On the Decomposition of Regional Events in Elementary Systems

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Abstract. We study the relation between labelled transition systems and the corresponding partial orders of regions. In particular, we focus on the sets of their potential events, or labels on the transitions, providing them with a structure so as to reason about concurrency from the perspective of the observable properties of these systems. This is achieved by introducing the notion of minimal events, as the generators of such a structure of labels. We show that these events are sufficient to synthesize a transition system, such that its Regional Partial Order is isomorphic to the one obtained with the full set of events.

1 Introduction

Elementary Net Systems [15], or ENS, represent a suitable paradigm to study the logical structure of observable properties of distributed systems. As a class of Petri Nets, they explicitly represent concurrency, and as elementary systems, their local states are interpretable as Boolean variables. Unlike in the sequential case, on a system depicting concurrency, these properties do not interact according to the classical Boolean logic. This work studies such logical variables, and their interactions.

ENS are formalized as bipartite directed graphs, the nodes of which belong in one of the following classes. A *condition*, or local state, can take the values true or false. An *event* can dynamically alter the values of the conditions which are connected to it by an edge. This change in the values is guided by the *firing rule* [12]. A global state of the system is determined by a total truth assignment on the conditions, it corresponds to a *marking*. The *case graph* of the net system is a graph depicting the global behaviour of the system. Its nodes are the global states of the system, and edges are now labelled with the events. As formalized in Section 2.1, an *Elementary Transition System*, or ETS, is the case graph of some Elementary Net System.

In this model, the labeling function induces an equivalence relation on the transitions. Such a relation, and the corresponding equivalence classes, were studied in the frame of *2-Structures* [10, 11], in a work that set the foundation of *Region Theory*. A *region* is a subset of global states, such that transitions entering, and those exiting it, are labelled consistently. In our view, regions correspond to local properties of the system; they are determined by the behaviour of a single sequential component. Due to the firing rule of ENS, there is a correspondence between conditions and regions, by requiring consistency in the labelling of transitions with respect to the latter. Given a condition of an ENS the set of global states for which it is true is a region of the corresponding ETS. Furthermore, it was shown in [10, 11], that for each region of its case graph, a condition can be added to an ENS, without effects on its global behaviour. Actually, this fact is at the core of the solution to the Elementary Synthesis Problem ([9]): given an ETS, obtain an ENS such that its case graph is isomorphic to it. An ENS with a condition for each region provides a solution to this problem.

Regions, as subsets of global states, can be ordered by inclusion. The set of regions of an ETS is closed under set complement, and union of disjoint elements. Such structures, orthomodular partial orders, were well studied in the first attempts to axiomatize quantum mechanics [7], and were therefore called *quantum logics*. This notion is introduced in Section 2.2. This subsection also introduces a set of relations one can define on quantum logics, so as to express some properties of the subclass of such logics, that arise in our framework. This work essentially deals with the relations between quantum logics and ETS. However, net systems are considered implicitly throughout the sections, in the interpretation of local and global states, as conditions and markings.

By identifying regions and conditions, we stress the duality between global states, seen as subsets of conditions, and regions, seen as subsets of global states. This duality was exploited in [4] to present a construction which, given a quantum logic, provides an ETS. A canonical ETS is obtained, in which states are particular subsets of the logic. A transition is considered between each ordered pair of states. Events are characterized by their local effect, and so the labels on the transitions are determined by the set of regions in which the two states differ. We already studied such a construction in [6], where we showed that some of these events are redundant when computing regions. In that work we focused on *Condition/Event Transition Systems*, further on CETS, which mainly differ from ETS in the existence of an initial state.

In this work, we continue that study by selecting a specific subclass of the *events* of the synthesized system that guarantees both connectedness and preservation of the regional structure: one can consider the regions of the canonical CETS obtained from a logic. The canonical CETS, and the alternative CETS we here propose will carry the same set of regions. The main contribution of this work relies on the properties of the regions which are minimal for the partial order defining the logic [3]. These regions, henceforth *atomic regions*, or simply *atoms*, were shown to be sufficient to solve the Synthesis Problem. In fact, with the appropriate relations defined on them, atoms are sufficient to describe the whole logical structure, as we report in Section 2.4.

This rather technical result will allow us to develop on the relation between the logic of regions, and the associated canonical CETS. It was already pointed out, as a remark in [8], that the events built as subsets of regions form a partial group. In this work, we formalize this notion, thus endowing the set of events with a structure. In Section 3.1, we introduce a partial composition operation on the events of the logic, with neutral element, and inverse. We also identify concurrency between events with the commutativity of their composition. In Section 3.2, such commutativity is characterized in relational terms over the atomic structure of the logic. This will allow us to determine whether an event is the concurrent composition of other two, solely on the base of the logical structure. This fact is then further developed in Section 3.3 to define a partial order of events, and conclude that events which are minimal with respect to this order, are sufficient to convey all information regarding concurrency of the synthesized CETS. We also show that when considering only these events, the resulting CETS is connected.

2 Condition/Event Systems, Definitions and Notation

2.1 Transition Systems

Definition 1. *A* labelled transition system *is a structure* $A = (Q, E, T)$*, where* Q *is a set of states,* E *is a set of* events *and* $T \subseteq Q \times E \times Q$ *is a set of* transitions *such that:*

1. *the underlying graph of the* labelled transition system *is connected;*
2. $\forall (q_1, e, q_2) \in T \quad q_1 \neq q_2$*;*
3. $\forall (q, e_1, q_1)(q, e_2, q_2) \in T \quad q_1 = q_2 \Rightarrow e_1 = e_2$*;*
4. $\forall e \in E \quad \exists (q_1, e, q_2) \in T$*.*

We will write $s\ [e\rangle$ to mean that there is s' such that $(s, e, s') \in T$, in this case we may also write $s\ [e\rangle\ s'$. *Transition system* will mean *labelled transition system* in what follows.

Note: we will always use finite structures in this contribution.

Regions were introduced in [10] and [11], as a tool for solving the *synthesis problem* [9]: construct a Petri net from a specification of its intended behaviour in terms of a *transition system*. The *synthesis problem* is solved for several classes of nets and corresponding classes of transition systems [1].

