On strong negation as linear duality

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Abstract

In [5] it is shown how a model for the logic of constructible duality is a symmetric monoidal closed category with products, that is, a model of linear logic. Remarkably, Nelson's strong negation $\sim$ behaves as linear duality $^{\perp}$. We compare it with some previous results from [4] regarding stable models in a linear logic setting, and suggest some further lines to explore.

1. From $\mathcal{N}_{cd}$ to linear logic

We began by reviewing the steps taken by Patterson et al [5] for the construction of an algebraic model for the logic of *constructible duality* (named here $\mathcal{N}_{cd}$), which is a conservative extension of the paraconsistent version of Nelson's logic constructible falsity, $\mathcal{N}_c^-$ (also named $\mathcal{N}4$ in [1, 2]).

Starting with the logic of constructible falsity $\mathcal{N}3$, (that is, positive intuitionistic logic with the involutory strong negation $\sim$), one obtains $\mathcal{N}4$ by dropping the axiom $\Gamma, \phi, \sim\phi \vdash \Delta$ and keeping the constants for falsity $\mathbf{0}$ and truth $\mathbf{1}$. The formal system of $\mathcal{N}4$ is obtained from that of intuitionistic logic plus the rules for $\sim$, as depicted in Figure 1.

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, \sim\sim\phi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, \sim\sim\phi}$$

$$\frac{\Gamma, \sim\phi \vdash \Delta \qquad \Gamma, \sim\psi \vdash \Delta}{\Gamma, \sim(\phi \wedge \psi) \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \sim\phi, \sim\psi}{\Gamma \vdash \Delta, \sim(\phi \wedge \psi)}$$

$$\frac{\Gamma, \phi, \sim\psi \vdash \Delta}{\Gamma, \sim(\phi \to \psi) \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \phi \qquad \Gamma \vdash \Delta, \sim\psi}{\Gamma \vdash \Delta, \sim(\phi \to \psi)}$$

$$\frac{}{\Gamma, \sim\mathbf{1} \vdash \Delta} \qquad \frac{}{\Gamma \vdash \Delta, \sim\mathbf{0}}$$

Figure 1. Fragment of the sequent system for $\mathcal{N}4$.

The first step consists in extending $\mathcal{N}4$ by defining the implication $\multimap$ as

$$\phi \multimap \psi = (\phi \to \psi) \wedge (\sim\psi \to \sim\phi)$$

(where $\to$ is the intuitionistic implication). In contrast to $\to$, the new implication forms a congruence on the equivalence class of interderivable formula. This implication provides an ordering on the corresponding algebra. A definable operator, which plays the rôle of a conjunction for $\multimap$ and hence providing a deduction theorem in the algebraic setting, is given by

$$\phi \otimes \psi = \sim(\phi \multimap \sim\psi).$$

This tensor operator satisfies

$$(\phi \otimes \psi) \multimap \gamma = \phi \multimap (\psi \multimap \gamma)$$

(that is, $\otimes$ is the right adjoint to $\multimap$). Since $\sim$ is involutory, $\otimes$ has a DeMorgan dual $\phi \parr \psi = \sim(\sim\phi \otimes \sim\psi)$.

The second step consists in adding a new constant $\mathbf{I}$ to $\mathcal{N}4$, which provides the unit for $\otimes$. Such constant is axiomatized as $\phi \to \mathbf{I}$ (for all formula ϕ), and $\sim\mathbf{I} = \mathbf{I}$. Since this unit is equal to its own negation, it is interpreted as maximally *overdefined* (both true and false), and every formula implies it. Its opposite, $\perp$ *undefined*, captures the notion of a proposition that is neither true nor false. Like $\mathbf{I}$, $\perp$ is its own negation, but implies every formula. The added constants $\mathbf{I}$ and $\perp$ are different from $\mathbf{0}$ and $\mathbf{1}$. The rules for the constants are shown in Figure 2. The logic $\mathcal{N}4$ with these constants is called *constructible duality*, we use the shorthand $\mathcal{N}_{cd}$.

