Generalized metrics with applications to ratings and formal concept analysis

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Abstract. We introduce an order theoretic approach to generalized metrics that covers various concepts of distance. In particular, we point out the role of supermodular mappings on lattices, which we then apply in diverse settings such as comparison of ratings and formal concept lattices.

Keywords: Generalized metric, supermodular mappings, comparison of ratings, semimodular lattices, formal concept analysis

1 Introduction

Generalized metrics recently have become of increased interest for modeling *directed distances* with values in qualitative measurement spaces including ordered monoids and lattices. In [8] generalized metrics are proposed which turned out to be relevant for formal concept analysis and closure operators (see [4], [9], [10]).

In this paper, we will apply generalized metrics in order to compare *ratings*, that is comparing the rating methodologies of different rating agencies with different result scales. We analyze suitable result scales for the rating process and show that ratings are not limited to chain lattices but can as well use certain *semimodular lattices* as target. The paper also considers applications to formal concept analysis covering the extensional as well as the intensional point of view.

For our approach *supermodularity* plays an important role, which goes beyond ideas of measurement associated with Dempster-Shafer-Theory (see [10]).

2 A prior result on generalized metrics

In this section we recall a theorem on generalized metrics (compare [8]). We start with

Definition 1 ([8]) *$\mathcal{M} = (M, *, \varepsilon, \leq)$ is an* **ordered monoid** *if $\mathbb{M} := (M, *, \varepsilon)$ is a monoid and $(M, \leq)$ is a poset such that $a \leq b$ implies $c * a \leq c * b$ and $a * c \leq b * c$, for all $a, b, c \in M$.*

The class of ordered monoids is quite large. Examples are:

- $(\mathbb{R}, +, 0, \leq)$ and $(\mathbb{R}_+, *, 1, \leq)$ under the natural ordering of the real numbers;
- for any set E, $(\mathcal{P}(E), \cup, \subseteq, \emptyset)$ and $(\mathcal{P}(E), \cap, \subseteq, E)$;
- a meet-semilattice $(L, \wedge, 1_L, \leq_L)$ bounded from above by 1_L and a join-semilattice $(L, \vee, 0_L, \leq_L)$ bounded from below by 0_L.

In order to distinguish the respective order relations, in the following we will use the symbol "$\leq_{\mathbb{P}}$" for the order relations of a given poset $\mathbb{P}$ and "$\leq$" of a given ordered monoid $\mathcal{M}$, respectively:

Definition 2 ([8]) *Let $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a poset and $\mathcal{M} = (M, *, \varepsilon, \leq)$ be an ordered monoid. A mapping*

$$\Delta\colon \leq_{\mathbb{P}} \longrightarrow M$$

is called **functorial w. r. t.** $(\mathbb{P}, \mathcal{M})$*, if*

- *for all $p \in P$: $\quad \Delta(p, p) = \varepsilon$,*
- *for all $p, t, q \in P$ with $p \leq_{\mathbb{P}} t \leq_{\mathbb{P}} q$: $\quad \Delta(p, t) * \Delta(t, q) = \Delta(p, q)$.*

Furthermore, Δ is called **weakly positive***, if $\varepsilon \leq \Delta(p, q)$ for all $(p, q) \in\ \leq_{\mathbb{P}}$.*

In case $\mathbb{P} = (P, \leq_{\mathbb{P}})$ is a lattice, Δ is called **supermodular** *w. r. t. $(\mathbb{P}, \mathcal{M})$ (resp.* **modular***), if $\Delta(p \wedge q, q) \leq \Delta(p, p \vee q)$ (resp. equality) holds for all $p, q \in P$.*

So far, functorial mappings are only defined on the order relation $\leq_{\mathbb{P}} \subseteq P \times P$. In order to extend functorial mappings from the ordering $\leq_{\mathbb{P}}$ to its superset $P \times P$, we need

Definition 3 ([8]) *Let P be a set, and $\mathcal{M} = (M, *, \varepsilon, \leq)$ be an ordered monoid. A function $d\colon P \times P \longrightarrow M$ is called* **generalized quasi-metric (GQM) w. r. t.** $(P, \mathcal{M})$*, if*