Definition 2. *A* region *of a transition system* $A = (Q, E, T)$ *is a subset* r *of* Q *such that every event* crosses r uniformly*, namely:*
$\forall e \in E, \forall (q_1, e, q_2), (q_3, e, q_4) \in T$*:*

1. *(*$q_1 \in r$ *and* $q_2 \notin r$*) implies (*$q_3 \in r$ *and* $q_4 \notin r$*)* and
2. *(*$q_1 \notin r$ *and* $q_2 \in r$*) implies (*$q_3 \notin r$ *and* $q_4 \in r$*).*

Given a transition system A, its set of regions will be denoted by $\mathcal{R}(A)$; given a state $q \in Q$, the set of regions containing q will be denoted by $\mathcal{R}_q(A)$ and, when the transition system is clear from the context, simply by $\mathcal{R}_q$. Note that the set of regions $\mathcal{R}(A)$ of a transition system $A = (Q, E, T)$ cannot be empty since at least the whole set of states Q is a region. We say an event $e \in E$ is *orthogonal* to a region $r \in \mathcal{R}(A)$, whenever $\forall (q_1, e, q_2) \in T : q_1 \in r \Leftrightarrow q_2 \in r$.

Definition 3. *Let $A = (Q, E, T)$ be a transition system. The* pre-set *and* post-set *operations, denoted respectively by the operators $^\bullet(\,.\,)$ and $(\,.\,)^\bullet$, applied to regions $r \in \mathcal{R}(A)$ and events $e \in E$ are defined by:*

1. $^\bullet r = \{e \in E \mid \exists\,(q_1, e, q_2) \in T$ *such that* $q_1 \notin r$ *and* $q_2 \in r\}$;
2. $r^\bullet = \{e \in E \mid \exists\,(q_1, e, q_2) \in T$ *such that* $q_1 \in r$ *and* $q_2 \notin r\}$;
3. $^\bullet e = \{r \in \mathcal{R}(A) \mid e \in r^\bullet\}$;
4. $e^\bullet = \{r \in \mathcal{R}(A) \mid e \in {}^\bullet r\}$.

Elementary systems require an initial global state. In this work, deal with a similar class of systems, in which only simple connectedness of the underlying graph is required. *Condition/Event transition systems*, in the context of the *synthesis problem*, have been introduced as the class of transition systems representing the behaviour of *Condition/Event Net Systems*, one of the basic classes of Petri nets [12]. Figure 1 shows a transition system of this class.

Definition 4. *A* Condition/Event Transition System *— CETS — is a transition system such that the following conditions are satisfied:*

1. $\forall\, q_1, q_2 \in Q \quad \mathcal{R}_{q_1} = \mathcal{R}_{q_2} \Rightarrow q_1 = q_2$;
2. $\forall\, q_1 \in Q\; \forall\, e \in E \quad {}^\bullet e \subseteq \mathcal{R}_{q_1} \Rightarrow \exists\, q_2 \in Q$ *such that* $(q_1, e, q_2) \in T$;
3. $\forall\, q_1 \in Q\; \forall\, e \in E \quad e^\bullet \subseteq \mathcal{R}_{q_1} \Rightarrow \exists\, q_2 \in Q$ *such that* $(q_2, e, q_1) \in T$.

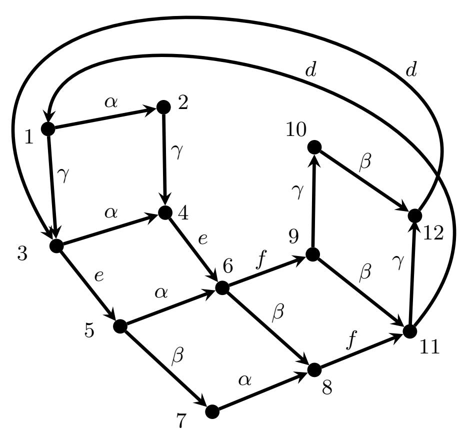

Fig. 1. A Transition System which is both CETS and ETS, with inital state 1.

We will use the following basic properties of regions of CETS:

Proposition 1. [3] *Let $A = (Q, E, T)$ be a CETS and $\mathcal{R}(A)$ its set of regions, then:*

1. $\emptyset, Q \in \mathcal{R}(A)$;
2. $r \in \mathcal{R}(A) \Rightarrow Q \setminus r \in \mathcal{R}(A)$;
3. $r_1, r_2 \in \mathcal{R}(A) \Rightarrow (r_1 \cap r_2 \in \mathcal{R}(A) \Leftrightarrow r_1 \cup r_2 \in \mathcal{R}(A))$.

2.2 Logics of Regions as Quantum Logics

Using the basic properties of regions recalled in Proposition 1, the set $\mathcal{R}(A)$ can be endowed with an algebraic structure: it can be partially ordered by set inclusion, and a complement can be defined as set-complement operation. More precisely, if $A = (Q, E, T)$ is a CETS, then the *logic* $L(A) = \langle \mathcal{R}(A), \subseteq, (.)', \emptyset, Q \rangle$, where $(r)' = Q \setminus r$, can be defined. We will call the logic $L(A)$ *regional logic*. This logic has been shown to be a *quantum logic* [4]:

Definition 5. *([13], definition 1.1.1) A* quantum logic *$\langle L, \leq, (.)', 0, 1 \rangle$ is a set L endowed with a partial order $\leq$ and a unary operation $(\,.\,)'$ (called orthocomplement), such that the following conditions are satisfied:*

1. *L has a least and a greatest element (respectively 0 and 1) and $0 \neq 1$;*
2. $\forall\, x, y \in L \quad x \leq y \Rightarrow y' \leq x'$;
3. $\forall\, x \in L \quad (x')' = x$;
4. *if $\{x_i \,|\, i \in I\}$ is a countable subset of L such that $i \neq j \Rightarrow x_i \leq x_j'$, then $\bigvee_{i \in I} x_i$ exists in L;*
5. $\forall\, x, y \in L \quad x \leq y \Rightarrow y = x \vee (x' \wedge y)$.

This latter condition is sometimes referred to as orthomodular law*.*

The symbols $\wedge$ and $\vee$ refer, when defined, to the ordinary meet and join operations. We say that two elements x and y in a logic are *orthogonal*, and write $x \perp y$, if $x \leq y'$. Hence, Proposition 1 implies that two regions are orthogonal whenever they are disjoint subsets of states.

Partial orders of regions are a subclass of quantum logics: they have been shown verify a set of properties which may not hold in an arbitrary quantum logic. One of them involves the notion of *state*.

