ and $x \in Q^+_{\Psi_y}$. Hence, due to the assertion (2) of Lemma 2, we obtain $y - \bar{x} \in P^+_{F_u}$, that is, $F_u(y - \bar{x}) > y - \bar{x}$. But this means, by Lemma 1, that $u \in U \setminus D$, as required.

Now we can obtain a characterization of the set D in terms of the maps (23).

Corollary 1. *Suppose that F is concave on $\mathbb{R}^q_+$, irreducible at zero and the condition* (14) *is satisfied. If $u \in U$, then $u \in D$ if and only if $\lambda((\Psi_{\bar{x}_u})_0) \geq 1$.*

The following statement gives a sufficient condition for an admissible vector not to belong to the set D.

Theorem 2. *Suppose that F is concave on $\mathbb{R}^q_+$, globally irreducible and the condition* (14) *is satisfied. If $u \in \overline{U}$ and $\lambda((\Phi_x)_0) > 1$ for some $x \in N_u$, then $u \in U \setminus D$.*

Proof. By (10) and Lemma 3, there exists $\bar{x} > 0$ such that $\Phi_x(\bar{x}) > \bar{x}$. It then follows, since assertion (1) of Lemma 2, that $F_u(x') > x'$ for $x' = y + \bar{x}$. But this means, by Lemma 1, that $u \in U \setminus D$. The proof is complete.

Corollary 2. *Suppose that F is concave on $\mathbb{R}^q_+$, globally irreducible and the condition* (14) *is satisfied. If $u \in D$ and $y \in N_u$ then $\lambda((\Phi_y)_0) \leq 1$.*

Combining Theorem 1, Corollary 1 and Corollary 2, we obtain the following characteristic of U for the differentiable map F.

Corollary 3. *Suppose that F is concave, differentiable on $\mathbb{R}^q_+$, and the condition* (14) *is satisfied. If the map F is also irreducible at zero, then*

$$\lambda(F'(\bar{x}_u)) < 1 \ (\forall u \in U \setminus D). \tag{26}$$

If, in addition, the map F is globally irreducible, then

$$\lambda(F'(\bar{x}_u)) = 1 \ (\forall u \in U \cap D). \tag{27}$$

Proof. If $u \in U \setminus D$, then the inequality (26) follows from Theorem 1 due to (25). If $u \in U \cap D$, then, by Corollary 1, we obtain $\lambda(F'(\bar{x}_u)) \geq 1$. On the other hand, according to Corollary 2, we have $\lambda(F'(\bar{x}_u)) \leq 1$, so $\lambda(F'(\bar{x}_u)) = 1$. The proof is complete.

Using the property 5 of Lemma 2 and Lemma 3, we obtain the following important property about the impossibility of a partial coincidence for coordinates of fixed points of F_u in the case of the globally irreducible map F:

$$\forall x, y \in N_u : x \lneqq y \Rightarrow x < y. \tag{28}$$

We will give now one more characteristic feature of the elements of N_u for $u \in U \setminus D$. The following assertion supplements the conclusion of Theorem 2.

Theorem 3. *Suppose that F is concave on $\mathbb{R}^q_+$, global irreducible and the condition* (14) *is satisfied. Then the following properties are valid:*

$$\lambda((\Phi_{\bar{x}_u})_0) < 1, \quad \lambda((\Phi_x)_0) > 1 \quad (\forall\, u \in U \setminus D,\ x \in N_u \setminus \{\bar{x}_u\}).$$

Proof. The first inequality follows from Theorem 1 and inequalities (24). Further, if $x \in N_u \setminus \{\bar{x}_u\}$, then, by assertion 5 of Lemma 2, $x' = \bar{x}_u - x$ is the fixed point of Φ_x that is positive by (28). Hence we obtain $\lambda((\Phi_x)_0) \geq 1$ [10, Theorem 10.3].

