Dual ordered structures of binary relations

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Abstract. The theory of ordered structures like a (lattice) ordered semigroups is applied to graphs and automatons as well as to coding, programming and artificial intelligence. In this paper an algebraic structure on an underlying set of binary relations is considered. The structure includes the operations of Boolean algebra, inverse and composition. It is defined a dual semigroup to the binary relations ordered semigroup, and then the general properties of dual operations are studied.

1. Introduction

Abstract theory of algebraic structures (sometimes called universal algebra) forms the basis for various applications [1-8]. Semigroups and lattices are the simplest structures but not the least ones.

Let's recall some definitions:

The semigroup is a set with single binary operation $*$ satisfying associative low. A semigroup with neutral (identity) element is called a monoid;

The semiring is a set with couple of binary operations – addition and multiplication - satisfying associative lows. There are neutral elements for both of them and addition is commutative. Also multiplication distributes over addition and multiplication by zero annihilates semiring;

The lattice (as an algebraic structure) is a set with pair of binary operations – join and meet - satisfying associative lows, commutative lows, and absorption lows. A distributive lattice is a lattice in which the operations of join and meet distribute over each other. A bounded lattice is a lattice with neutral elements. The lattice's bottom is a neutral element for the join operation and the lattice's top is a neutral element for the meet operation;

The lattice (as a poset) is a partial ordered set such that each finite-elements subset has *supremum* (join) and *infimum* (meet). A bounded lattice is a lattice with bottom and top elements;

The ordered semigroup is a semigroup together with a partial order $\prec$ that is compatible with the semigroup operation i.e. $\forall u, v, w\, u \prec v \to w * u \prec w \circ v \wedge u * w \prec v * w$. The bounded semigroup is an ordered semigroup with bottom and top elements.

It's well known that any ordered semigroup is isomorphic to a subsemigroup of binary relations ordered by subset relation. In this paper we deal with a left composition of binary relation as a semigroup operation, i.e. we set

$$R_1 \circ R_2 = \{(u_1, u_2) \mid \exists u_3\, (u_1, u_3) \in R_1 \wedge (u_3, u_2) \in R_2\} \tag{1}$$

At first, we denote a universe as U and consider a power set of Cartesian square $2^{U\times U}$ as a collection of binary relations on U. The traditional approach to studying binary relations leads to ordered semigroup $S_{\mathrm{R}} = \langle 2^{U\times U}, (\circ, \subseteq)\rangle$ and bounded distributive lattice $L_{\mathrm{R}} = \langle 2^{U\times U}, (\cup, \cap)\rangle$. In this way we don't take into account a complement operation

$$\overline{R} = \{(u_1,u_2) \mid (u_1,u_2) \notin R\} \tag{2}$$

However, it's very convenient to use a complement element. For example, we can write the trichotomy low for relation R in several forms. First, we can write it as in equation (3)

$$I_{\mathrm{R}} \cup R \cup R^{-1} = 1_{\mathrm{R}} = U \times U = \overline{\varnothing} = \overline{0_{\mathrm{R}}} \tag{3}$$

Then, we can rewrite it in alternative form as antisymmetric low for the complement $\bar{R}$ as in equation (4)

$$\bar{R} \cap \bar{R}^{-1} \subseteq I_{\mathrm{R}} = \{(u,u) \mid u \in U\} \tag{4}$$

In this case and below we use the notations 1_{R}, 0_{R} and I_{R} for complete relation, empty relation and identity relation respectively. Note that 1_{R} and 0_{R} are top and bottom elements for lattice L_{R}. Also we denote the inverse relation of R as

$$R^{-1} = \{(u_2,u_1) \mid (u_1,u_2) \in R\} \tag{5}$$

2. Algebraic structure h_{R}

Let's consider an algebraic structure $h_{\mathrm{R}} = \left\langle 2^{U\times U}, \left(\cup,\cap,\circ,\bar{\ },^{-1},\subseteq,0_{\mathrm{R}},1_{\mathrm{R}},I_{\mathrm{R}}\right)\right\rangle$. It's easy to prove properties (6)-(41):