Definition 6. *([13], definition 2.1.1) A* two-valued state *on a quantum logic L is a mapping $s : L \rightarrow \{0, 1\}$ such that:*

1. $s(1) = 1$;
2. *if $\{x_i \,|\, i \in I\}$ is a set of mutually orthogonal elements in L, then $s(\bigvee_{i \in I} x_i) = \Sigma_{i \in I} s(x_i)$.*

Note: a two-valued state s, seen as a characteristic set function, always selects either an element $x \in L$ or its orthocomplement x': $\forall x \in L, s(x) = 1 \Leftrightarrow s(x') = 0$. An immediate consequence is that, if s_1 and s_2 are distinct states, then $s_1 \setminus s_2$, seen as a set, is not empty. In this work, we will refer to two-valued states simply as states, and mainly consider them as the subsets they are formally characteristic functions of.

Given a logic L, we denote by $\mathcal{S}(L)$ the set of all (two-valued) states on L and by $\mathcal{S}_x$ the set $\{s \in \mathcal{S}(L) \mid s(x) = 1\}$. In general, logics could, by their structure, admit no state at all. Logics having "enough" states in such a way that the order relation can be re-constructed by their states are called *rich*:

Definition 7. *([13], definition 2.1.14) Let L be a logic and $x, y \in L$. L is* rich *if:*

$$\mathcal{S}_x \subseteq \mathcal{S}_y \ \Rightarrow \ x \leq y.$$

Intuitively, a rich logic is faithfully represented by its set of states. A theorem due to Stanley Gudder ([13], 2.2.1) shows that each element of a rich logic can be seen as the subset of states that contain it. Such subsets form, with inclusion and set complement, a new logic which is isomorphic to the original one. In this logic, meet and join correspond respectively to intersection and union, and two elements are orthogonal whenever they are disjoint. This theorem also shows that, under the assumptions of Proposition 1, inclusion and set complement provide a rich quantum logic. Thus, regional logics are rich.

It is well known [1] that given an elementary transition system A, its regions will constitute the places of an elementary net system whose case graph is isomorphic to A. In this view, the states of a regional logic correspond to the potential markings of the associated net system. The following definition will allow us to identify the sequential components of such a net system, directly on the regional logic. We introduce the notion of *compatibility* among regions, which translates precisely the idea that they belong to the same sequential component:

Definition 8. *We say that two elements a and b of a logic L are* compatible *if, and only if, there exist three mutually orthogonal elements $\hat{a}$, $\hat{b}$ and c in L such that $a = \hat{a} \vee c$ and $b = \hat{b} \vee c$.*

The relation of *compatibility* between a and b is noted as $a \ \$ \ b$. Its opposite relation is noted by $x \not\$ y$. Note that two ordered or orthogonal elements are always compatible. A maximal subset of *pairwise compatible* elements in L, with the partial order on L, is a sublogic of L. It is also a Boolean algebra: it is closed under complement $(\cdot)'$, meet $(\wedge)$, and join $(\vee)$, and the last two distribute over each other. We may refer to these as *maximal Boolean sublogics* or *blocks* of L.

Intuitively, the blocks of L will correspond to the sequential components in the net system view. This motivates the idea that there should be some consistency among compatible elements of the logic. The well studied property of *regularity*, accurately translates this notion.

Definition 9. *([13], definition 1.3.26) A logic L is called* regular *if, for any set $\{a, b, c\} \subseteq L$ of pairwise compatible elements, $a \ \$ \ (b \vee c)$.*

The main consequence of regularity is precisely that it will allow us to interpret compatibility of two elements as their belonging to the same sequential component.

Proposition 2. *([13], Proposition 1.3.29) A logic L is* regular *if and only if every pairwise compatible subset of L admits an enlargement to a* Boolean sublogic *of L.*

Indeed, it has been shown [4] that every regional logic is regular. In [4] regular logic were called *coherent* logics.

The dual notion, that incompatibility of two regions should translate their belonging to different sequential components, was studied in [5]. Two conditions belonging to different sequential components are potentially marked independently from each other. Hence, the underlying logic allows for four states, corresponding to the possible combinations of markings of the two conditions, as places of the net system. This property of quantum logics, called ETI, was shown to hold in every regional logic.

2.3 Synthesis of Saturated Transition Systems

Given a *regional logic* $L(A) = \langle \mathcal{R}(A), \subseteq, (.)', \emptyset, Q \rangle$ for some CETS A, we can construct a new transition system from $L = L(A)$. The transition system defined by L is constructed with $\mathcal{S}(L)$ as its set of states. Its events are pairs of symmetric differences between states.

Definition 10. *Let $e = [s_1, s_2]$ denote the pair of set differences $\langle s_1 \setminus s_2, s_2 \setminus s_1 \rangle$. Define ${}^\bullet e = s_1 \setminus s_2$, and $e^\bullet = s_2 \setminus s_1$, so that $e = \langle {}^\bullet e, e^\bullet \rangle$. Let*

$$E = \{[s_1, s_2] = \langle s_1 \setminus s_2, s_2 \setminus s_1 \rangle \mid s_1, s_2 \in \mathcal{S}(L), s_1 \neq s_2\}.$$

Transitions and their labels can now be defined as

$$T = \{(s_1, [s_1, s_2], s_2) \mid s_1, s_2 \in \mathcal{S}(L), s_1 \neq s_2\}.$$

We can now define the transition system $A(L) = (\mathcal{S}(L), E, T)$. In this frame, $s_1\ [e\rangle\ s_2 \Rightarrow s_2 = (s_1 \setminus {}^\bullet e) \cup e^\bullet$. We will refer to this binding as firing rule.

It was shown in [4] that, if L is rich, then $A(L)$ is an Elementary Transition System. It was also shown that A embeds into $A(L)$, but note that $A(L)$ might have more states than A. By construction, every pair of states in $A(L)$ is linked by a transition in each direction. Its underlying graph is complete. We will hence say that it is *saturated* with both states, and events.

Note that the regions of $A(L)$ form again a regional logic, $\mathcal{R}(A(L))$. In general, we know [4] that $L(A)$ embeds into $\mathcal{R}(A(L))$.

2.4 Block Diagrams, Cliques of Atoms and Examples

A compact graphical representation of finite logics can be obtained with a technique due to Richard Greechie and reported in [14] for the case of finite orthomodular lattices.

Remark 1. The set of elements L of a logic, or any of its subsets S, together with a symmetric relation such as $\$$, $\not\$$, or $\perp$ can be considered as an undirected graph. All notion of graph theory as in [2] can then be applied. For instance, we will call a *clique of the relation* R, any subset of L such that each pair of its elements is in R. Let S be clique of R, then it is *maximal* if $\forall x \notin S : (\exists y \in S : x \not R\, y)$.