Suppose, by way of contradiction, that $\lambda((\Phi_x)_0) = 1$. Then due to [13, Theorem 2.2.11], we get $(0, 1) \subset M_{\Phi_x}(x')$ (see (12)–(13)). We show that the condition $\alpha \in M_{\Phi_x}(x')$ is satisfied if and only if the following equality is true:

$$F((1-\alpha)x + \alpha(x + x')) = (1-\alpha)F(x) + \alpha F(x + x') \ (\forall \alpha \in (0, 1)). \tag{29}$$

Indeed, we have: $\alpha \in M_{\Phi_x}(x') \Leftrightarrow \Phi_x(\alpha x') = \alpha \Phi_x(x') \Leftrightarrow F(\alpha x' + x) - F(x) = \alpha(F(x' + x) - F(x)) \Leftrightarrow F(\alpha x' + x) = (1-\alpha)F(x) + \alpha F(x + x')$. Since $\alpha x' + x = (1-\alpha)x + \alpha(x + x')$, we obtain the equality (29). Taking into account that $x + x' = \bar{x}_u$, we obtain from here the equality

$$F((1-\alpha)x + \alpha \bar{x}_u) = (1-\alpha)F(x) + \alpha F(\bar{x}_u) \ (\forall \alpha \in (0, 1)). \tag{30}$$

By assertion 6 of Lemma 2, x' is a fixed point of $\Psi_{\bar{x}_u}$ too. We show now that, moreover, if $\lambda((\Phi_x)_0) = 1$, then all points $\alpha x'$ ($\alpha \in (0, 1]$) are also fixed points of the map $\Psi_{\bar{x}_u}$.

Indeed, since $\bar{x}_u - \alpha x' = (1-\alpha)\bar{x}_u + \alpha x$, we have $\Psi_{\bar{x}_u}(\alpha x') = F(\bar{x}_u) - F(\bar{x}_u - \alpha x') = F(\bar{x}_u) - F((1-\alpha)\bar{x}_u + \alpha x)$. The equality (30) holds for all $\alpha \in (0, 1)$, so we get: $\Psi_{\bar{x}_u}(\alpha x') = F(\bar{x}_u) - (1-\alpha)F(\bar{x}_u) - \alpha F(x) = \alpha(\bar{x}_u + u) - \alpha(x + u) = \alpha x'$, so that really $\Psi_{\bar{x}_u}(\alpha x') = \alpha x'$ ($\forall \alpha \in (0, 1]$). These equalities mean that $(\Psi_{\bar{x}_u})_0(x') = x'$, thus $\lambda((\Psi_{\bar{x}_u})_0) \geq 1$ [10, Theorem 10.3]. But then it follows, by Corollary 1, that $u \in D$. This contradiction shows that $\lambda((\Phi_x)_0) > 1$. The proof is complete.

In conclusion, we give several assertions about a cardinality of set N_u with $u \in D$. If $\bar{x}_u$ is not unique in N_u, then it follows from (28) that $\bar{x}_u > x_u$ ($\forall x_u \in N_u \setminus \{\bar{x}_u\}$). In this case, we can define the set

$$L_u = \mathrm{co}\{x_u, \bar{x}_u\},$$

where $\operatorname{co} M$ is the convex hull of a set M. We note that, as follows from the assertion 5 of Lemma 2 and from Theorem 3, for $u \in U \setminus D$ the set N_u cannot contain the entire segment L_u, along with points $x_u, \bar{x}_u$. But for the D' the situation is different. The following assertion shows that N_u with $u \in D$ is either a singleton or an infinite set.

Theorem 4. *Suppose that F is concave on $\mathbb{R}^q_+$, global irreducible and the condition (14) is satisfied. Then the following property holds:*

$$u \in D, x_u \in N_u \setminus \{\bar{x}_u\} \Rightarrow N_u \supseteq L_u.$$

Proof. Due to the assertion 5 of Lemma 2, the map Φ_{x_u} has a non-zero fixed point $\bar{x} = \bar{x}_u - x_u$. The map Φ_{x_u} is global irreducible (see Lemma 3), therefore, this fixed point is positive. Due to (8), we get: $(\Phi_{x_u})_0(\bar{x}) \geq \Phi_{x_u}(\bar{x}) = \bar{x}$, hence $\lambda((\Phi_{x_u})_0) \geq 1$ [10, Theorem 10.3]. But, the Corollary 2 gives the opposite inequality, so that $\lambda((\Phi_{x_u})_0) = 1$. In this case, the equation $\alpha_{\Phi_{x_u}} = 0$ must be fulfilled, where $\alpha_{\Phi_{x_u}}$ is defined by (13) (see [13, Theorem 2.2.11]). This means that $\Phi_{x_u}(\alpha\bar{x}) = \alpha\Phi_{x_u}(\bar{x})$ $(\forall \alpha \in [0,1])$. Therefore, we obtain for the set of positive fixed points of Φ_{x_u} that $N^+_{\Phi_{x_u}} \supseteq \{\alpha\bar{x} \mid \alpha \in [0,1]\}$. By assertion 3 of Lemma 2 this means that $N_u \supseteq \{x_u + \alpha\bar{x} \mid \alpha \in [0,1]\} = \{x_u + \alpha(\bar{x}_u - x_u) \mid \alpha \in [0,1]\} = L_u$. The proof is complete.

