Mathematical Models of Biophysical Processes Taking Into Account Memory Effects and Self-Similarity

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Abstract. Constructed are mathematical models of deformation-relaxation processes in biophysical materials under conditions of nonisothermic moisture transfer, taking into account fractional integro-differential apparatus. A two-dimensional mathematical model of nonisothermic moisture transfer in biophysical materials with fractal structure is synthesized. The relations in the differential and integral forms are given to present one-dimensional Maxwell's, Kelvin's and Voigt's fractional rheological models. The analytical expressions for describing the stress component in relation to deformations for fractal models are found and on the basis of them is generalized a mathematical rheological model of two-dimensional visco-elastic deformation Presented are difference schemes to obtain numerical results of the study on the processes of visco-elastic deformation and heat-and-mass transfer. The algorithmic aspects and results of the identification of fractal parameters are shown. The results of the adapted method for splitting fractional-differential two-dimensional creep kernel are presented. Determined are the patterns of stress, deformation and heat exchange processes for different types of biophysical materials with fractal structure.

Keywords: mathematical models, biomaterials, fractal structure, integro-differentiation of fractional order, memory effects and spatial correlations.

1 Introduction

Construction of mathematical models of physical and mechanical behavior of biological materials taking into account effects of memory and self-organization allows obtaining new data in relation to the state and dynamics of changes of their properties, as well as improving the accuracy of diagnostics of the functioning of biophysical processes. In recent years, there has been a strong interest in using fractional differential equations for simulating biophysical processes. The publications are concerned with mathematical issues of the study on differential equations with derivatives of fractional order, analytical methods for their solution, the existence of solutions, as well as issues related to the geometric and physical interpretations of fractional derivatives [1, 2]. In comparison with the traditional topics of research [3], only a small number of works are devoted to the problem of synthesis of mathematical models of visco-elastic deformation under conditions of nonisothermic moisture transfer in biophysical materials with a fractal structure, taking into account non-locality of processes and multiphase nature of the system. Not entirely solved remains the problem of the correct and physically-meaningful formulation of the boundary and initial conditions for nonlocal mathematical models of nonequilibrium processes with regard to the fractal structure of the medium.

Typically, to describe nonstationary processes, the operators of integration and differentiation are used which cause the imposition of certain conditions on the processes underway and generalize their properties. The use of the mathematical apparatus of integro-differentiation of fractional order makes it possible to take into account the properties of a material which is characterized by biological variability of rheological properties, structural inhomogeneity, the complex nature of spatial correlations, the presence of "memory" effects, self-organization, and deterministic chaos. One of the special advantages of using fractal analysis is the possibility to more fully describe the processes of the real world. The fractional-differential index indicates the share of the system's states that persist throughout the whole process of its functioning. The research is devoted to the construction of mathematical models and software of physical and mechanical fields in biophysical materials with fractal structure. Such fractional order models [4, 5] describe the evolution of physical systems with residual memory and the self-similarity of a fractal structure that occupy an intermediate position between Markov's systems and systems that are characterized by complete memory.

The fractional integro-differentiation apparatus complicates mathematical models and requires the improvement of numerical methods for their implementation, since analytical methods are of limited application. This work is concerned with the use of finite-difference schemes to find a numerical solution to differential equations with fractional order derivatives and the creation of algorithmic support for numerical simulation of non-isothermic moisture transfer in biophysical materials with fractal structure [6, 7, 8].

2 Problem formulation

A two-dimensional mathematical model of the nonstationary process of heat-and-moisture transfer in biophysical materials is described by an interconnected system of differential equations in partial derivatives with a fractional order in time t and spatial coordinates x_1 and x_2:

$$c\rho \frac{\partial^{\alpha} T}{\partial t^{\alpha}} = \lambda_1 \frac{\partial^{\beta'} T}{\partial x_1^{\beta'}} + \lambda_2 \frac{\partial^{\beta'} T}{\partial x_2^{\beta'}} + \varepsilon \rho_0 r \frac{\partial^{\alpha} U}{\partial t^{\alpha}}, \quad (1)$$