An element of a quantum logic L is called an *atom* whenever it is minimal for the partial order when one excludes 0. Let $\mathcal{A}(L)$ be the set of such elements. When restricted to atoms, the compatibility relation $\$$ coincides with orthogonality relation $\perp$. Hence, the atoms of a maximal Boolean sublogic will form a maximal clique of $\perp$. Note that, in the net system interpretation, orthogonality acts as mutual exclusion. Indeed, two atomic conditions belonging in the same sequential component cannot be marked simultaneously. From point 3 in Definition 5, it is clear that any element of the logic can be retrieved as the join of a subset of such a clique, and actually, the pair $(\mathcal{A}(L), \perp|_{\mathcal{A}(L)\times\mathcal{A}(L)})$ is sufficient to recover the whole structure of the logic.

The *block diagram* of a logic exploits this fact, and depicts only the atoms of the logic. Instead of representing all orthogonality dependencies, maximal cliques of $\perp$ are represented by straight lines. In this way, the *block diagram* of a Boolean algebra will be composed only by one maximal clique of $\perp$ while at least two *blocks* are needed for the representation of a non-Boolean logic L. In Figure 2, a simple non-Boolean logic L is represented. The two *blocks* of atoms in $\mathcal{A}(L)$ composing its *block diagram* are drawn on the right.

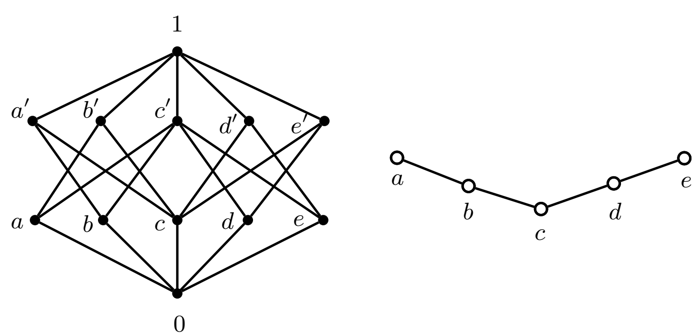

Fig. 2. A simple non-Boolean logic composed by two *blocks*.

We will now present a result that will allow us to characterize the states of the logic on its block diagram. Informally, this result states that a state must contain exactly one atom per maximal Boolean sublogic. In order to formalize this, we shall define some notation.

Definition 11. *Let $\mathcal{C}_\perp(\mathcal{A}(L))$ be the set of maximal cliques of the $\perp$ relation. Let $\mathcal{C}_{\text{NC}}(\mathcal{A}(L))$ be the set of maximal cliques of $\not\$$. By $CL(\mathcal{A}(L))$ we denote the set of elements $\alpha \in \mathcal{C}_{\text{NC}}(\mathcal{A}(L))$ such that $\forall \beta \in \mathcal{C}_\perp(\mathcal{A}(L))$, $|\alpha \cap \beta| = 1$. Finally, for any subset $S \subseteq \mathcal{A}(L)$, the* up-closure *of S is defined as $\uparrow S := \{s' \in L \mid \exists s \in S : s \leq s'\}$.*

In [4], it was shown that states of a regular logic can be characterized in terms of sets of atoms of the logic (see Proposition 29, p. 649). We restate here the result, fitting it to the terminology used in this paper (in [4], *coherent* was used instead of *regular*, *prime filter* instead of *state*, and the result was stated for transitive partial Boolean algebras, a class coinciding with regular logics).

Theorem 1. *Let $\langle L, \leq, (.)', 0, 1\rangle$ be a regular logic, $\mathcal{A}(L)$ its set of atoms, $s \in S(L)$ and $CL(\mathcal{A}(L))$ as in definition 11 above, then $(s \cap \mathcal{A}(L)) \in CL(\mathcal{A}(L))$. Moreover, if $\alpha \in CL(\mathcal{A}(L))$ then $\uparrow\alpha$ is a state in L.*

Example 1. With reference to Figure 2 the states of L are: $\uparrow\{a, d\}$, $\uparrow\{a, e\}$, $\uparrow\{b, d\}$, $\uparrow\{b, e\}$, $\uparrow\{c\}$. A maximal clique of $\not\$$ is not, in general, sufficient for the definition of a state. The requirement, as in definition 11, that each maximal clique of $\not\$$ meets each maximal clique of $\perp$ is essential. With reference to Figure 3, the up-closure of the maximal clique $\{a_1, b_2, g_1\}$ of $\not\$$, $\uparrow\{a_1, b_2, g_1\}$ is not a state since an element from the block $\{c_1, c_2, c_3\}$ is missing.

Example 2. The regional logic L of the transition system in Figure 1 is represented, as *block diagram* of its atoms, in Figure 3. The minimal regions in L, corresponding to the atoms, are: $a_2 = \{2, 4, 6, 8\}$, $a_1 = \{1, 3, 5, 7\}$, $g_1 = \{1, 2, 9, 11\}$, $g_2 = \{3, 4, 10, 12\}$, $b_2 = \{7, 8, 11, 12\}$, $b_1 = \{5, 6, 9, 10\}$, $c_1 = \{9, 10, 11, 12\}$, $c_2 = \{1, 2, 3, 4\}$ and $c_3 = \{5, 6, 7, 8\}$. Its representation as *Net System* is in Figure 4 where only one representative of the class of markings is drawn. This marking is composed by the regions $c_2 = \{1, 2, 3, 4\}$, $a_1 = \{1, 3, 5, 7\}$ and $g_1 = \{1, 2, 9, 11\}$.

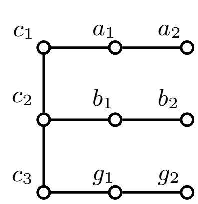

Fig. 3. The regional logic of the transition system in Figure 1

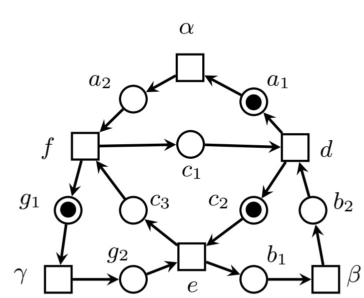

Fig. 4. Net System representation for the Transition System in Figure 1, the marking corresponds to state 1

We now show that the events of a logic are fully characterized by the atoms in their neighbourhoods. This allows us to characterize events on the block diagram of L.