Taking into account properties (20), (28), we obtain from the Theorem 4 the following assertion.

Corollary 4. *Suppose that F is concave on $\mathbb{R}^q_+$, global irreducible and the condition (14) is satisfied. Then the following properties hold:*

$$|N_u| \in \{1, +\infty\} \ (\forall u \in D'), \quad |N_u| = 1 \ (\forall u \in D''). \tag{31}$$

If F is also strictly concave on $[0, \bar{x}_F]$, then $|N_u| = 1$ $(\forall u \in D)$.

In conclusion, let us give an example showing the essentiality of the requirement of global irreducibility in the above statements, which used this assumption. This example uses a generalization of so-called Leslie's model [14] of the following form:

$$x^{(t+1)}_{i,1} = f_i(a_t), \quad x^{(t+1)}_{i,j+1} = \alpha_{i,j} x^{(t)}_{i,j} \quad (i \in \overline{1,m}, j \in \overline{1,n-1}), \tag{32}$$

where $a_t = \sum_{i=1}^m \sum_{j=1}^n \beta_{i,j} x^{(t)}_{i,j}$, $\alpha_{i,j} > 0$, $\beta_{i,j} \geq 0$, $x^{(t)}_{i,j} \geq 0$ $(\forall i \in \overline{1,m}, j \in \overline{1,n})$. The functions $f_i(a)$ $(i \in \overline{1,m})$ will be assumed to be nonnegative and concave on $\mathbb{R}_+$. The step operator $F(x) = (f_{i,j}(x))$ of this model has the following components:

$$f_{i,1}(x) = f_i(a(x)), \quad f_{i,j+1}(x) = \alpha_{i,j} x_{i,j} \ (i \in \overline{1,m},\ j \in \overline{1,n-1}), \tag{33}$$

where $a(x) = \sum_{i=1}^m \sum_{j=1}^n \beta_{i,j} x_{i,j}$. Thus, the map F inherits a concavity and monotonicity of functions $f_i(a)$.

The admissible set $\overline{U}$ in this case, in addition to the requirement of non-negativity of variables, is given by the following constraints:

$$x_{i,1} = f_i(a(x)) - u_{i,1}, \quad x_{i,j+1} = \alpha_{i,j} x_{i,j} - u_{i,j+1} \; (i \in \overline{1,m},\; j \in \overline{1,n-1}), \tag{34}$$

Let us introduce the following notation:

$$\sigma(a) = \sum_{i=1}^{m} \sigma^{(i)} f_i(a), \quad q(u) = (q,u), \quad \mu(a) = \sigma(a) - a, \tag{35}$$

where $(\cdot\,,\cdot)$ denotes the scalar product, $q_j^{(i)} = \sum\limits_{k=j}^{n} \beta_{i,k} \prod\limits_{\ell=j}^{k-1} \alpha_{i,\ell}$ (by convention $\prod\limits_{\ell=r}^{s} a_\ell = 1$ for $i > k$), $\sigma^{(i)} = q_1^{(i)}$, $q = (q^{(1)}, q^{(2)}, \ldots, q^{(m)})$, $q^{(i)} = (q_1^{(i)}, q_2^{(i)}, \ldots, q_n^{(i)})$ $(i \in \overline{1,m},\; j \in \overline{1,n})$ $(k, \ell, r, s \in \mathbb{Z})$.

Multiplying equalities (34) by $\beta_{i,j}$ and summing, we obtain for $a = a(x)$ the equation

$$q(u) = \mu(a). \tag{36}$$

Thus, if $x_u \in N_u$, then (36) holds, which for a given u can be regarded as an equation for a. This equation, due to the concavity of $\mu(a)$, can have no more than two solutions for $\mu(a) \neq \mu^* = \max_a \mu(a)$.