$$R_1 \cup (R_2 \cup R_3) = (R_1 \cup R_2) \cup R_3 \tag{6}$$

$$R_1 \cap (R_2 \cap R_3) = (R_1 \cap R_2) \cap R_3 \tag{7}$$

$$R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3 \tag{8}$$

$$R_1 \cup R_2 = R_2 \cup R_1 \tag{9}$$

$$R_1 \cap R_2 = R_2 \cap R_1 \tag{10}$$

$$0_{\mathrm{R}} \cup R = R \tag{11}$$

$$1_{\mathrm{R}} \cap R = R \tag{12}$$

$$I_{\mathrm{R}} \circ R = R \circ I_{\mathrm{R}} = R \tag{13}$$

$$1_{\mathrm{R}} \cup R = 1_{\mathrm{R}} \tag{14}$$

$$0_{\mathrm{R}} \cap R = 0_{\mathrm{R}} \tag{15}$$

$$0_{\mathrm{R}} \circ R = R \circ 0_{\mathrm{R}} = 0_{\mathrm{R}} \tag{16}$$

$$1_{\mathrm{R}} \circ 1_{\mathrm{R}} = 1_{\mathrm{R}} \tag{17}$$

$$R_1 \cup (R_2 \cap R_3) = (R_1 \cup R_2) \cap (R_1 \cup R_3) \tag{18}$$

$$R_1 \cap (R_2 \cup R_3) = (R_1 \cap R_2) \cup (R_1 \cap R_3) \tag{19}$$

$$R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3) \tag{20}$$

$$(R_2 \cup R_3) \circ R_1 = (R_2 \circ R_1) \cup (R_3 \circ R_1) \tag{21}$$

$$R_1 \circ (R_2 \cap R_3) \subseteq (R_1 \circ R_2) \cap (R_1 \circ R_3) \tag{22}$$

$$(R_2 \cap R_3) \circ R_1 \subseteq (R_2 \circ R_1) \cap (R_3 \circ R_1) \tag{23}$$

$$R_1 \cup (R_1 \cap R_2) = R_1 \tag{24}$$

$$R_1 \cap (R_1 \cup R_2) = R_1 \tag{25}$$

$$R \cup R = R \tag{26}$$

$$R \cap R = R \tag{27}$$

$$\overline{\overline{R}} = R \tag{28}$$

$$\overline{1_{\mathrm{R}}} = 0_{\mathrm{R}} \tag{29}$$

$$\overline{0_{\mathrm{R}}} = 1_{\mathrm{R}} \tag{30}$$

$$\overline{R_1 \cup R_2} = \overline{R}_1 \cap \overline{R}_2 \tag{31}$$
$$\overline{R_1 \cap R_2} = \overline{R}_1 \cup \overline{R}_2 \tag{32}$$
$$\left(R_1 \cup R_2\right)^{-1} = R_1^{-1} \cup R_2^{-1} \tag{33}$$
$$\left(R_1 \cap R_2\right)^{-1} = R_1^{-1} \cap R_2^{-1} \tag{34}$$
$$\left(R_1 \circ R_2\right)^{-1} = R_2^{-1} \circ R_1^{-1} \tag{35}$$
$$\overline{R^{-1}} = \overline{R}^{-1} \tag{36}$$
$$R_1 \subseteq R_2 \Leftrightarrow R_1 \cup R_2 = R_2 \Leftrightarrow R_1 \cap R_2 = R_1 \tag{37}$$
$$R_1 \subseteq R_2 \Rightarrow R_1 \cup R_3 \subseteq R_2 \cup R_3 \tag{38}$$
$$R_1 \subseteq R_2 \Rightarrow R_1 \cap R_3 \subseteq R_2 \cap R_3 \tag{39}$$
$$R_1 \subseteq R_2 \Rightarrow R_1 \circ R_3 \subseteq R_2 \circ R_3 \wedge R_3 \circ R_1 \subseteq R_3 \circ R_2 \tag{40}$$
$$R_1 \subseteq R_2 \Leftrightarrow \overline{R}_1 \supseteq \overline{R}_2 \tag{41}$$

The typical algebraic structures we can obtain by restriction of structure $h_{R(U\times U)}$ are as follows:

The bounded lattices of binary relations $LO_{\mathrm{R}}^1 = \left\langle 2^{U\times U}, \left(\subseteq, 0_{\mathrm{R}}, 1_{\mathrm{R}}\right)\right\rangle$ and $LO_{\mathrm{R}}^2 = \left\langle 2^{U\times U}, \left(\supseteq, 1_{\mathrm{R}}, 0_{\mathrm{R}}\right)\right\rangle$;

The bounded monoids of binary relations $M_{\mathrm{R}}^1 = \left\langle 2^{U\times U}, \left(\cup, \subseteq, 0_{\mathrm{R}}, 1_{\mathrm{R}}\right)\right\rangle$, $M_{\mathrm{R}}^2 = \left\langle 2^{U\times U}, \left(\cap, \subseteq, 0_{\mathrm{R}}, 1_{\mathrm{R}}\right)\right\rangle$, $M_{\mathrm{R}}^3 = \left\langle 2^{U\times U}, \left(\cup, \supseteq, 1_{\mathrm{R}}, 0_{\mathrm{R}}\right)\right\rangle$, $M_{\mathrm{R}}^4 = \left\langle 2^{U\times U}, \left(\cap, \supseteq, 1_{\mathrm{R}}, 0_{\mathrm{R}}\right)\right\rangle$ and $M_{\mathrm{R}} = \left\langle 2^{U\times U}, \left(\circ, \subseteq, I_{\mathrm{R}}, 0_{\mathrm{R}}, 1_{\mathrm{R}}\right)\right\rangle$;