Definition 12. *For each $e \in E$, the* neighbourhood *of e is $\nu(e) = {}^\bullet e \cup e^\bullet$; the* atomic neighbourhood *is $\nu_\mathcal{A}(e) = ({}^\bullet e \cap \mathcal{A}(L)) \cup (e^\bullet \cap \mathcal{A}(L))$*

Lemma 1. *Let $e_1, e_2 \in E$. If $\nu_\mathcal{A}(e_1) = \nu_\mathcal{A}(e_2)$, then $e_1 = e_2$.*

Proof. Let $e_1 = \langle s_1 \setminus s_2, s_2 \setminus s_1 \rangle$. Choose $x \in s_1 \setminus s_2$, and suppose that x is not an atom of L. Then there is at least one atom a of L in s_1, with $a < x$. From $x \notin s_2$, it follows that $a \notin s_2$. A symmetric argument applies to $s_2 \setminus s_1$, hence e_1 cannot be distinguished from e_2 because of a non-atomic element in their neighbourhoods. □

3 Structured Regional Events

The synthesis of the saturated transition system from a given logic, allows one to identify sets of transitions according to the modifications they apply on states, when these are seen as subsets of regions. These modifications, or symmetric differences of regions, are then taken as the set of labels of the transition system.

3.1 The Structure of Events

In this section, we exploit this fact to endow the set of events with a richer structure. We define a composition operation with inverse, and neutral element. The neutral element will just be $e_\emptyset := \langle \emptyset, \emptyset \rangle$ the empty event. This event should be considered only for structural purposes. Intuitively, it would just leave any state unchanged; hence no transition carries it as a label.

We first define the operation of sequential composition of two events. The resulting event will correspond to the consecutive occurrence of its operands. The existence of this new event will depend, however, on the existence of a state binding the operands in sequence. In our setting, events are defined over a fixed structure of states, making the following a partial operation.

Let L be a regional logic.

Definition 13. *We say that two events $e_1, e_2 \in E$ are sequentiable iff*

$$\exists s \in \mathcal{S}(L) : {e_1}^\bullet \subseteq s \wedge {}^\bullet e_2 \subseteq s \tag{1}$$

If two events are sequentiable, we define their sequential composition as:

$$\begin{aligned} e_1 \oplus e_2 &= \langle ({}^\bullet e_1 \setminus {e_2}^\bullet) \cup ({}^\bullet e_2 \setminus {e_1}^\bullet), ({e_2}^\bullet \setminus {}^\bullet e_1) \cup ({e_1}^\bullet \setminus {}^\bullet e_2) \rangle \\ &= \langle ({}^\bullet e_1 \cup {}^\bullet e_2) \setminus ({e_1}^\bullet \cup {e_2}^\bullet), ({e_1}^\bullet \cup {e_2}^\bullet) \setminus ({}^\bullet e_1 \cup {}^\bullet e_2) \rangle \end{aligned}$$

The definition does not depend on the choice of s. Condition 1 requires an occurrence of e_1 to bring the system to a state which enables e_2. The following result exemplifies why this is required, and shows that $e_1 \oplus e_2$ belongs to E.

Proposition 3. *Let $s_0, s, s_1 \in \mathcal{S}(L)$ be three distinct states, and $e_1, e_2 \in E$ be events such that $s_0\ [e_1\rangle\ s\ [e_2\rangle\ s_1$. Then $s_0\ [e_1 \oplus e_2\rangle\ s_1$*

Proof. By definition of E, $e_1 = \langle s_0 \setminus s, s \setminus s_0\rangle$, and $e_2 = \langle s \setminus s_1, s_1 \setminus s\rangle$. Then by definition of $\oplus$, $e_1 \oplus e_2 = \langle((s_0 \setminus s) \cup (s \setminus s_1)) \setminus ((s \setminus s_0) \cup (s_1 \setminus s)),$
$((s \setminus s_0) \cup (s_1 \setminus s)) \setminus ((s_0 \setminus s) \cup (s \setminus s_1))\rangle$. Straightforward set operations then lead to $e_1 \oplus e_2 = \langle s_0 \setminus s_1, s_1 \setminus s_0\rangle$. □

Associativity of the composition comes as a consequence of this last result. We now define the inverse of an event.

Definition 14. *Let $e \in E : e = \langle {}^{\bullet}e, e^{\bullet}\rangle$. Then $e^{-1} = \langle e^{\bullet}, {}^{\bullet}e\rangle$ is the inverse of e.*

The transition system synthesized from a logic is saturated with events. For any ordered pair of states, E contains an event which labels the corresponding transition. Hence, as stated in the following proposition, any event has an inverse.

Proposition 4. *Let $e \in E$ then $e^{-1} \in E$.*

Proof. $e \in E \Rightarrow \exists s, s' \in \mathcal{S}_L : s\ [e\rangle\ s'$. By definition, it holds that $e = \langle s \setminus s', s' \setminus s\rangle$. Now, $\exists t \in T : t = (s', [s', s], s)$, and clearly, it will carry as a label $\langle s' \setminus s, s \setminus s'\rangle = e^{-1} \in E$. □

The inverse is well defined for all event, and so is the composition $e \oplus e^{-1} = e^{-1} \oplus e = e_{\emptyset}$.

Whenever composition of two events is well defined, inversion behaves accordingly, in the sense that $(e_1 \oplus e_2)^{-1} = e_2^{-1} \oplus e_1^{-1}$ whenever they are defined.

The sequential composition operation is, of course, not commutative in general. We can formalise concurrency of two events as the commutativity of their sequential composition: the occurrence of any of them does not disable the other.

Definition 15. *We say two events $e_1, e_2 \in E$ are* independent *iff*

$$\nu_{\mathcal{A}}(e_1) \cap \nu_{\mathcal{A}}(e_2) = \emptyset$$

Note that this condition holds iff $({}^{\bullet}e_1 \cup e_1{}^{\bullet}) \cap ({}^{\bullet}e_2 \cup e_2{}^{\bullet}) = \emptyset$

We say they are concurrent *if they are independent, and there exists a state that enables both of them:*

$$\exists s \in \mathcal{S}(L) : {}^{\bullet}e_1 \cup {}^{\bullet}e_2 \subseteq s \wedge (e_1{}^{\bullet} \cup e_2{}^{\bullet}) \cap s = \emptyset$$

It follows from independence that, when two events are concurrent, their composition becomes simply $e_1 \oplus e_2 = \langle {}^{\bullet}e_1 \cup {}^{\bullet}e_2, e_1{}^{\bullet} \cup e_2{}^{\bullet}\rangle$.