$$\frac{\partial^{\alpha} U}{\partial t^{\alpha}} = a_1 \frac{\partial^{\beta'} U}{\partial x_1^{\beta'}} + a_2 \frac{\partial^{\beta'} U}{\partial x_2^{\beta'}} + a_1 \delta \frac{\partial^{\beta'} T}{\partial x_1^{\beta'}} + a_2 \delta \frac{\partial^{\beta'} T}{\partial x_2^{\beta'}}, \quad (2)$$

with the following initial conditions

$$T\big|_{t=0} = T_0(x_1, x_2), \; U\big|_{t=0} = U_0(x_1, x_2), \quad (3)$$

and boundary conditions of the third kind

$$\lambda_i \frac{\partial^{\gamma} T}{\partial x_i^{\gamma}}\bigg|_{x_i=0,l_i} + \rho_0 (1-\varepsilon) \beta^* \left(U\big|_{x_i=0,l_i} - U_p\right) = \alpha^* \left(T\big|_{x_i=0,l_i} - t_c\right), \quad (4)$$

$$a_i \delta \frac{\partial^{\gamma} T}{\partial x_i^{\gamma}}\bigg|_{x_i=0,l_i} + a_i \frac{\partial^{\gamma} U}{\partial x_i^{\gamma}}\bigg|_{x_i=0,l_i} = \beta^* \left(U_p - U\big|_{x_i=0,l_i}\right), \quad (5)$$

where $(t, x_1, x_2) \in D, D = [0, \tau_{\max}] \times [0, l_1] \times [0, l_2]$; T, U are required functions, where T is temperature, U is moisture content, $c(T,U)$ is specific heat capacity, $\rho\,(U)$ is density, ρ_0 is basic density, ε is phase transition coefficient, r is specific heat of vapour generation., $\lambda_i\,(T,U)\,(i=1,2)$ are coefficients of thermal conductivity, $a_i\,(T,U)\,(i=1,2)$ are coefficients of water conductivity, $\delta\,(T,U)$ is thermogradient coefficient, t_c is the ambient temperature value, $U_p\,(t_c, \varphi)$ is equilibrium moisture content, φ is relative moisture content of the drying agent, $\alpha^*\,(t_c, \upsilon)$ is coefficient of heat exchange, υ is the speed of the drying agent movement, $\beta^*\,(t_c, \phi, \upsilon)$ is coefficient of moisture exchange, α is fractional order of derivative in time, $(0 < \alpha \le 1)$, β', γ are fractional indices of the derivative for spatial coordinates $(1 < \beta' \le 2)$, $(0 < \gamma \le 1)$.

On the basis of the Volterra hypothesis on the hereditary elastic deformation solid and the method of structural modeling, fractional analogues of classical one-dimensional rheological models (Maxwell's, Voigt's, and Kelvin's models) are constructed by replacing the ordinary derivative with the fractional order of the derivative in the differential equations.

Maxwell's fractional-differential model:

$$\sigma(t) + \tau^{\alpha} D_t^{\alpha} \sigma(t) = E \tau^{\beta} D_t^{\beta} \varepsilon(t), \; 0 < \alpha, \beta < 1, \quad (6)$$

Voigt's fractional-differential model:

$$\sigma(t) = E(\tau^{\alpha} D_t^{\alpha} \varepsilon(t) + \tau^{\beta} D_t^{\beta} \varepsilon(t)),\ \ 0 < \alpha < \beta < 1, \tag{7}$$

Kelvin's fractional-differential model:

$$E_1 \tau^{\alpha} D_t^{\alpha} \sigma(t) + (E_1 + E_2)\sigma(t) = E_1 E_2 (\varepsilon(t) + \tau^{\beta} D_t^{\beta} \varepsilon(t)),\ \ 0 < \alpha, \beta < 1, \tag{8}$$

where τ is relaxation time, E is elastic modulus for Maxwell's and Voigt's models, E_1 is elastic modulus of Voigt's element for Kelvin's model, E_2 is elastic modulus for Kelvin's model, $\sigma(t)$ is stress, $\varepsilon(t)$ is deformation, D_t^{α}, D_t^{β} are fractional derivatives in time with order, respectively, α, β.