The bounded semirings (with multiplicative identity) of binary relations $SR_{\mathrm{R}}^1 = \left\langle 2^{U\times U}, \left(\cup, \cap, \subseteq, 0_{\mathrm{R}}, 1_{\mathrm{R}}\right)\right\rangle$, $SR_{\mathrm{R}}^2 = \left\langle 2^{U\times U}, \left(\cap, \cup, \supseteq, 1_{\mathrm{R}}, 0_{\mathrm{R}}\right)\right\rangle$, and $SR_{\mathrm{R}} = \left\langle 2^{U\times U}, \left(\cup, \circ, \subseteq, 0_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$;

The Boolean algebras of binary relations $B_{\mathrm{R}}^1 = \left\langle 2^{U\times U}, \left(\cup, \cap, {}^{-}, 0_{\mathrm{R}}, 1_{\mathrm{R}}\right)\right\rangle$ and $B_{\mathrm{R}}^2 = \left\langle 2^{U\times U}, \left(\cap, \cup, {}^{-}, 1_{\mathrm{R}}, 0_{\mathrm{R}}\right)\right\rangle$.

3. Dual semigroup to S_{R}

Let's consider a Boolean isomorphism $F(R) = \overline{R}$ from B_{R}^1 onto B_{R}^2. We define a binary operation • in accordance with duality principle $F\left(R_1 \bullet R_2\right) = F\left(R_1\right) \circ F\left(R_2\right)$, i.e. we set

$$R_1 \bullet R_2 = \overline{\overline{R_1} \circ \overline{R_2}} = \left\{\left(u_1, u_2\right) \mid \forall u_3 \left(u_1, u_3\right) \in R_1 \vee \left(u_3, u_2\right) \in R_2\right\}. \tag{42}$$

Note that $F\left(0_{\mathrm{R}}\right) = 1_{\mathrm{R}}$, $F\left(1_{\mathrm{R}}\right) = 0_{\mathrm{R}}$, $F\left(I_{\mathrm{R}}\right) = \overline{I_{\mathrm{R}}}$ and moreover

$$R_1 \circ R_2 = \overline{\overline{R_1} \bullet \overline{R_2}} = \left\{\left(u_1, u_2\right) \mid \exists u_3 \left(u_1, u_3\right) \in R_1 \wedge \left(u_3, u_2\right) \in R_2\right\} \tag{43}$$
$$R_1 \bullet \left(R_2 \bullet R_3\right) = \left(R_1 \bullet R_2\right) \bullet R_3 \tag{44}$$
$$\overline{I_{\mathrm{R}}} \bullet R = R \bullet \overline{I_{\mathrm{R}}} = R \tag{45}$$
$$1_{\mathrm{R}} \bullet R = R \bullet 1_{\mathrm{R}} = 1_{\mathrm{R}} \tag{46}$$
$$0_{\mathrm{R}} \bullet 0_{\mathrm{R}} = 0_{\mathrm{R}} \tag{47}$$
$$R_1 \bullet \left(R_2 \cap R_3\right) = \left(R_1 \bullet R_2\right) \cap \left(R_1 \bullet R_3\right) \tag{48}$$
$$\left(R_2 \cap R_3\right) \bullet R_1 = \left(R_2 \bullet R_1\right) \cap \left(R_3 \bullet R_1\right) \tag{49}$$
$$R_1 \bullet \left(R_2 \cup R_3\right) \supseteq \left(R_1 \bullet R_2\right) \cup \left(R_1 \bullet R_3\right) \tag{50}$$
$$\left(R_2 \cup R_3\right) \bullet R_1 \supseteq \left(R_2 \bullet R_1\right) \cup \left(R_3 \bullet R_1\right) \tag{51}$$
$$\left(R_1 \bullet R_2\right)^{-1} = R_2^{-1} \bullet R_1^{-1} \tag{52}$$
$$R_1 \supseteq R_2 \Rightarrow R_1 \bullet R_3 \supseteq R_2 \bullet R_3 \wedge R_3 \bullet R_1 \supseteq R_3 \bullet R_2 \tag{53}$$

By our construction semigroup S_{R} is isomorphic to semigroup $\overline{S}_{\mathrm{R}} = \left\langle 2^{U\times U}, (\bullet, \supseteq) \right\rangle$ as well as monoid M_{R} and semiring SR_{R} are isomorphic to $\overline{M}_{\mathrm{R}} = \left\langle 2^{U\times U}, \left(\bullet, \supseteq, \overline{I_{\mathrm{R}}}, 1_{\mathrm{R}}, 0_{\mathrm{R}}\right)\right\rangle$ and $\overline{SR}_{\mathrm{R}} = \left\langle 2^{U\times U}, \left(\cap, \bullet, \supseteq, 1_{\mathrm{R}}, \overline{I_{\mathrm{R}}}\right)\right\rangle$ respectively. In such cases, we'll say that the algebraic structures are dual.