We show that two events are concurrent if and only if their composition is commutative and some state enables both of them. This correspondence between concurrency among two events, and commutativity of their composition accurately translates the idea that concurrent events form diamonds in the saturated transition system. We exploit this fact to prove the following result.

Proposition 5. *Let $e_1, e_2 \in E$; then they are concurrent iff $e_1 \oplus e_2$ and $e_2 \oplus e_1$ are well defined and equal, and there exists a state which enables both.*

Proof. We first prove that if e_1 and e_2 are concurrent, then their composition is commutative. So let $s, s_1, s_2, s' \in \mathcal{S}(L) : s\ [e_1\rangle\ s_1 \wedge s_1\ [e_2\rangle\ s' \wedge s\ [e_2\rangle\ s_2$. The firing rule yields $s_1 = (s \setminus {}^{\bullet}e_1) \cup e_1{}^{\bullet}$, and $s' = (s_1 \setminus {}^{\bullet}e_2) \cup e_2{}^{\bullet}$, and it follows from independence of e_1 and e_2 that $s' = (s \setminus ({}^{\bullet}e_1 \cup {}^{\bullet}e_2)) \cup (e_1{}^{\bullet} \cup e_2{}^{\bullet}) = (s_2 \setminus {}^{\bullet}e_1) \cup e_1{}^{\bullet}$. Hence $s_2\ [e_1\rangle\ s'$.

For the converse, suppose $s, s_1, s_2, s' \in \mathcal{S}(L) : s\ [e_1\rangle\ s_1 \wedge s_1\ [e_2\rangle\ s' \wedge$ $s\ [e_2\rangle\ s_2 \wedge s_2\ [e_1\rangle\ s'$. Assume, as a contradiction hypothesis that $\exists a \in \nu_{\mathcal{A}}(e_1) \cap \nu_{\mathcal{A}}(e_2) \neq \emptyset$. We distinguish four cases. If $a \in {}^{\bullet}e_1 \cap {}^{\bullet}e_2$ then $a \notin s_1$ and so $s_1 \not[e_2\rangle$. If $a \in {}^{\bullet}e_1 \cap e_2{}^{\bullet}$ then $a \in s_2$, and so $s \not[e_2\rangle s_2$. Analogously $a \in {}^{\bullet}e_2 \cap e_1{}^{\bullet}$ implies that $s \not[e_1\rangle s_1$. Finally, If $a \in e_1{}^{\bullet} \cap e_2{}^{\bullet}$ then $s_1 \not[e_2\rangle$. □

The following example should clarify the last proof. The composition of two concurrent events is a diagonal of the diamond they form. In the net system interpretation, this corresponds to an event which has the same effect as firing the operands simultaneously. This view justifies the term of *step*.

Example 3. In Figure 5 we can see that $s_1\ [e_1\rangle\ s_5\ [e_2\rangle\ s_2$, and $s_1\ [e_2\rangle\ s_6\ [e_1\rangle\ s_2$. The fact that from s_1 the system reaches s_2 after firing e_1 and e_2 regardless of the order of firing, implies that $e_1 \oplus e_2 = e_2 \oplus e_1 = d$.

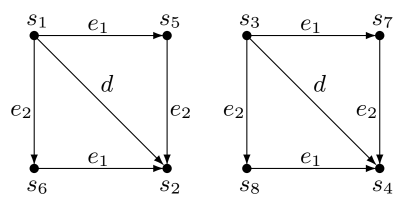

Fig. 5. Two diamonds

We say that an event is a *step* if it is the diagonal of some diamond. Whenever e_1, e_2 are concurrent events, we may write $e = e_1 \oplus e_2 = e_1$ **step** e_2, and say that e is the step $\{e_1, e_2\}$.

3.2 Induced Orthogonality Subgraphs

There is an essential difference between the non-commutative and the commutative composition of events. Indeed, commutative composition provides the diagonal of a diamond. Such a diagonal event should, in our view, be distinguished from other events. Provided that a system already depicts two concurrent events, their step provides no additional information regarding concurrency, or connectedness of the system. This section will be devoted to formalising, and proving this last statement.

In order to do so, we study the structure of diamonds as seen in the Greechie representation of the logic. In the following, we will consider states of the logic as the cliques of incompatibility of the atoms which intersect every block of the logic as in theorem 1. We will therefore see events as the symmetric differences of these. For the basic notions on graph theory, see [2].

Definition 16. *Let* $G_\perp(L) = (\mathcal{A}(L), \perp|_{\mathcal{A}(L)\times\mathcal{A}(L)})$ *be the* Orthogonality Graph *of a Regional Logic L. Given a subset of atoms* $A \subseteq \mathcal{A}(L)$*, we will call* $(A, \perp|_{A^2})$ *the* subgraph *of* $G_\perp(L)$ induced *by A.*

Recall that each maximal clique of $G_\perp(L)$ corresponds to the set of atoms of a block (maximal Boolean sublogic) of L.

A *bipartite graph* is a graph whose set of vertices can be partitioned in two classes, such that no two vertices in the same class are bound by an edge (see [2]). A state, seen as a subgraph of $G_\perp(L)$, has the property that no pair of its vertices is in $\perp$. We will see that this implies that, for each event e, its set of pre-conditions and its set of post-condition, form a bipartite graph $({}^\bullet e, e^\bullet)$, when considered as the subgraph of $G_\perp(L)$ induced by $\nu_\mathcal{A}(e)$.

Proposition 6. *Let* $e \in E$*, and* $G_\perp(L)$ *as in Definition 16. Then* $({}^\bullet e \cap \mathcal{A}(L), e^\bullet \cap \mathcal{A}(L))$ *is a bipartition of* $(\nu_\mathcal{A}(e), \perp|_{\nu_\mathcal{A}(e)^2})$.

Proof. $e \in E$ implies $\exists s, s' \in \mathcal{S}(L) : s\ [e\rangle\ s'$. Then ${}^\bullet e \subseteq s$ and $e^\bullet \subseteq s'$. ${}^\bullet e \subseteq s \Rightarrow \forall a_1, a_2 \in {}^\bullet e : a_1, a_2 \in s$. From Theorem 1, it follows that $a_1 \not\$\, a_2$, so in particular $(a_1, a_2) \notin \perp$. The result is analogous for $e^\bullet$ and s'. □

The converse, however, is not always the case. We here provide an example of a logic, in which there is a bipartite subgraph of $G_\perp(L)$ which does not correspond to an event.