Two-dimensional fractal rheological models in integral form can be written as follows:

Voigt's fractal model

$$\begin{aligned} \sigma_{ii} &= \frac{1}{\Gamma(1-\alpha)}(D_t \int_0^t (t-\xi)^{-\alpha} [p_1(\varepsilon_{11}(\xi) - \varepsilon_{T1}(\xi)) + \\ &+ p_2(\varepsilon_{22}(\xi) - \varepsilon_{T2}(\xi))] d\xi) + A, \end{aligned} \tag{9}$$

$$\begin{aligned} \sigma_{12} &= \frac{2C_{33}}{\Gamma(1-\alpha)} \left(D_t \int_0^t (t-\xi)^{-\alpha} (\varepsilon_{12}(\xi) - \varepsilon_{T3}(\xi)) d\xi \right) + \\ &+ \frac{\mathrm{E}\tau^{\beta}}{\Gamma(1-\beta)} \left(D_t \int_0^t (t-\xi)^{-\beta} (\varepsilon_{12}(\xi) - \varepsilon_{T3}(\xi)) d\xi \right), \end{aligned} \tag{10}$$

Maxwell's fractal model

$$\begin{aligned} \sigma_{ii} &= \frac{1}{\tau^{\alpha}} \int_0^t F(t-\xi)[p_1(\varepsilon_{11}(\xi) - \varepsilon_{T1}(\xi)) + \\ &+ 2\mathrm{E}\tau^{\beta} D_t^{\beta} (\varepsilon_{11}(\xi) - \varepsilon_{T1}(\xi))] d\xi + \\ &+ \frac{1}{\tau^{\alpha}} \int_0^t F(t-\xi)[p_2(\varepsilon_{22}(\xi) - \varepsilon_{T2}(\xi)) + \\ &+ 2\mathrm{E}\tau^{\beta} D_t^{\beta} (\varepsilon_{22}(\xi) - \varepsilon_{T2}(\xi))] d\xi, \end{aligned} \tag{11}$$

$$\begin{aligned} \sigma_{12} &= \frac{1}{\tau^{\alpha}} \int_0^t F(t-\xi)[2C_{33}(\varepsilon_{12}(\xi) - \varepsilon_{T3}(\xi)) + \\ &+ \mathrm{E}\tau^{\beta} D_t^{\beta} (\varepsilon_{12}(\xi) - \varepsilon_{T3}(\xi))] d\xi, \end{aligned} \tag{12}$$

Kelvin's fractal model

$$\sigma_{ii} = A_1\int_0^t G(t-\xi)[p_1(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))+ \\ +\frac{2\mathrm{E}_1\mathrm{E}_2\tau^\beta}{(\mathrm{E}_1+\mathrm{E}_2)}D_t^\beta(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))]d\xi+ \\ +A_1\int_0^t G(t-\xi)[p_2(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))+ \\ +\frac{2\mathrm{E}_1\mathrm{E}_2\tau^\beta}{(\mathrm{E}_1+\mathrm{E}_2)}D_t^\beta(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))]d\xi, \tag{13}$$

$$\sigma_{12} = C_3G(t)+A\int_0^t G(t-\xi)[2C_{33}(\varepsilon_{12}(\xi)-\varepsilon_{T3}(\xi))+ \\ +\frac{\mathrm{E}_1\mathrm{E}_2\tau^\beta}{(\mathrm{E}_1+\mathrm{E}_2)}D_t^\beta(\varepsilon_{12}(\xi)-\varepsilon_{T3}(\xi))]d\xi, \tag{14}$$