Now we use the previous definitions to argue the following logical consequences:

$$\begin{gathered}
u_1 R_1 \circ (R_2 \bullet R_3) u_2 \Leftrightarrow \exists u_3\; u_1 R_1 u_3 \wedge u_3 (R_2 \bullet R_3) u_2 \Leftrightarrow \exists u_3\; u_1 R_1 u_3 \wedge (\forall u_4\; u_3 R_2 u_4 \vee u_4 R_3 u_2) \\
\Leftrightarrow \exists u_3 \forall u_4\; u_1 R_1 u_3 \wedge (u_3 R_2 u_4 \vee u_4 R_3 u_2) \Leftrightarrow \exists u_3 \forall u_4\; (u_1 R_1 u_3 \wedge u_3 R_2 u_4) \vee (u_1 R_1 u_3 \wedge u_4 R_3 u_2) \\
\Rightarrow \forall u_4 \exists u_3\; (u_1 R_1 u_3 \wedge u_3 R_2 u_4) \vee (u_1 R_1 u_3 \wedge u_4 R_3 u_2) \Leftrightarrow \forall u_4\; (\exists u_3\; u_1 R_1 u_3 \wedge u_3 R_2 u_4) \vee (\exists u_3\; u_1 R_1 u_3 \wedge u_4 R_3 u_2) \\
\Leftrightarrow \forall u_4\; u_1 R_1 \circ R_2 u_4 \vee (\exists u_3\; u_1 R_1 u_3 \wedge u_4 R_3 u_2) \Leftrightarrow \forall u_4\; (u_1 R_1 \circ R_2 u_4 \vee u_4 R_3 u_2) \wedge (u_1 R_1 \circ R_2 u_4 \vee \exists u_3\; u_1 R_1 u_3) \\
\Leftrightarrow (\forall u_4\; u_1 R_1 \circ R_2 u_4 \vee u_4 R_3 u_2) \wedge (\forall u_4\; u_1 R_1 \circ R_2 u_4 \vee \exists u_3\; u_1 R_1 u_3) \\
\Leftrightarrow u_1 R_1 \circ (R_2 \bullet R_3) u_2 \wedge (\forall u_4\; u_1 R_1 \circ R_2 u_4 \vee \exists u_3\; u_1 R_1 u_3) \\
\Leftrightarrow u_1 R_1 \circ (R_2 \bullet R_3) u_2 \wedge ((\forall u_4 \exists u_3\; u_1 R_1 \circ R_2 u_4) \vee (\exists u_3\; u_1 R_1 u_3)) \Rightarrow u_1 R_1 \circ (R_2 \bullet R_3) u_2 \wedge (\exists u_3\; u_1 R_1 u_3) \\
\Leftrightarrow u_1 R_1 \circ (R_2 \bullet R_3) u_2 \wedge \left(u_1 \in D_{R_1}\right)
\end{gathered}$$

Let's denote a domain of R_1 as $D_{R_1} = \{u_1 \mid \exists u_3\; u_1 R_1 u_3\} \subseteq U$ and then consider a binary relation $E\left(D_{R_1}, 1_{\mathrm{R}}\right) = \left\{(u_1, u_2) \mid u_1 \in D_{R_1} \wedge u_2 \in U\right\} = D_{R_1} \times U \subseteq 1_{\mathrm{R}} = U \times U$. Now we can write

$$R_1 \circ (R_2 \bullet R_3) \subseteq (R_1 \circ R_2) \bullet R_3 \cap E\left(D_{R_1}, 1_{\mathrm{R}}\right) \tag{54}$$

Note that

$u_1 R_1 \circ 1_{\mathrm{R}} u_2 \Leftrightarrow \exists u_3\; u_1 R_1 u_3 \wedge u_3 1_{\mathrm{R}} u_2 \Leftrightarrow \exists u_3\; u_1 R_1 u_3 \wedge u_2 \in U \Leftrightarrow u_1 \in D_{R_1} \wedge u_2 \in U \Leftrightarrow u_1 E\left(D_{R_1}, 1_{\mathrm{R}}\right) u_2$

and so we obtain

$$R_1 \circ (R_2 \bullet R_3) \subseteq (R_1 \circ R_2) \bullet R_3 \cap R_1 \circ 1_{\mathrm{R}} \tag{55}$$

Similarly, we get

$$(R_2 \bullet R_3) \circ R_1 \subseteq R_2 \bullet (R_3 \circ R_1) \cap 1_{\mathrm{R}} \circ R_1 \tag{56}$$

$$R_1 \bullet (R_2 \circ R_3) \supseteq (R_1 \bullet R_2) \circ R_3 \cup R_1 \bullet 0_{\mathrm{R}} \tag{57}$$

$$(R_2 \circ R_3) \bullet R_1 \supseteq R_2 \circ (R_3 \bullet R_1) \cup 0_{\mathrm{R}} \bullet R_1 \tag{58}$$