(A0) *for all $(p, q) \in P \times P$: $\quad \varepsilon \leq d(p, q)$*

(A1) *for all $p \in P$: $\quad d(p, p) = \varepsilon$*

(A2) *for all $p, t, q \in P$: $\quad d(p, q) \leq d(p, t) * d(t, q)$*

If in addition, (A3) holds, d is a **generalized metric (GM) w. r. t.** $(P, \mathcal{M})$*:*

(A3) *for all $(p, q) \in P \times P$: $\quad d(p, q) = \varepsilon = d(q, p) \implies p = q$*

It is not quite obvious if functorial maps can be extended from $\leq_{\mathbb{P}}$ to the superset $P \times P$, which gives rise to the following

Question: For a given $\Delta\colon\ \leq_{\mathbb{P}} \longrightarrow M$*, does there exist a generalized quasi-metric* $d\colon P \times P \longrightarrow M$ *w. r. t.* $(P, \mathcal{M})$ *which extends* Δ *such that* $d|_{\leq_{\mathbb{P}}} = \Delta$*?*

We find a positive answer and sufficient conditions in the following

Theorem 1 ([8]) *Let* $\mathbb{P} = (P, \leq_{\mathbb{P}})$ *be a lattice and let* $\mathcal{M} = (M, *, \varepsilon, \leq)$ *be an ordered monoid. If a map* $\Delta\colon\ \leq_{\mathbb{P}}\ \longrightarrow M$ *is weakly positive, supermodular and functorial w. r. t.* $(\mathbb{P}, \mathcal{M})$*, then*

$$d\colon P \times P \longrightarrow M, (p, q) \mapsto \Delta(p \wedge q, q)$$

is a GQM w. r. t. $(P, \mathcal{M})$*.*

3 Application to ratings

In this section we formalize the *rating process* and show how to compare ratings from different sources.

Let O be a finite set of *objects* to be rated, prominent examples are financial entities which issue debt. There are different (credit) rating agencies applying different ratings, where a (credit) *rating* is a mapping $A : O \to C(n) := \{0, \ldots, n\}$. "0" represents the lowest (credit) quality, "n" the highest, and $C(n)$ is called *rating scale*. It is clear that $C(n)$ is a complete lattice, naturally and totally ordered by "$\leq$", and n is called *length* of the *chain* $C(n)$. Our goal is to compare the results of two different rating agencies. The two agencies rate the same objects but they apply different rating methodologies, which leads to the

Question: Given two ratings A and B from different sources, which one is more progressive?

Progressive in this context means systematically giving a better rating to the same set of objects. Such "optimism" might lead to an underestimation of the underlying risks compared to the less progressive view since the more progressive view tends to ask for a lower risk premium.

Input: O a set, a finite chain $S := C(n) = \{0, \ldots, n\}$, two ratings $A, B : O \to S$

Definition 4 (Rating B is progressive given rating A)

$$D^{+}(A, B) := \sum_{o \in O\colon\ A(o) \leq B(o)} \mathit{rank}\ B(o) - \mathit{rank}\ A(o)$$

where the natural rank function in a chain *is given by rank* $:= s,\ s \in S$

We immediately notice:

– $D^+(A,B)$ is well defined and finite if O is finite: since there are only finitely many objects to be rated, we do not need to worry about non finite or even non countable sets.

– $D^+(A,B) \geq 0$

– $D^+(A,B) = 0$ iff $\forall o \in O : B(o) \leq A(o)$

A little less obvious is the following property: $D^+(A,B)$ is "triangular", i.e. $\forall E \colon O \to S$: $D^+(A,B) \leq D^+(A,E) + D^+(E,B)$. To see this we apply Theorem 1 as follows:

The set $\mathcal{O}$ of all ratings $O : A \to S$ is endowed with a natural order: $A \leq_{\mathbb{O}} B$ if $A(o) \leq B(o)$ for all $o \in O$. We write $\mathbb{O} = (\mathcal{O}, \leq_{\mathbb{O}})$. $\mathbb{O}$ is even a lattice where $(A \vee B)(o) = \max(A(o), B(o))$ and $(A \wedge B)(o) = \min(A(o), B(o))$.