where $\varepsilon^T=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{12})$, $\sigma^T=(\sigma_{11},\sigma_{22},\sigma_{12})$ is the vector of deformation and stress, respectively, whose components depend on time t and spatial variables x_1 and x_2, $\Gamma(\cdot)$ is Gamma-function, D_t is integer derivative of the first order in time, $\varepsilon_T=(\varepsilon_{T1},\varepsilon_{T2},\varepsilon_{T3})^T$ is the vector of deformations whose components are determined by temperature change ΔT and moisture content ΔU:
$\varepsilon_{T1}=\alpha_{11}\Delta T+\beta_{11}\Delta U,\ \varepsilon_{T2}=\alpha_{22}\Delta T+\beta_{22}\Delta U,\ \varepsilon_{T3}=0,\ \alpha_{11},\alpha_{22},\beta_{11},\beta_{22}$ are coefficients of thermal expansion and drying shrinkage; C_{ij} are components of the elasticity tensor of an orthotropic body, and with *i=1*, then $p_1=C_{11}, p_2=C_{12}$, at *i=2* will be $p_1=C_{21}, p_2=C_{22}$

$G(t)=t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha)$, $A_1=\frac{(\mathrm{E}_1+\mathrm{E}_2)}{\mathrm{E}_1\tau^\alpha}, F(t)=t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{t^\alpha}{\tau^\alpha}\right)$,

$$A=\frac{2\mathrm{E}\tau^\beta}{\Gamma(1-\beta)}(D_t\int_0^t(t-\xi)^{-\beta}[(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))+ \\ +(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))]d\xi).$$

Taking into account the relation for models (9)-(14), the general fractal mathematical model of two-dimensional visco-elastic deformation is described by means of equilibrium equations with fractional order γ $(0<\gamma\le 1)$ for spatial coordinates x_1 and x_2:

$$C_{11}\left(\bar{R}_{11}\frac{\partial^{\gamma}\varepsilon_{11}}{\partial x_1^{\gamma}}-\tilde{R}_{11}\right)+C_{12}\left(\bar{R}_{12}\frac{\partial^{\gamma}\varepsilon_{22}}{\partial x_1^{\gamma}}-\tilde{R}_{12}\right)+ \\ +2C_{33}\left(\bar{R}_{33}^{2}\frac{\partial^{\gamma}\varepsilon_{12}}{\partial x_2^{\gamma}}-\tilde{R}_{33}^{2}\right)=0, \tag{15}$$

$$C_{21}\left(\bar{R}_{21}\frac{\partial^{\gamma}\varepsilon_{11}}{\partial x_2^{\gamma}}-\tilde{R}_{21}\right)+C_{22}\left(\bar{R}_{22}\frac{\partial^{\gamma}\varepsilon_{22}}{\partial x_2^{\gamma}}-\tilde{R}_{22}\right)+ \\ +2C_{33}\left(\bar{R}_{33}^{1}\frac{\partial^{\gamma}\varepsilon_{12}}{\partial x_1^{\gamma}}-\tilde{R}_{33}^{1}\right)=0, \tag{16}$$

where $\bar{R}_{ij}$, $\tilde{R}_{ij}$ $(i,j=1,2,3)$ are the corresponding values of the integrals:

$$\int_0^t R_{ij}\left(t-z,T,U\right)dz=\bar{R}_{ij},\int_0^t R_{ij}\left(t-z,T,U\right)\frac{\partial^{\gamma}\varepsilon_{T1,T2}}{\partial x_k^{\gamma}}dz=\tilde{R}_{ij}.$$

3 Numerical methods of solving mathematical models

The numerical method for solving the problem (1)-(5) is based on the use of the predictor-corrector method which in turn is implemented on the difference approximations of fractional derivatives, namely: the difference approximation of the fractional derivative *a* in the time interval $\left[t^k,t^{k+1}\right]$, taking into account the Riemann-Liouville formula [2], this can be written as follows:

$$\left.\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right|_{t^k}\approx\frac{u^{k+1}-\alpha u^k}{\Gamma\left(2-\alpha\right)\Delta t^{\alpha}},\ \left(\Delta t=t^{k+1}-t^k\right), \tag{17}$$

Using the Grunwald-Letnikov formula [1], the difference approximation of the fractional derivative β' for the spatial coordinate x_1 will be written accordingly:

$$\left.\frac{\partial^{\beta}u}{\partial x_1^{\beta'}}\right|_{x_{1(n)}}\approx\frac{1}{h_1^{\beta'}}\sum_{j=0}^{n}q_j u_{n-j+1} \tag{18}$$

where

$$q_0 = 1, q_j = (-1)^j \frac{\beta'(\beta'-1)\ldots(\beta'-j+1)}{j!}, h_1 = x_{1(n+1)} - x_{1(n)}.$$