In the latter, we have taken into account the following equalities:

$$\begin{gathered}
E\left(1_{\mathrm{R}}, D_{R_1^{-1}}\right) = U \times D_{R_1^{-1}} = 1_{\mathrm{R}} \circ R_1 \\
E\left(\overline{D_{\overline{R_1}}}, 1_{\mathrm{R}}\right) = \overline{D_{\overline{R_1}}} \times U = R_1 \bullet 0_{\mathrm{R}} \\
E\left(1_{\mathrm{R}}, \overline{D_{\overline{R_1}^{-1}}}\right) = U \times \overline{D_{\overline{R_1}^{-1}}} = 0_{\mathrm{R}} \bullet R_1
\end{gathered}$$

4. Extension of algebraic structure h_{R}

At first, we denote $O_{\mathrm{R}} = \overline{I_{\mathrm{R}}}$ and then we consider $H_{\mathrm{R}} = \left\langle 2^{U\times U}, \left(\cup, \cap, \bullet, \circ, ^{-}, ^{-1}, \subseteq, 0_{\mathrm{R}}, 1_{\mathrm{R}}, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$ as an extension of algebraic structure h_{R} . It is clear that all of the properties (6)-(58) are true for structure H_{R} .

Let's rewrite (57)-(58) in the form

$$\begin{gathered}
(R_2 \bullet R_3) \circ R_1 \cup R_2 \bullet 0_{\mathrm{R}} \subseteq R_2 \bullet (R_3 \circ R_1) \\
R_1 \circ (R_2 \bullet R_3) \cup 0_{\mathrm{R}} \bullet R_3 \subseteq (R_1 \circ R_2) \bullet R_3
\end{gathered}$$

So we can rewrite (55)-(58) as:

$$R_1 \circ (R_2 \bullet R_3) \subseteq (R_1 \circ R_2) \bullet R_3 \cap R_1 \circ 1_{\mathrm{R}} \tag{59}$$

$$R_1 \circ (R_2 \bullet R_3) \cup 0_{\mathrm{R}} \bullet R_3 \subseteq (R_1 \circ R_2) \bullet R_3 \tag{60}$$

$$(R_2 \bullet R_3) \circ R_1 \subseteq R_2 \bullet (R_3 \circ R_1) \cap 1_{\mathrm{R}} \circ R_1 \tag{61}$$

$$(R_2 \bullet R_3) \circ R_1 \cup R_2 \bullet 0_{\mathrm{R}} \subseteq R_2 \bullet (R_3 \circ R_1) \tag{62}$$

Obviously, for all binary relations $R_1, R_2, R_3 \in 2^{U\times U}$ we have

$$R_1 \circ (R_2 \bullet R_3) \subseteq (R_1 \circ R_2) \bullet R_3 \tag{63}$$

$$(R_2 \bullet R_3) \circ R_1 \subseteq R_2 \bullet (R_3 \circ R_1) \tag{64}$$

This is immediate from the inclusions (59)-(62).

Properties like the (55)-(58), (63)-(64) we'll call the laws of semi-compatibility. Now we are interested in cases of compatibility (low) of dual operation with each other

$$R_1 \circ (R_2 \bullet R_3) = (R_1 \circ R_2) \bullet R_3 = R_1 \circ R_2 \bullet R_3 \tag{65}$$

$$(R_2 \bullet R_3) \circ R_1 = R_2 \bullet (R_3 \circ R_1) = R_2 \bullet R_3 \circ R_1 \tag{66}$$

Note that we won't find algebraic substructures of H_{R} satisfying (65)-(66). Indeed, from (16), (17), (46), (47) we obtain

$$0_{\mathrm{R}} \circ (0_{\mathrm{R}} \bullet 1_{\mathrm{R}}) = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = (0_{\mathrm{R}} \circ 0_{\mathrm{R}}) \bullet 1_{\mathrm{R}} \tag{67}$$

$$0_{\mathrm{R}} \circ (1_{\mathrm{R}} \bullet 1_{\mathrm{R}}) = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = (0_{\mathrm{R}} \circ 1_{\mathrm{R}}) \bullet 1_{\mathrm{R}} \tag{68}$$

$$(1_{\mathrm{R}} \bullet 0_{\mathrm{R}}) \circ 0_{\mathrm{R}} = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = 1_{\mathrm{R}} \bullet (0_{\mathrm{R}} \circ 0_{\mathrm{R}}) \tag{69}$$

$$(1_{\mathrm{R}} \bullet 1_{\mathrm{R}}) \circ 0_{\mathrm{R}} = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = 1_{\mathrm{R}} \bullet (1_{\mathrm{R}} \circ 0_{\mathrm{R}}) \tag{70}$$

Hence, we have to restrict structure H_{R} to find algebraic substructures satisfying (65)-(66) and we'll call them compatible (sub)structures. Let's consider H_{R} without 0_{R} or 1_{R}.