For $A \leq_{\mathbb{O}} E$ define $\Delta^+ \colon \leq_{\mathbb{O}} \to \mathbb{N} \cup \{0\}$ via $\Delta^+(A,E) := \sum_{o \in O} \mathit{rank}\ E(o) - \mathit{rank}\ A(o)$. Δ^+ is functorial, since $\Delta^+(A,E) = \Delta^+(A,B) + \Delta^+(B,E)$ for the totally ordered triple $A \leq_{\mathbb{O}} B \leq_{\mathbb{O}} E$. Since $\min(a,b) + \max(a,b) = a + b$ for all real numbers a, b, Δ^+ is even a modular map.

Applying Theorem 1, thus D^+, which is the the extension of Δ^+, is triangular.

Usually $D^+(A,B) \neq D^+(B,A)$, i.e. D^+ is not symmetric. If $D^+(A,B) > D^+(B,A)$ then A is more *conservative* than B, and B is more progressive than A. In order to measure a symmetric distance between ratings, we proceed as follows:

Input: O a finite set, a finite chain $C(n)$, two ratings $A, B : O \to C(n)$

Definition 5 (Distance between ratings A and B)

$$D(A,B) := D^+(A,B) + D^+(B,A)$$

Being the L^1-distance of the rankings, D is symmetric: $D(A,B) = D(B,A)$, and $D(A,B) = 0 = D(B,A)$ if and only if $A = B$.

We will use D to derive a brute-force algorithm to solve the following issue: Rating scales do not need to be identical since different raters might use different rating scales:

Question: how can we compare ratings in case the rating scales are of different length?

The algorithm we propose will use embeddings (order preserving injections) of one chain into the other and minimize the distance D over all possible embeddings. For example, there are 3 possibilities of embedding $C(1)$ into $C(2)$, and 6 possibilities of embedding $C(1)$ into $C(3)$.

Input: O a finite set, ratings $A : O \to C(k)$, $B : O \to C(n)$, $k, n \in \mathbb{N}$ with $k \leq n$

Algorithm 1 (Scaling with minimal distance) –

– *Run through all embeddings* $E_i : C(k) \to C(n)$

– *Calculate* $E_i \circ A$ *and* $D(B,\ E_i \circ A)$ *for each embedding* E_i

– *Pick (one of) the* E_i *with minimal distance* $D(B,\ E_i \circ A)$

Comments: This algorithm is based on the implicit assumption, that both rating agencies are subject matter experts and "know what they are doing", which is reflected in building the minimum of the distances over all possible embeddings. No (subjective) expert opinion or management discretion is needed to decide before hand on the best possible embedding: instead, the algorithm increases objectivity in the sense that the best embedding is chosen purely based on the input data.

4 Generalized targets for ratings

So far, we only have used finite chains - i.e. totally ordered sets - for the rating process. In this section we will generalize the target sets of the rating process, another application of Theorem 1 will help us to answer the following

Questions: Are we limited to totally ordered sets? What about more general lattices as target of ratings? Which lattices will work?

The idea is to use Theorem 1 to "extend" the distances defined above, which essentially compares positions in a finite chain. To this end, we need

Definition 6 (Jordan-Dedekind chain condition) *A poset* P *is said to satisfy the* Jordan-Dedekind chain condition *if any two maximal chains between the same elements of* P *have the same finite length, where a chain* $C \subseteq P$ *is called* maximal *if, for any chain* $D \subseteq P$*,* $C \subseteq D$ *implies* $C = D$*.*

If $p, q \in P$ with $p \leq q$, then p, q are contained in at least one chain in P. In order to measure a distance Δ between p and q using the natural rank function as introduced in Definition 4, we can take any *maximal chain* between p and q, and the Jordan-Dedekind chain condition makes sure that this procedure is independent of choice of the maximal chain, and thus the following is well defined: $\Delta(p, q) := \text{length}(C) = \text{rank}(q)$ (in C) for any maximal chain C with $p, q \in C$.

The lattice depicted in Figure 1 violates the Jordan-Dedekind chain condition and serves as counter example: the chain on the left side yields $\Delta(x \wedge y, x \vee y) = 2$, the chain on the right would yield 3 as distance Δ between $x \wedge y$ and $x \vee y$.