There is a representation theorem for $\mathcal{N}_{cd}$ with respect to $\mathcal{H} \times \mathcal{H}^{op}$, for $\mathcal{H}$ a Heyting algebra. That is, every point in $\mathcal{H} \times \mathcal{H}^{op}$ is representable as an equivalence class of interderivable formulae in $\mathcal{N}_{cd}$. The remarkable insight of Patterson et al [5], is that the algebra $\mathcal{H} \times \mathcal{H}^{op}$ captures truth

$$\frac{\Gamma, \mathbf{I} \vdash \Delta}{\Gamma, {\sim}\mathbf{I} \vdash \Delta} \qquad \frac{\Gamma, \bot \vdash \Delta}{\Gamma, {\sim}\bot \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, \mathbf{I}}{\Gamma \vdash \Delta, {\sim}\mathbf{I}} \qquad \frac{\Gamma \vdash \Delta, \bot}{\Gamma \vdash \Delta, {\sim}\bot}$$

$$\frac{}{\Gamma \vdash \Delta, \mathbf{I}} \qquad \frac{}{\Gamma, \bot \vdash \Delta}$$

Figure 2. Rules for the constants of $\mathcal{N}_{cd}$.

propagating on one direction whereas falsity propagates in the opposite direction.

$Units$	$Formulae$
$h(\mathbf{1}) = (1, 0)$	$\hat{h}(\phi \wedge \psi) = \hat{h}(\phi) \wedge \hat{h}(\psi)$
$h(\mathbf{0}) = (0, 1)$	$\hat{h}(\phi \vee \psi) = \hat{h}(\phi) \vee \hat{h}(\psi)$
$h(\mathbf{I}) = (1, 1)$	${\sim}\hat{h}(\phi) = {\sim}\hat{h}(\phi)$
$h(\bot) = (0, 0)$	$\hat{h}(\phi \rightarrow \psi) = \hat{h}(\phi) \rightarrow \hat{h}(\psi)$

$Atoms$
$(x, x') \wedge (y, y') = (x \wedge y, x' \vee y')$
$(x, x') \vee (y, y') = (x \vee y, x' \wedge y')$
${\sim}(x, x') = (x', x)$
$(x, x') \rightarrow (y, y') = (x \rightarrow y, x' \wedge y')$

For P the set of propositional symbols, $h : P \longrightarrow \mathcal{H} \times \mathcal{H}^{op}$ is the valuation function, $\hat{h}$ is the unique homomorphic extension of h on formulae, and $x, x', y, y' \in \mathcal{H}$.

Figure 3. Interpretation of $\mathcal{N}_{cd}$ in $\mathcal{H} \times \mathcal{H}^{op}$.

The interpretation of $\mathcal{N}_{cd}$ in terms of $\mathcal{H} \times \mathcal{H}^{op}$ is described in Figure 3. We use the same symbols for the logical connectives and the operators in the algebra. Remember that $\rightarrow$ is modelled as the pseudo complement in the algebra, that is $\hat{h}(\phi \rightarrow \psi) = \hat{h}(\phi) \rightarrow \hat{h}(\psi)$ which is the element $\bigvee\{y \mid x \wedge y \leq z\}$, for $x = \hat{h}(\phi)$ and $z = \hat{h}(\psi)$ in the algebra $\mathcal{H}$. Note how the interpretation of strong negation is given by switching the coordinates of the point in the algebra $\mathcal{H} \times \mathcal{H}^{op}$. This interpretation of the strong negation is simpler than eg. [7, 2].

A formula ϕ of $\mathcal{N}_{cd}$ is algebraically valid ($\models \phi$) if and only if for all Heyting algebras $\mathcal{H}$ and for all valuation function $h : P \longrightarrow \mathcal{H} \times \mathcal{H}^{op}$ we have $\hat{h}(\phi) = (1, *)$ (where $*$ stands for any valuation in $\mathcal{H}$). A formula ϕ of $\mathcal{N}_{cd}$ is derivable in $\mathcal{N}_{cd}$ ($\vdash \phi$) if and only if it is algebraically valid.