We are studying the simplest cases of subsets of $2^{U\times U}$ as an underlying set for the operations from H_{R} below.

Let's denote the collection of (partial) functions from U to U as $U_{+1}^{U} \subseteq 2^{U\times U}$. It's easy that $\left\langle U_{+1}^{U}, \left(\circ, \subseteq, I_{U\times U}, 0_{U\times U}\right)\right\rangle$ is a bounded below submonoid of M_{R}.

We want to prove that U_{+1}^{U} is closed under the dual operation $\bullet$.

At first, we suppose that U contains only one element. In this case $U_{+1}^{U} = 2^{U\times U} = \{I_{\mathrm{R}}, 0_{\mathrm{R}}\}$, where I_{R} is identity function and 0_{R} is empty function. So U_{+1}^{U} is closed under $\bullet$ because $2^{U\times U}$ is closed.

Let now U contains more than one element. Suppose $R_1, R_2 \in U_{+1}^{U}$, but $R_1 \bullet R_2 \notin U_{+1}^{U}$. Hence, there are $u_1, u_2, u_3 \in U$ such that $u_2 \neq u_3$ and $\left((u_1,u) \in R_1 \vee (u,u_2) \in R_2\right) \wedge \left((u_1,u) \in R_1 \vee (u,u_3) \in R_2\right)$ for all $u \in U$. However, $R_1 \in U_{+1}^{U}$ and so there is no more than one $u_0 \in U$ satisfying relation $(u_1,u_0) \in R_1$. Whence $u_2 \neq u_3 \wedge (u,u_2) \in R_2 \wedge (u,u_3) \in R_2$ for all $u \in U \setminus \{u_0\} \neq \varnothing$. The latter is in contradiction with $R_2 \in U_{+1}^{U}$.

The proof is complete.

Let's consider the scale of sets $U_0^U \subseteq U^U \subseteq U_{+1}^{U}$, where U^U is a collection of total functions and U_0^U is a collection of total bijections from U to U.

We assume U to be a two-element set and denote the cardinality of $U = \{u_1, u_2\}$ as $|U|$. Obviously, $\left|2^{U\times U}\right| = 16$, $\left|U_{+1}^{U}\right| = 9$, $\left|U^U\right| = 4$, $\left|U_0^U\right| = |U| = 2$. Note $1_{\mathrm{R}} \notin U_{+1}^{U}$, $0_{\mathrm{R}} \in U_{+1}^{U}$, $0_{\mathrm{R}} \notin U^U$.

We have simulated some interesting cases of algebraic substructures to check irregularities in (65)-(66). Table 1 contains statistics on the incompatibility of dual operations.

Table 1. A total amount of incompatibility of $\circ$ and $\bullet$.

$R_1, R_2, R_3 \in W$	$R_1 \circ (R_2 \bullet R_3) \neq (R_1 \circ R_2) \bullet R_3$	$(R_2 \bullet R_3) \circ R_1 \neq R_2 \bullet (R_3 \circ R_1)$
$W = 2^{U \times U} \setminus \{0_{\mathrm{R}}\}$	706 from 3375	706 from 3375
$W = 2^{U \times U} \setminus \{1_{\mathrm{R}}\}$	706 from 3375	706 from 3375
$W = U_{+1}^{U}$	90 from 729	20 from 729
$W = U^{U}$	0 from 64	0 from 64
$W = U^{U} \cup \{0_{\mathrm{R}}\}$	10 from 75	4 from 75
$W = U^{U} \cup \{1_{\mathrm{R}}\}$	4 from 75	10 from 75
$W = U_0^{U}$	0 from 8	0 from 8
$W = U_0^{U} \cup \{0_{\mathrm{R}}\}$	0 from 27	0 from 27
$W = U_0^{U} \cup \{1_{\mathrm{R}}\}$	0 from 27	0 from 27

Cayley tables 2 and 3 describes the dual operations on the set $U_0^U \cup \{0_{\mathrm{R}}, 1_{\mathrm{R}}\}$.

Table 2. A Cayley table for $\circ$ on the set $U_0^U \cup \{0_{\mathrm{R}}, 1_{\mathrm{R}}\}$.

$\circ$	O_{R}	I_{R}	0_{R}	1_{R}
O_{R}	I_{R}	O_{R}	0_{R}	1_{R}
I_{R}	O_{R}	I_{R}	0_{R}	1_{R}
0_{R}	0_{R}	0_{R}	0_{R}	0_{R}
1_{R}	1_{R}	1_{R}	0_{R}	1_{R}

Table 3. A Cayley table for $\bullet$ on the set $U_0^U \cup \{0_{\mathrm{R}}, 1_{\mathrm{R}}\}$.