Example 4. In Figure 3, we can consider $(\{a_1, b_1, g_1, a_2, b_2, g_2\})$ as an induced subgraph. In this case, $(\{a_1, b_1, g_1\}, \{a_2, b_2, g_2\})$ forms a bipartition, but there is no event $e = \langle\{a_1, b_1, g_1\}, \{a_2, b_2, g_2\}\rangle$, since no state enables it. Such a state would be a maximal clique of $\not\$$, but as seen in Example 2, $\{a_1, b_1, g_1\}$ does not intersect every block of L, and is therefore not a state.

We wish to study the properties of diamonds, as seen on $G_\perp(L)$. The next result will exhibit a property that any pair of concurrent events must verify.

Proposition 7. *Let* e_1 *and* e_2 *be two concurrent events. Then* $\forall a_1 \in {}^\bullet e_1, \forall a_2 \in {e_2}^\bullet : a_1 \not\$\, a_2$*, and symmetrically* $\forall a_3 \in {e_1}^\bullet, \forall a_4 \in {}^\bullet e_2 : a_3 \not\$\, a_4$.

Proof. Since e_1 and e_2 are concurrent, they must form a diamond. Namely, $\exists s, s_1, s_2, s' \in \mathcal{S}(L) : s\ [e_1\rangle\ s_1\ [e_2\rangle\ s' \wedge s\ [e_2\rangle\ s_2\ [e_1\rangle\ s'$. Hence, ${e_1}^\bullet \cup {}^\bullet e_2 \subseteq s_1$, and ${e_2}^\bullet \cup {}^\bullet e_1 \subseteq s_2$. Theorem 1 then yields the result. □

This last result might become clearer when looking at a Greechie diagram.

Example 5. Consider the events $e_1 = \langle\{a_1\}, \{a_2\}\rangle$, and $e_2 = \langle\{b_1\}, \{b_2\}\rangle$ from Figure 3. They are both enabled at state $s = \{a_1, b_1, c_3\}$. The existence of the state $s_1 = \{a_2, b_1, c_3\}$ implies that $a_2 \in {e_1}^\bullet$, and $b_1 \in {}^\bullet e_2$ verify $a_2 \not\$\, b_1$. Symmetrically, the state $s_2 = \{a_1, b_2, c_3\}$ provides $a_1 \not\$\, b_2$.

We shall now prove the much stronger converse result. Namely, we show that given an event, seen as the bipartite induced subgraph of $G_\perp(L)$, we can determine whether it is the step of two concurrent events, and if so, actually reconstruct the corresponding diamond.

Lemma 2. *Connected components of the subgraph of $G_\perp(L)$ induced by an event e, form a set of pairwise concurrent events, the step of which is e.*

Proof. Let $e \in E$, and suppose that there is a partition $\{a_i\}_{i \leq n}$ of ${}^\bullet e$, and a partition $\{b_i\}_{i \leq n}$ of $e^\bullet$ such that $a_i \cup b_i$ are the connected components of $(\nu_\mathcal{A}(e), \perp|_{\nu_\mathcal{A}(e)^2})$, for $i \leq n$. We proceed by induction over the states. For the base case, consider $s, s' \in \mathcal{S}(L) : s\ [e\rangle\ s'$. Now, let $j \leq n$, then $(s \setminus a_j) \cup b_j$ must be a clique of $\not\$$, since otherwise there would be an $r_1 \in \bigcup_{i \neq j} a_i$ and an $r_2 \in b_j$ such that $r_1 \perp r_2$, contradicting that $a_j \cup b_j$ is a connected component for $\perp$.

Now suppose $(s \setminus a_j) \cup b_j$ is not a state. Then there must be a block B of $G_\perp(L)$ such that $((s \setminus a_j) \cup b_j) \cap B = \emptyset$. $s \in \mathcal{S}(L)$ implies that $\exists r \in s \cap B$, so it must be $r \in a_j$. Analogously, $s' \in \mathcal{S}(L)$ implies that $\exists r' \in s' \cap B$, so it must be $r' \in \bigcup_{i \neq j} b_i$. Clearly, $r, r' \in B \to r\ \$\ r'$, and ${}^\bullet e \cap e^\bullet = \emptyset \to r \neq r'$, so $r \perp r'$. This contradicts the fact that $a_j \cup b_j$ is a connected component. Then $s_j = (s \setminus a_j) \cup b_j$ must be a state, and there must be an $e_j = \langle a_j, b_j \rangle \in E$ such that $s\ [e_j\rangle\ s_j$.

The induction step is absolutely analogous. One only needs to consider the event $e'_j = \langle \bigcup_{i \neq j} a_i, \bigcup_{i \neq j} b_j \rangle$, noting that $s_j\ [e'_j\rangle\ s'$.

The result holds for any choice of $j \leq n$ at any induction step, hence all permutations of n provide a sequential decomposition of the event e. Note that, by construction, and for any $j \leq n$, the events e_j are pairwise independent, and the existence of the state s provides that they must be concurrent. It should be clear that $\forall j_1 \neq j_2 : e_{j_1} \oplus e_{j_2} = e_{j_2} \oplus e_{j_1} \in E$. It follows from associativity of $\oplus$ that $\bigoplus_{i \leq n} e_i = e$. □

This Lemma represents the main technical contribution of this work, and most of the results we henceforth present are consequences of it. Indeed, it is a powerful tool which will allow us to decompose events, as the step of a family of pairwise concurrent events.

In order to formalise the notion of minimality of the events which are no further decomposable in this way, we introduce the following partial order of events. We shall not extend our discourse on this last construction in this paper, but we find it justifies the intuitive idea that such events are in some sense minimal.

Definition 17. *Let E be a set of events. We define $<\subseteq E^2$ as $\forall e_1, e \in E : e_1 < e$ iff $\exists e_2 \in E : e_1 \oplus e_2 = e_2 \oplus e_1 = e$.*

Proposition 8. *$(E, <)$ is a strict partial order*

Proof. $<$ is irreflexive since, by convention, we do not consider $e_\emptyset \in E$. $<$ is transitive, as a consequence of Proposition 7, and independence of concurrent events. $<$ is antisymmetric, since $e_1 \oplus e_2 = e_2 \oplus e_1 = e$, and $e_3 \oplus e = e \oplus e_3 = e_1$ imply that $e_1 \oplus e_2 \oplus e_3 = e_1$, and so $e_3 = e_2^{-1}$. Hence $s\ [e\rangle$ implies that $s\ [e_2\rangle$ and

$s\ [e_2^{-1}\rangle$, but then there must be an $s' \in \mathcal{S}(L) : s'\ [e_2\rangle\ s\ [e_2\rangle$, which contradicts the separation axioms in Definition 4. □

We will say that an event is *minimal* if it is minimal with respect to this partial order. From Lemma 2, it should be clear that an event is minimal if and only if it is connected, as a bipartite induced subgraph of $G_\perp(L)$.