Example 1. Consider the system of constraints

$$x_{i,1} = f_i(a) - u_{i,1}, \quad x_{i,2} = \frac{1}{2} x_{i,1} - u_{i,2},$$

$$a = x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2}, \quad x_{i,j} \geq 0,\; u_{i,j} \geq 0 \; (i,j = 1,2),$$

where $f_1(a) = 6a(1+a)^{-1}$, $f_2(a) = \min\{a, 1\}$ are non-negative monotone increasing concave functions of a nonnegative argument. We find from (35):

$$\sigma(a) = \frac{3}{2} f_1(a) + \frac{3}{2} f_2(a) = \frac{9a}{1+a} + \frac{3}{2} \min\{a, 1\},$$

$$q(u) = \frac{3}{2} u_{1,1} + u_{1,2} + \frac{3}{2} u_{2,1} + u_{2,2}, \quad \mu(a) = \frac{9a}{1+a} + \frac{3}{2} \min\{a, 1\} - a.$$

As shown in [14], condition (14) for the model (32) is equivalent to condition

$$\sigma'(+\infty) < 1 < \sigma'(0).$$

It is easy to see that in our particular case this condition is satisfied. Next, here $F \in \{\mathbb{R}_+^4 \mapsto \mathbb{R}_+^4\}$ has the following form: $F(x) = (f_1(a(x)), \frac{1}{2} x_{1,1}, f_2(a(x)), \frac{1}{2} x_{2,1})$. This map clearly satisfies all assumptions of previous propositions, with the exception of the global irreducibility. The function $\mu(a)$ reaches its maximum value $\mu^* = 11/2$ for $a^* = 2$.

We take for illustration $u = (16/5, 0; 1/10, 0)$ and show that $\in D''$. Calculate $q(u) = 27/5$ for this vector. Solving the equation $\mu(a) = 27/5$, we obtain solutions $a_1 = 3/2$, $a_2 = 13/5$. Further, we find $x(a_1, u) = (2/5, 1/5; 9/10, 0)$, $x(a_2, u) = (17/15, 17/30; 9/10, 0)$, so that $N_u = \{x(a_1, u), x(a_2, u)\}$. Therefore, $\bar{x}_u = x(a_2, u)$, and from (19) it follows that $u \in D''$. Thus, we get $|N_u| = 2$ for $u \in D''$, so that the second part of the conclusion (31) of Corollary 4 is not satisfied. In addition, we see that the property (28) is also violated for these vectors.

The matrix $F'(x)$ has the form of a generalized Leslie's matrix:

$$L(a) = \begin{bmatrix} f_1'(a) & f_1'(a) & f_1'(a) & f_1'(a) \\ 1/2 & 0 & 0 & 0 \\ f_2'(a) & f_2'(a) & f_2'(a) & f_2'(a) \\ 0 & 0 & 1/2 & 0 \end{bmatrix},$$

where $a = a(x)$ (see (33)). To compare the dominant eigenvalue of this matrix with unity, it is not necessary to find this value, because the followig equation is valid [14]:

$$\operatorname{sgn}(\lambda(L(a)) - 1) = \operatorname{sgn}(\sigma'(a) - 1),$$

where $\operatorname{sgn}(x) = |x|/x$ for $x \neq 0$, and $\operatorname{sgn}(0) = 0$.

For $x = x(a_1, u)$ we obtain $\lambda(F'(x)) = \lambda(L(a_1)) > 1$ as $\sigma(a_1) > 1$. We see that the conclusions of Theorem 2 and of Corollary 2 for the chosen vector $u \in D$ also do not hold.

Thus, we have demonstrated that the requirement of the global irreducibility in the above statements is essential.

4 Conclusion

In this paper we investigated the properties of the admissible controls of the problem (3) that can be optimal with appropriate presetting of the objective function. In the context of the interpretation of this problem, from which we proceeded, the question of presence of positive equilibrium x for the iterative process (2) with given admissible control u is essential. We showed that such controls must be contained in the set D' defined by (18). In the case of the differentiability of the map F, these controls are distinguished from the others, by Corollary 3, with simple characteristic property (27). It turns out that the dominant eigenvalue at the equilibrium point $\bar{x}_u$ should equal unity. In addition, as Corollary 4 shows, only for $u \in D'$ the set N_u can contain an infinite number of elements.

The question whether in our assumptions this set is non-empty remains generally open. But in some cases, for some known (and sufficiently general) mathematical models used in the modeling of biological communities, it is sometimes possible to establish that $D' \neq \varnothing$. Preliminary studies show that this takes place for the above generalization of Leslie's model, as was the case in Example 1. It can be shown that in the assumption of concavity of all functions $f_i(x)$, the set D' in this model is always nonempty. Particularly important and, to a degree, unexpected is the fact that the set D' here is part of some hyperplane. Other key properties of the admissible set of problem (3) for this and other models call for further investigations.

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