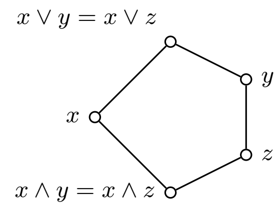


Fig. 1: A lattice violating the Jordan-Dedekind chain condition

Slightly more general formulated, in a poset P with Jordan-Dedekind chain condition which has a smallest element 0_P we can define rank(q) in the same way for every element $q \in P$ as length(C) for any maximal chain containing 0_P and q. This rank function $\Delta\colon P \to \mathbb{N} \cup \{0\}$ is weakly positive and functorial. If Δ happens to be also supermodular, then applying Theorem 1 all together we get

Corollary 1 *Let P be a lattice with Jordan-Dedekind chain condition and supermodular rank function Δ. Then*

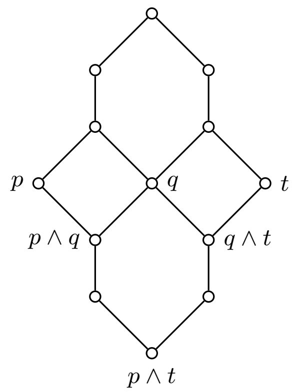
$$\boxed{d\colon P \times P \longrightarrow M, \quad (p, q) \mapsto \Delta(p \wedge q, q)}$$

is a GQM w. r. t. $(P, \mathbb{N} \cup \{0\})$. Furthermore, given two ratings $A, B : O \to P$,

$$D^{+}(A, B) := \sum_{o \in O\colon\, A(o) \leq B(o)} rank\ B(o) - rank\ A(o)$$

is a also a GQM w. r. t. $(O, \mathbb{N} \cup \{0\})$.

Remark: The Jordan-Dedekind chain condition per se is not enough, as we can deduct from the lattice depicted in Figure 2.



$$d(p,t) = rank(p) - rank(p \wedge t) = 3 - 0 = 3$$
$$d(p,q) = rank(p) - rank(p \wedge q) = 3 - 2 = 1$$
$$d(q,t) = rank(q) - rank(q \wedge t) = 3 - 2 = 1$$

Hence, $d(p,t) = 3 > 2 = d(p,q) + d(q,t)$, and d is <u>not</u> a triangular metric.

Fig. 2: A complete lattice satisfying the Jordan-Dedekind chain condition but bearing a non triangular metric based on the rank function

So with the help of Corollary 1 we can give a positive answer: not only simple chains are suitable targets for the rating process, much more, there is the huge class of lattices which allow for a (finite) Jordan-Dedekind chain condition together with a supermodular rank function as rating targets. In particular, modular lattices of finite length will work very well, where a lattice is called *modular* if it does not contain a sublattice of the form in Figure 1.

But we are not limited to modular lattices. In Figure 3 there is an example of a *lower semimodular* lattice which is not modular.

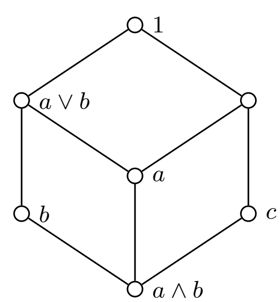


Fig. 3: A non modular lattice satisfying the Jordan-Dedekind chain condition

A lattice L is called *lower semimodular* if $\forall a, b \in L\colon\ b \lessdot a \vee b \Rightarrow a \wedge b \lessdot a$, where we write $a \lessdot b$ if $a < b$ and $a < x \leq b$ implies $x = b$. Every modular is lower semimodular, but the converse is obviously not true. In particular, the rank function of the lattice depicted in Figure 3 is only supermodular but not modular since $rank(b)+rank(c)=2<3=rank(1)=rank(b\vee c)+rank(b\wedge c)$.

This behavior of the rank function is somewhat typical, as we can see by the following characterization of lower semimodular lattices:

Theorem 2 *Let L be a lattice bounded from below such that any chain between any two elements of L is finite. L is lower semimodular if and only if L possesses a rank function r such that $\forall x,y\in L$:*

$$rank(x)+rank(y)\leq rank(x\vee y)+rank(x\wedge y).$$

L is modular if and only if $\forall x,y\in L$:

$$rank(x)+rank(y)=rank(x\vee y)+rank(x\wedge y).$$

Proof: this is the dual version of Theorem 2.27 from [1].

So lower semimodular lattices, bounded from below such that any chain between any two elements is finite, are exactly the appropriate class of lattices for our purposes.

Furthermore, we can generalize the scaling Algorithm 1 to this class of lattices using *rank preserving* mappings, where a mapping φ between two lattices L and L', which both possess a well-defined rank function, is called *rank preserving* if $rank(u)\leq rank(v)$ implies $rank(\varphi(u))\leq rank(\varphi(v))$ for all $u,v\in L$.