$\bullet$	O_{R}	I_{R}	0_{R}	1_{R}
O_{R}	O_{R}	I_{R}	0_{R}	1_{R}
I_{R}	I_{R}	O_{R}	0_{R}	1_{R}
0_{R}	0_{R}	0_{R}	0_{R}	1_{R}
1_{R}	1_{R}	1_{R}	1_{R}	1_{R}

The cases of incompatibility on the set $U_0^U \cup \{0_{\mathrm{R}}, 1_{\mathrm{R}}\}$ are listed below apart from (67)-(70). $R_1 \circ (R_2 \bullet R_3) \neq (R_1 \circ R_2) \bullet R_3$:

$$1_{\mathrm{R}} \circ (I_{\mathrm{R}} \bullet 0_{\mathrm{R}}) = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = (1_{\mathrm{R}} \circ I_{\mathrm{R}}) \bullet 0_{\mathrm{R}}$$
$$1_{\mathrm{R}} \circ (O_{\mathrm{R}} \bullet 0_{\mathrm{R}}) = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = (1_{\mathrm{R}} \circ O_{\mathrm{R}}) \bullet 0_{\mathrm{R}}$$
$$0_{\mathrm{R}} \circ (I_{\mathrm{R}} \bullet 1_{\mathrm{R}}) = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = (0_{\mathrm{R}} \circ I_{\mathrm{R}}) \bullet 1_{\mathrm{R}}$$
$$0_{\mathrm{R}} \circ (O_{\mathrm{R}} \bullet 1_{\mathrm{R}}) = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = (0_{\mathrm{R}} \circ O_{\mathrm{R}}) \bullet 1_{\mathrm{R}}$$

$(R_2 \bullet R_3) \circ R_1 \neq R_2 \bullet (R_3 \circ R_1)$:

$$(0_{\mathrm{R}} \bullet I_{\mathrm{R}}) \circ 1_{\mathrm{R}} = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = 0_{\mathrm{R}} \bullet (I_{\mathrm{R}} \circ 1_{\mathrm{R}})$$
$$(0_{\mathrm{R}} \bullet O_{\mathrm{R}}) \circ 1_{\mathrm{R}} = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = 0_{\mathrm{R}} \bullet (O_{\mathrm{R}} \circ 1_{\mathrm{R}})$$
$$(1_{\mathrm{R}} \bullet I_{\mathrm{R}}) \circ 0_{\mathrm{R}} = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = 1_{\mathrm{R}} \bullet (I_{\mathrm{R}} \circ 0_{\mathrm{R}})$$
$$(1_{\mathrm{R}} \bullet O_{\mathrm{R}}) \circ 0_{\mathrm{R}} = 0_{\mathrm{R}} \neq 1_{\mathrm{R}} = 1_{\mathrm{R}} \bullet (O_{\mathrm{R}} \circ 0_{\mathrm{R}})$$

It's easy to see that $U_0^U, \left(\bullet, O_{\mathrm{R}}\right)$ and $\left\langle U_0^U, \left(\circ, I_{\mathrm{R}}\right)\right\rangle$ are abelian groups. Moreover, $\left\langle U_0^U \cup \{0_{\mathrm{R}}\}, \left(\cap, \bullet, \circ, ^{-1}, \subseteq, 0_{\mathrm{R}}, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$ is a bounded below compatible algebraic structure and $\left\langle U_0^U \cup \{1_{\mathrm{R}}\}, \left(\cup, \bullet, \circ, ^{-1}, \subseteq, 1_{\mathrm{R}}, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$ is a bounded above compatible algebraic structure.

Let's give other examples.

Let $F_1 \subseteq U_{+1}^U$ be a set of partial and total functions are listed as $O_{\mathrm{R}} = \{(u_1, u_2), (u_2, u_1)\}$, $I_{\mathrm{R}} = \{(u_1, u_1), (u_2, u_2)\}$, $0_{\mathrm{R}} = \varnothing$, $f_1 = \{(u_1, u_1)\}$, $f_2 = \{(u_1, u_2)\}$, $f_3 = \{(u_2, u_1)\}$, $f_4 = \{(u_2, u_3)\}$.

Cayley tables 4 and 5 describes the dual operations on the F_1.

Table 4. A Cayley table for $\circ$ on the set F_1.

$\circ$	O_{R}	I_{R}	0_{R}	f_1	f_2	f_3	f_4
O_{R}	I_{R}	O_{R}	0_{R}	f_3	f_4	f_1	f_2
I_{R}	O_{R}	I_{R}	0_{R}	f_1	f_2	f_3	f_4
0_{R}	0_{R}	0_{R}	0_{R}	0_{R}	0_{R}	0_{R}	0_{R}
f_1	f_2	f_1	0_{R}	f_1	f_2	0_{R}	0_{R}
f_2	f_1	f_2	0_{R}	0_{R}	0_{R}	f_1	f_2
f_3	f_4	f_3	0_{R}	f_3	f_4	0_{R}	0_{R}
f_4	f_3	f_4	0_{R}	0_{R}	0_{R}	f_3	f_4

Table 5. A Cayley table for $\bullet$ on the set F_1.