The difference scheme for the numerical implementation of a system of differential equations (1), (2) considering the approximation expressions (17), (18) can be written as:

$$c\rho \frac{T_{n,m}^{k+1} - \alpha T_{n,m}^{k}}{\Gamma(2-\alpha)\Delta\tau^{\alpha}} = \frac{\lambda_1}{h_1^{\beta'}} \sum_{j=0}^{n} q_j T_{n-j+1,m}^{k+1-\omega} + \frac{\lambda_2}{h_2^{\beta'}} \sum_{j=0}^{m} q_j T_{n,m-j+1}^{k+1-\omega} + \\ + \varepsilon\rho_0 r \frac{U_{n,m}^{k+1} - \alpha U_{n,m}^{k}}{\Gamma(2-\alpha)\Delta\tau^{\alpha}} \tag{19}$$

$$\frac{U_{n,m}^{k+1} - \alpha U_{n,m}^{k}}{\Gamma(2-\alpha)\Delta\tau^{\alpha}} = \frac{a_1}{h_1^{\beta'}} \sum_{j=0}^{n} q_j U_{n-j+1,m}^{k+1-\omega} + \frac{a_2}{h_2^{\beta'}} \sum_{j=0}^{m} q_j U_{n,m-j+1}^{k+1-\omega} + \\ + \frac{a_1\delta}{h_1^{\beta'}} \sum_{j=0}^{n} q_j T_{n-j+1,m}^{k+1-\omega} + \frac{a_2\delta}{h_2^{\beta'}} \sum_{j=0}^{m} q_j T_{n,m-j+1}^{k+1-\omega} \tag{20}$$

In the case when $\omega = 1$, we obtain an explicit finite-difference scheme, and when $\omega = 0$ - an implicit scheme.

To determine the stability conditions of the obtained difference equations of the connected heat-and-mass transfer, the method of conditional assignment of some known functions of the system is used, according to which the following relation is found:

$$\Delta t^{\alpha} \left(\frac{C_1}{h_1^{\beta'}} + \frac{C_2}{h_2^{\beta'}} \right) \le \frac{(\alpha+1) C_3}{(2+\beta')\Gamma(2-\alpha)} \tag{21}$$

where $C_1 = \lambda_1, a_1; C_2 = \lambda_2, a_2; C_3 = (c\rho - \varepsilon\rho_0 r), (1+\delta)^{-1}$.

Supposing that fractal parameters α, β' take integer values, an analysis and comparison have been made, according to which the obtained stability condition (21) coincides with the condition of stability for classical equations of thermal conductivity.

The system of equations (15)-(16) will correspond to the following finite-difference scheme:

$$\frac{C_{11}}{\Gamma(2-\gamma)h_1^{\gamma}}\bar{R}_{11}\left(\varepsilon^k_{11(n+1,m)}-\gamma\varepsilon^k_{11(n,m)}\right)-C_{11}\tilde{R}_{11}+$$
$$+\frac{C_{12}}{\Gamma(2-\gamma)h_1^{\gamma}}\bar{R}_{12}\left(\varepsilon^k_{22(n+1,m)}-\gamma\varepsilon^k_{22(n,m)}\right)-C_{12}\tilde{R}_{12}+ \qquad (22)$$
$$+\frac{2C_{33}}{\Gamma(2-\gamma)h_2^{\gamma}}\bar{R}^2_{33}\left(\varepsilon^k_{12(n,m+1)}-\gamma\varepsilon^k_{12(n,m)}\right)-2C_{33}\tilde{R}^2_{33}=0,$$