Input: O a finite set, ratings $A:O\to L$, $B:O\to L'$ for lower semimodular lattices L,L', where the finite number of elements of L' is denoted by n, and k denotes the number of elements of L such that $k\leq n$.

Algorithm 2 (Extended scaling with minimal distance) –

- *Run through all rank preserving injections $E_i:L\to L'$*
- *Calculate $E_i\circ A$ and $D(B,\ E_i\circ A)$ for each embedding E_i*
- *Pick (one of) the E_i with minimal distance $D(B,\ E_i\circ A)$*

Actually, this algorithm is the same as Algorithm 1, but applied to rank preserving injections instead of order preserving embeddings.

An example with only two rank preserving injections is depicted in Figure 4. Should we pick $\varphi(1)=a$ and $\varphi(2)=b$ or should we opt for the other possibility $\varphi(1)=b$ and $\varphi(2)=a$? Based on data, we would pick the possibility with the minimum distance.

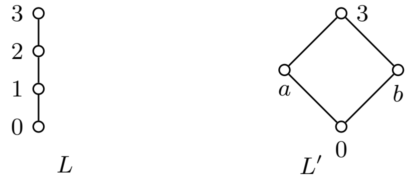


Fig. 4: Lattices L, L' which allow only for two rank preserving injections $\varphi\colon L \to L'$

5 Application to concept lattices

In order to keep the paper self-contained, we give a very short summary of formal concept lattices:

A *formal context* is a triple $\mathbb{K} = (G, M, I)$, where G is a set of objects, M is a set of attributes, and $I \subseteq G \times M$ is a binary incidence relation that expresses which objects have which attributes. For subsets $X \subseteq G$ of objects and subsets $Y \subseteq M$ of attributes, one defines the following mappings between the power sets of G and M:

- $G \supseteq X \mapsto X^{\triangleright} = \{m \in M\colon (x, m) \in I \text{ for every } x \in X\}$, and dually
- $M \supseteq Y \mapsto Y^{\triangleleft} = \{g \in G\colon (g, y) \in I \text{ for every } y \in Y\}$.

Clearly, $X_1 \subseteq X_2$ implies $X_1^{\triangleright} \supseteq X_2^{\triangleright}$ and $Y_1 \supseteq Y_2$ implies $Y_1^{\triangleleft} \subseteq Y_2^{\triangleleft}$. By a *formal concept* of the context $\mathbb{K}$ is understood a pair (X, Y) with $X \subseteq G$, $Y \subseteq M$ such that $X^{\triangleright} = Y$ and $Y^{\triangleleft} = X$. The set X is called the *extent* of the concept, and the set Y is referred to as *intent* of the concept. (X_1, Y_1) is called a *subconcept* of (X_2, Y_2) if $X_1 \subseteq X_2$, and we write $(X_1, Y_1) \preceq (X_2, Y_2)$. The class $\mathfrak{B}\mathbb{K}$ of all formal concepts of a given context $\mathbb{K}$ turns out to be ordered by $\preceq$, and even to be a complete lattice (cfr. Theorem 3 in chapter 1 of [9]), where supremum resp. infimum of two formal concepts are defined by

- $(X_1, Y_1) \vee (X_2, Y_2) = ((X_1 \cup X_2)^{\triangleright\triangleleft}, Y_1 \cap Y_2)$, resp.
- $(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \cap X_2, (Y_1 \cup Y_2)^{\triangleleft\triangleright})$

The lattice $\mathfrak{B}\mathbb{K}$ is called *concept lattice* of the context $\mathbb{K} = (G, M, I)$.

One consequence is that the mapping $G \supseteq X \mapsto X^{\triangleright\triangleleft} \subseteq G$ is a *closure mapping*, and therefore $\#(X) \leq \#(X^{\triangleright\triangleleft})$, where $\#(A)$ is the *count measure* of a set A, i.e. counting the number of elements of A. After these preparations we can derive

Proposition 1 *For $\alpha \coloneqq (A_1, A_2)$, $\beta \coloneqq (B_1, B_2) \in \mathfrak{B}\mathbb{K}$ with $\alpha \leq \beta$, the map*

$$\varDelta\colon\ \leq_{\mathfrak{B}\mathbb{K}} \longrightarrow \mathbb{N} \cup \{0\}, \quad (\alpha, \beta) \mapsto \varDelta(\alpha, \beta) \coloneqq \#(B_1 - A_1). \tag{1}$$

is functorial, weakly positive and supermodular.