The algebra $\mathcal{H} \times \mathcal{H}^{op}$ can be seen as having a copy of intuitionistic logic and a copy of Browerian logic. $\mathcal{N}_{cd}$ can be decomposed by means of the exponentials ! (schriek) and ? (why) which satisfy all the rules of an exponential in linear logic. The first operator, !, is defined as $!\phi = \phi \wedge \mathbf{I}$. The contents of the operation is to extract the positive information of ϕ. That is $\vdash \psi \rightarrow {\sim}!\phi$, for all ψ. As expected, we have $\phi \rightarrow \psi = !\phi \multimap \psi$. The exponential obeys the rules $!\phi \multimap \phi$ and $!!\phi \multimap !\phi$. The DeMorgan dual of ! is defined as $?\phi = {\sim}!{\sim}\phi = \phi \vee \mathbf{I}$, which obeys the rules $\phi \multimap ?\phi$, $?\phi = ??\phi$ and $\phi \multimap ?\psi = {\sim}\psi \rightarrow \phi$.

Algebraically, we have

$$\mathcal{H} \times \mathcal{H}^{op} \longrightarrow^{!} \mathcal{H} \times 1 \cong \mathcal{H}$$

that is, ! is a projection from the algebra of $\mathcal{N}_{cd}$ (a paraconsistent logic) to a Heyting algebra (a consistent logic). Note that ! is an interior operation on the poset $(\mathcal{H} \times \mathcal{H}^{op}, \multimap)$. Similarly, $\mathcal{H} \times \mathcal{H}^{op} \longrightarrow^{?} 1 \times \mathcal{H}^{op} \cong \mathcal{H}^{op}$. Additionally, other projections can be defined, which are defined in terms of the undefined unit $\bot$,

$$\hat{!}\phi = \phi \vee \bot \qquad \hat{?}\phi = \phi \wedge \bot,$$

and hence

$$\mathcal{H} \times \mathcal{H}^{op} \longrightarrow^{\hat{!}} 0 \times \mathcal{H}^{op} \qquad \mathcal{H} \times \mathcal{H}^{op} \longrightarrow^{\hat{?}} \mathcal{H} \times 0$$

respectively. These exponentials do not occur in linear logic.

Patterson et al [5] showed (via the Chu construction) that $\mathcal{H} \times \mathcal{H}^{op}$ is a model of linear logic, with comonads arising from the exponentials ! and ?. This produces a sound translation ^{t} of linear logic into $\mathcal{N}_{cd}$, as show in the table[1] below.

Linear logic	$\mathcal{N}_{cd}$
$\mathbf{0}^t$	$\mathbf{0}$
$\top^t$	$\mathbf{1}$
$\mathbf{1}^t$	$\mathbf{I}$
$\bot^t$	$\mathbf{I}$
$(\phi \otimes \psi)^t$	${\sim}((\phi^t \rightarrow {\sim}\psi^t) \wedge (\psi^t \rightarrow {\sim}\phi^t))$
$(\phi \multimap \psi)^t$	$(\phi^t \rightarrow \psi^t) \wedge ({\sim}\psi^t \rightarrow {\sim}\phi^t)$
$(\phi \parr \psi)^t$	$({\sim}\phi^t \rightarrow \psi^t) \wedge ({\sim}\psi^t \rightarrow \phi^t))$
$(\phi \& \psi)^t$	$(\phi^t \wedge \psi^t)$
$(\phi \oplus \psi)^t$	$(\phi^t \vee \psi^t)$
$(\phi^\bot)^t$	${\sim}\phi^t$
$(!_\ell \phi)^t$	$\phi^t \wedge \mathbf{I} = !\phi^t$
$(?_\ell \phi)^t$	$\phi^t \vee \mathbf{I} = ?\phi^t$

Therefore, if a formula ϕ is valid in linear logic, then ϕ^t is valid in $\mathcal{N}_{cd}$.

As a result, the exponentials of linear logic can be divided into two parts, one dealing with counting the occurrence of assumptions (typically, considered as consumable resources), and the other captures the embedding of intuitionistic logic into a paraconsistent logic.