$\bullet$	O_{R}	I_{R}	0_{R}	f_1	f_2	f_3	f_4
O_{R}	O_{R}	I_{R}	0_{R}	f_1	f_2	f_3	f_4
I_{R}	I_{R}	O_{R}	0_{R}	f_3	f_4	f_1	f_2
0_{R}	0_{R}	0_{R}	0_{R}	0_{R}	0_{R}	0_{R}	0_{R}
f_1	f_1	f_2	0_{R}	0_{R}	0_{R}	f_1	f_2
f_2	f_2	f_1	0_{R}	f_1	f_2	0_{R}	0_{R}
f_3	f_3	f_4	0_{R}	0_{R}	0_{R}	f_3	f_4
f_4	f_4	f_3	0_{R}	f_3	f_4	0_{R}	0_{R}

It's easy to see that $\left\langle F_1, \left(\cap, \bullet, \circ, ^{-1}, \subseteq, 0_{\mathrm{R}}, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$ is a bounded below compatible algebraic structure.

Now let $F_2 \subseteq U^U$ be a set of total functions are listed as O_{R}, I_{R}, $g_1 = \{(u_1, u_1), (u_2, u_1)\}$, $g_2 = \{(u_1, u_2), (u_2, u_2)\}$.

Cayley tables 6 and 7 describes the dual operations on the F_2.

Table 6. A Cayley table for $\circ$ on the set F_2.

$\circ$	O_{R}	I_{R}	g_1	g_2
O_{R}	I_{R}	O_{R}	g_1	g_2
I_{R}	O_{R}	I_{R}	g_1	g_2
g_1		g_1	g_1	g_2
	g_1		g_1	g_2

Table 7. A Cayley table for $\bullet$ on the set F_2.

$\bullet$	O_{R}	I_{R}	g_1	g_2
O_{R}	O_{R}	I_{R}	g_1	g_2
I_{R}	I_{R}	O_{R}	g_1	g_2
g_1	g_1	g_2	g_1	g_2
g_2	g_2	g_1	g_1	g_2

In this case $\left\langle F_2, \left(\cap, \bullet, \circ, ^{-1}, \subseteq, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$ is unbounded compatible algebraic structure. Taking into account (42)-(43) we can write

$$R_1 \circ \left(R_2 \bullet R_3\right) = \left(R_1 \circ R_2\right) \bullet R_3 \Leftrightarrow \overline{R_1} \bullet \left(\overline{R_2} \circ \overline{R_3}\right) = \left(\overline{R_1} \bullet \overline{R_2}\right) \circ \overline{R_3} \tag{71}$$

$$\left(R_2 \bullet R_3\right) \circ R_1 = R_2 \bullet \left(R_3 \circ R_1\right) \Leftrightarrow \left(\overline{R}_2 \circ \overline{R}_3\right) \bullet \overline{R}_1 = \overline{R}_2 \circ \left(\overline{R}_3 \bullet \overline{R}_1\right) \tag{72}$$

Let's denote the sets of relations are complement of functions from F_1 and F_2 as $\overline{F_1} = \left\{O_{\mathrm{R}}, I_{\mathrm{R}}, 1_{\mathrm{R}}, \overline{f_1}, \overline{f_2}, \overline{f_3}, \overline{f_4}\right\}$ and $\overline{F_2} = \left\{O_{\mathrm{R}}, I_{\mathrm{R}}, \overline{g_1}, \overline{g_2}\right\}$. In accordance with (71)-(72) we get ordered (not bounded and not lattice) compatible algebraic structures $\left\langle \overline{F_1}, \left(\cup, \bullet, \circ, ^{-1}, \subseteq, 1_{\mathrm{R}}, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$ and $\left\langle \overline{F_2}, \left(\bullet, \circ, \subseteq, O_{\mathrm{R}}, I_{\mathrm{R}}\right)\right\rangle$.

Of cause, the list of examples can be continued.

5. Conclusion

We have studied non-traditional algebraic structures on the underlying set of binary relations. Starting from left composition, inclusion and Boolean isomorphism we defined dual ordered semigroups. Then we extended them to the more general ordered algebraic structure with a couple of dual operations. We have proved that these operations satisfy the semi-compatibility laws. This is notable and important fact. We paid special attention to the algebraic substructures satisfying the compatibility laws. So we have considered interesting examples of compatible algebraic structures.

The results will be useful for graphs and automatons as well as for coding, programming and artificial intelligence.

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