Proof. Let $\gamma \coloneqq (C_1, C_2)$ such that $\alpha \leq \beta \leq \gamma$.

- Firstly, we can calculate

$$\begin{aligned}\Delta(\alpha,\gamma) &= \#C_1 - \#A_1 \\ &= (\#C_1 - \#B_1) + (\#B_1 - \#A_1) \\ &= \Delta(\beta,\gamma) + \Delta(\alpha,\beta).\end{aligned}$$

  Secondly, we see that

$$\Delta(\alpha,\alpha) = \#(A_1 - A_1) = \#\emptyset = 0.$$

  Hence, Δ is functorial.
- Δ is weakly positive, since $0 \leq \Delta(\alpha,\beta)$ holds for all $\alpha, \beta \in \mathfrak{B}\mathbb{K}$.
- To show that Δ is supermodular, we start to calculate $\Delta(\alpha \wedge \beta, \beta)$ and $\Delta(\alpha, \alpha \vee \beta)$ separately:

$$\begin{aligned}\Delta(\alpha\wedge\beta,\beta) &= \Delta\big((A_1 \cap B_1, (A_2 \cup B_2)^{\lhd\rhd}), (B_1, B_2)\big) \\ &= \#\big(B_1 - (A_1 \cap B_1)\big) \\ &= \#B_1 - \#(A_1 \cap B_1).\end{aligned}$$

$$\begin{aligned}\Delta(\alpha,\alpha\vee\beta) &= \Delta\big((A_1, A_2), (A_1 \cup B_1)^{\rhd\lhd}, A_2 \cap B_2)\big) \\ &= \#\big((A_1 \cup B_1)^{\rhd\lhd} - A_1)\big) \\ &= \#\big(A_1 \cup B_1)^{\rhd\lhd}\big) - \#A_1.\end{aligned}$$

  Consequently, since $X \mapsto X^{\rhd\lhd}$ is a closure mapping:

$$\begin{aligned}\Delta(\alpha\wedge\beta,\beta) &= \#B_1 - \#(A_1 \cap B_1) \\ &= \#(A_1 \cup B_1) - \#A_1 \\ &\leq \#\big(A_1 \cup B_1)^{\rhd\lhd}\big) - \#A_1 \\ &= \Delta(\alpha, \alpha \vee \beta).\end{aligned}$$

  Hence, Δ is supermodular. $\diamond$

All together, we now can introduce a *generalized metric* for concept lattices as follows:

Theorem 3 *Let Δ be as in (1) and $\alpha \coloneqq (A_1, A_2)$, $\beta \coloneqq (B_1, B_2) \in \mathfrak{B}\mathbb{K}$.*

Then the map

$$d\colon \mathfrak{B}\mathbb{K} \times \mathfrak{B}\mathbb{K} \longrightarrow \mathbb{N} \cup \{0\}, \quad (\alpha,\beta) \mapsto d(\alpha,\beta) \coloneqq \Delta(\alpha \wedge \beta, \beta)$$

is a GM.

Proof. This is a consequence of Proposition 1 together with Theorem 1. ◇

In our considerations we have focused on the extent. Likewise, there is also an intensional point of view for generalized metrics. In general, there are always two types of generalized metrics:

① $d_{ext}(\alpha, \beta) \coloneqq \#(B_1 - A_1)$

② $d_{int}(\alpha, \beta) \coloneqq \#(B_2 - A_2)$

6 Conclusions

- In order to compare ratings, we propose a sound directed metric in order to measure how **progressive** or **conservative** ratings are.
- **Scaling:** For chains S, S' of different size we propose an algorithmic solution.
- **Posets as target:** As target other then simply chains there is the huge class of lattices which allow for a (finite) Jordan-Dedekind chain condition together with a supermodular rank function. In particular, lower semimodular lattices of finite length will work very well. Also, our scaling algorithm based on minimal distances **extends to this class of lattices**.
- **Formal concept analysis:** Our concept of generalized metrics carries over to concept lattices, where we can cover the extensional as well as the intensional point of view.

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