What is especially attractive is the identification of Nelson's strong negation $\sim$ in $\mathcal{N}_{cd}$ as the linear duality $^\bot$.

[1] Since $\mathbf{1}^\bot = \bot$ in linear logic, it is clear why we have $\mathbf{1}^t = \bot^t = \mathbf{I}$.

2. Discussion: towards an equivalent characterisation

Osorio et al shown in [4] a characterisation of stable models for augmented programs formed by a linear part (for modelling consumable resources) and augmented clauses for representing knowledge (which involves strong negation). Such characterisation is based on the Girard's embedding of intuitionistic logic into linear logic, together with a series of results which characterise stable models as provability in intutionistic logic [6]. Such results are later extended to Nelson's logic $\mathcal{N}3$ and super intuitionistic logics [3]. It is important to mention that strong negation $\sim$ and linear duality $^\perp$ were considered as different operators in such references. What seems clear is that Nelson logic was enough for the purposes intended.

The general approach of [4] is summarised as follows.

Let Π an augmented program and M a set of $\sim$-literals, that is $M \subseteq \mathcal{L}^*_\Pi = \{\sim a \mid a \in \mathcal{L}_\Pi\} \cup \mathcal{L}_\Pi$. Let $\widetilde{M} = M - \mathcal{L}^*_\Pi$. M is a *stable model* of Π if and only if

$$\Pi \cup \neg\neg\widetilde{M} \cup \neg\neg M \Vdash M \qquad (1)$$

that is,

- Π is consistent, that is, for no atom $a \in \mathcal{L}_\Pi$, $\Pi \vdash a$ and $\Pi \vdash \neg a$, (where $\neg a = a \to \mathbf{0}$),
- Π is complete, that is, $\Pi \vdash l$ or $\Pi \vdash \neg l$ for all $\sim$-literal $l \in \mathcal{L}^*_\Pi$,
- $\Pi \vdash M$.

As mentioned, the characterisation (1) above works for intuitionistic logic (IL), Nelson's logic ($\mathcal{N}3$) and certain super-intuitioinistics logics, appropriately adapted in each case. When the logic for provability $\vdash$ considered in [4] is linear logic (LL), a number of translations are necessary. The first is the well-know Girard's embedding from IL to LL:

$$\begin{array}{ll} a^G = a \; (a \in P) & (\phi \to \psi)^G = \phi^G \multimap \psi^G \\ (\phi \wedge \psi)^G = \phi^G \& \psi^G & (\phi \vee \psi)^G = !\phi^G \oplus !\psi^G \\ (\neg\phi)^G = !\phi^G \multimap \mathbf{0}. & \end{array}$$

The second translation deals recursively with the strong negation:

$$\begin{array}{l} a^\circ := (!a) \otimes \top \\ (\phi \vee \psi)^\circ := \phi^\circ \parr \psi^\circ \\ (\phi \wedge \psi)^\circ := \phi^\circ \& \psi^\circ \\ (\phi \to \psi)^\circ := !(\phi^{\neg\circ} \parr \psi^\circ) \otimes \top \\ (\neg\phi)^\circ := (!\phi^{\neg\circ}) \otimes \top \\ (\sim \phi)^\circ := \phi^{\sim\circ} \end{array}$$

$$\begin{array}{l} a^{\sim\circ} := (!(a \parr 0)^\perp) \otimes \top \\ (\sim \phi)^{\sim\circ} := \phi^\circ \\ (\neg\phi)^{\sim\circ} := \phi^\circ \\ (\phi \vee \psi)^{\sim\circ} := \phi^{\sim\circ} \& \psi^{\sim\circ} \\ (\phi \wedge \psi)^{\sim\circ} := \phi^{\sim\circ} \parr \psi^{\sim\circ} \\ (\phi \to \psi)^{\sim\circ} := \phi^\circ \& \psi^{\sim\circ} \end{array}$$