3.3 The Subset of Minimal Events

The main result that we present here is that it is sufficient to consider minimal events when synthesizing a transition system, in order for it to be connected, but also to depict all the concurrency encoded in the logic. This last notion will be formalized by comparing the set of regions of such a transition system, and that of the canonical synthesized system, which is saturated with events.

Definition 18. *We call $M \subseteq E$ the set of events which are minimal in $(E, <)$.*

We will start showing that M is sufficient to generate the regions of the transition system synthesized from L. The idea is rather simple. When two concurrent events are in E, then the separation axioms (see Definition 4) corresponding to their step are redundant.

Definition 19. *Let $A = (Q, E, T)$ be a transition system, and $G \subseteq E$ be a subset of events of A. Define T_G as the set of all transitions of A labelled by some element of G. Then*

$$A \setminus G = (Q, E \setminus G, T \setminus T_G)$$

Clearly, $A \setminus G$ is a, possibly non-connected, transition system.

Lemma 3. *Let $A = (Q, E, T)$ be a CETS, $s_0, s_1, s_2, s_3 \in Q$, and $e_1, e_2, d \in E$, such that $s_0\ [e_1\rangle\ s_1\ [e_2\rangle\ s_3$, $s_0\ [e_2\rangle\ s_2\ [e_1\rangle\ s_3$, and $s_0\ [d\rangle\ s_3$. Then A and $A \setminus \{d\}$ have the same set of regions.*

Proof. Any region of A is also a region of $A \setminus \{d\}$, since the latter has been obtained by removing an event. Suppose now that r is a region of $A \setminus \{d\}$, but not of A. This implies that there are two transitions in A, say (s_1, d, s_2) and (s_3, d, s_4), with different crossing relations with respect to r. Suppose that $s_1 \in r$, $s_2, s_3, s_4 \notin r$ (all other combinations can be dealt with in a similar way). By hypothesis, ${}^\bullet d = {}^\bullet e_1 \cup {}^\bullet e_2$, and $d^\bullet = e_1{}^\bullet \cup e_2{}^\bullet$. Hence, e_1 and e_2 are enabled at s_1 and at s_3, and form two diamonds, as shown in Figure 5.

Suppose that s_5 is in r. Then, e_1 is orthogonal to r, and $e_2 \in {}^\bullet r$; but this leads to a contradiction, since $s_7 \notin r$, and (s_7, e_2, s_4) does not cross the border of r. On the other hand, $e_5 \notin r$ contradicts the hypothesis that $s_3 \notin r$, because in this case e_1 should leave r. □

This last result implies, in particular, that minimal events convey all the information regarding concurrency, all other events being steps of these. This is formalized in the following theorem.

Theorem 2. *Let L be a rich, and regular logic. Let $A = (\mathcal{S}(L), E, T)$ be the saturated transition system synthesized from L. Let M be the set of minimal events in $(E, <)$. Then the Regional logic $\mathcal{R}(A)$ is isomorphic to $\mathcal{R}((\mathcal{S}(L), M, T_M))$.*

Proof. We reason by induction over the set of states. Starting from A, we apply Lemma 3 to show that for any diagonal d, $\mathcal{R}(A) \simeq \mathcal{R}(A \setminus \{d\})$. In the inductive step, the hypothesis will be that $\mathcal{R}(A \setminus D) \simeq \mathcal{R}(A)$, then for any step d in $E \setminus D$, Lemma 3 provides $\mathcal{R}(A \setminus (D \cup \{d\})) \simeq \mathcal{R}(A)$. Clearly, when D contains all the steps of E, then $E \setminus D = M$. □

Theorem 3. *Let M be a set of minimal events in $(E, <)$, and let T_M be the set of all transitions carrying some label in M, then the graph representation of the transition system $(\mathcal{S}(L), M, T_M)$ is connected.*

Proof. By construction, the graph associated to the saturated synthesized transition system $(\mathcal{S}(L), E, T)$ is a complete graph. It is therefore connected. Now let $s, s' \in \mathcal{S}(L)$. We proceed again by induction. Suppose $[s, s']$ is not a minimal event. Then Lemma 2 shows that $\exists s_{1/2} \in \mathcal{S}(L) : s\ [[s, s_{1/2}]\rangle\ s_{1/2}\ [[s_{1/2}, s']\rangle\ s'$. For the inductive step, assume there is a path $s\ [e_1\rangle\ s_1\ [e_2\rangle \ldots [e_i\rangle\ s_i\ [e_{i+1}\rangle\ s'$. Now, from Lemma 2 it holds that $\forall e_{j+1} \notin M : \exists s_{j1/2}$ such that $s_j\ [[s_j, s_{j1/2}]\rangle\ s_{j1/2}\ [[s_{j1/2}, s_{j+1}]\rangle\ s_{j+1}$. Clearly,$s\ [e_1\rangle\ s_1\ [e_2\rangle \ldots s_j\ [[s_j, s_{j1/2}]\rangle\ s_{j1/2}\ [[s_{j1/2}, s_{j+1}]\rangle\ s_{j+1} \ldots [e_i\rangle\ s_i\ [e_{i+1}\rangle\ s'$ is a path connecting s and s'. The induction will then end with a path connecting s and s' where all arcs correspond to minimal events. □

A direct consequence of this last theorem, is that the transition system depicting only minimal events verifies the first axiom of Definition 1.

4 Conclusions and Further Research

In the synthesis of a transition system from a quantum logic, the state of the art provides a canonical system which is saturated with events. All possible events are considered. In this work, we have endowed this set of events with a structure, so as to provide an alternative canonical system with less events, by considering only the ones which do not correspond to the step firing of any other two. We have shown that they then have the same set of regions. Together with the connectedness of the underlying graph in the new canonical system, we have gathered technical results which seem promising towards the main aim in this line of research. Our goal is to show that, under suitable conditions, the regional logic of the synthesized transition system is isomorphic to the original one.

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