$$\begin{array}{l} a^{\neg\circ} := ?(a \parr 0)^\perp \\ (\neg\phi)^{\neg\circ} := ?(\phi^\circ) \\ (\sim \phi)^{\neg\circ} := !\phi^{\neg\sim\circ} \otimes \top \\ (\phi \vee \psi)^{\neg\circ} := \phi^{\neg\circ} \& \psi^{\neg\circ} \\ (\phi \wedge \psi)^{\neg\circ} := \phi^{\neg\circ} \parr \psi^{\neg\circ} \\ (\phi \to \psi)^{\neg\circ} := \phi^\circ \& \psi^{\neg\circ} \end{array}$$

$$\begin{array}{l} a^{\neg\sim\circ} := ?a \\ (\neg\phi)^{\neg\sim\circ} := \phi^{\neg\circ} \\ (\sim \phi)^{\neg\sim\circ} := \phi^{\neg\circ} \\ (\phi \vee \psi)^{\neg\sim\circ} := \phi^{\neg\sim\circ} \parr \psi^{\neg\sim\circ} \\ (\phi \wedge \psi)^{\neg\sim\circ} := \phi^{\neg\sim\circ} \& \psi^{\neg\sim\circ} \\ (\phi \to \psi)^{\neg\sim\circ} := \phi^{\neg\circ} \parr \psi^{\neg\sim\circ} \end{array}$$

and so, the characterisation (1) of asp as provability in LL takes the form

- $!(\Pi)^G \otimes \phi \otimes (M^\perp_\&)^G) \multimap \mathbf{0}$ is not provable in LL,
- $!(\Pi)^G \otimes \phi \multimap q \otimes (M_\&)^G$ is provable in LL,

where ϕ is any formula in linear logic (standing for resources), q is an atom from $atoms(\phi)$ (which works as a query), and $M_\&$ stands for the set of atoms of M connected by &.

Despite being purely syntactical translations, they don't seem to be very enlightening as to provide insights into a model theoretic (or algebraic) justification.

Evidently, one expects that the characterisation (1) based on provability be equivalent to the model theoretic counterpart. That is, for a valuation function $h : P \longrightarrow \mathcal{N}$ in a Nelson algebra $\mathcal{N}$,

$$\{\hat{h}(\phi) \mid \phi \in \Pi\} \cup h[\neg\neg\widetilde{M}] \cup h[\neg\neg M] \models h[M] \qquad (2)$$

(where $h[X]$ is the usual direct image of h on the set X). This seems an interesting path to explore.

In the previous section we have seen how the algebra $\mathcal{H} \times \mathcal{H}^{op}$ is a model of linear logic and furthermore, a model for $\mathcal{N}_{cd}$. Hence, we can use a valuation $h : P \to \mathcal{H} \times \mathcal{H}^{op}$ in the characterisation (2). Furthermore, linear duality $^{\perp}$ is captured by strong negation $\sim$, so the contents of the second translation of [4] is not necessary in $\mathcal{H} \times \mathcal{H}^{op}$. Intuitively, the approach followed in [4] essentially moves from $\mathcal{N}3$ to LL via syntactical translations, whereas Patterson et al enriches $\mathcal{N}4$ (which is a conservative extension of $\mathcal{N}3$) as to obtain $\mathcal{N}_{cd}$. The representation of$\mathcal{N}_{cd}$ in terms of $\mathcal{H} \times \mathcal{H}^{op}$ provides the necessary algebraic support.

It is necessary to explore the advantages of $\mathcal{N}_{cd}$ for the characterisation of stable models along the lines previously summarised, that is, as provability in $\mathcal{N}_{cd}$ as well as validity in $\mathcal{H} \times \mathcal{H}^{op}$. This approach would lead to a finer treatment of stable models for preferential fragments of linear logic, eg MLL (involving only $\otimes$, $\parr$, $\multimap$ and $^{\perp}$) and the extensions MALL, MELL, etc. Also, much insight can be gained from the use of topological spaces (particulary frames and locales) as representation of Heyting algebras in the construction of $\mathcal{H} \times \mathcal{H}^{op}$. Alternatives to topological spaces, like algebraic domains, as well as the relation of morphisms for such domains and proofs in $\mathcal{N}_{cd}$ remain to be seen.

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