$$\frac{C_{21}}{\Gamma(2-\gamma)h_2^{\gamma}}\bar{R}_{21}\left(\varepsilon^k_{11(n,m+1)}-\gamma\varepsilon^k_{11(n,m)}\right)-C_{21}\tilde{R}_{21}+$$
$$+\frac{C_{22}}{\Gamma(2-\gamma)h_2^{\gamma}}\bar{R}_{22}\left(\varepsilon^k_{22(n,m+1)}-\gamma\varepsilon^k_{22(n,m)}\right)-C_{22}\tilde{R}_{22}+ \qquad (23)$$
$$+\frac{2C_{33}}{\Gamma(2-\gamma)h_1^{\gamma}}\bar{R}^1_{33}\left(\varepsilon^k_{12(n+1,m)}-\gamma\varepsilon^k_{12(n,m)}\right)-2C_{33}\tilde{R}^1_{33}=0.$$

To the relations (22), (23), the boundary and initial conditions are added in finite-difference form:

$$\varepsilon^{\;k}_{11,22,12(\varsigma,m)}=0,\;\; \varepsilon^{\;k}_{11,22,12(n,\varsigma)}=0,\;\; \varsigma=1,N,M \qquad (24)$$

$$\varepsilon^{\;0}_{11,22,12(n,m)}=0. \qquad (25)$$

4 The results of splitting two-dimensional fractal creep kernel, identification of fractional-differential parameters and results of numerical simulation

Considering the complexity of identification of two-dimensional fractal parameters of kernels, we present the results of the adapted method for splitting two-dimensional kernel for rheological models taking into account the previously found functions of longitudinal and transverse creep. The shear creep kernels for Voigt's, Kelvin's, and Maxwell's models, respectively:

$$\Pi_{sh(F)}(t-\xi)=\frac{(t-\xi)^{\beta-1}}{2\mathrm{E}\tau^{\beta}(1+\nu_0)}(E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}}{2\mathrm{E}\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta-\alpha}\right)+$$
$$+\nu_0E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}\nu_2}{2\mathrm{E}\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta-\alpha}\right)), \qquad (26)$$

$$\Pi_{sh(K)}\left(t-\xi\right)=\frac{\left(\mathrm{E}_1+\mathrm{E}_2\right)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}\left(1+\nu_0\right)}\left(t-\xi\right)^{\beta-1}(E_{\beta,\beta}\left(-\frac{\mathrm{E}_{11}\left(\mathrm{E}_1+\mathrm{E}_2\right)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)+$$
$$+\nu_0 E_{\beta,\beta}\left(-\frac{\nu_2\mathrm{E}_{11}\left(\mathrm{E}_1+\mathrm{E}_2\right)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)), \quad (27)$$

$$\Pi_{sh(M)}\left(t-\xi\right)=\frac{1}{2\mathrm{E}_2\tau^{\beta}\left(1+\nu_0\right)}\left(t-\xi\right)^{\beta-1}(E_{\beta,\beta}\left(-\frac{\mathrm{E}_{11}}{2\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)+$$
$$+\nu_0 E_{\beta,\beta}\left(-\frac{\nu_2\mathrm{E}_{11}}{2\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)). \quad (28)$$

where ν_0,ν_1,ν_2 are Poisson's ratios

The function of volumetric creep rate for fractional-differential rheological models will take the following form:

$$\Pi_{vl(F)}\left(t-\xi\right)=\frac{\left(t-\xi\right)^{\beta-1}}{2\mathrm{E}\tau^{\beta}\left(1-2\nu_0\right)}(E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}}{2\mathrm{E}\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta-\alpha}\right)-$$
$$-2\nu_0 E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}\nu_2}{2\mathrm{E}\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta-\alpha}\right)), \quad (29)$$

$$\Pi_{vl(K)}\left(t-\xi\right)=\frac{\left(\mathrm{E}_1+\mathrm{E}_2\right)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}\left(1-2\nu_0\right)}\left(t-\xi\right)^{\beta-1}(E_{\beta,\beta}\left(-\frac{\mathrm{E}_{11}\left(\mathrm{E}_1+\mathrm{E}_2\right)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)-$$
$$-2\nu_0 E_{\beta,\beta}\left(-\frac{\nu_2\mathrm{E}_{11}\left(\mathrm{E}_1+\mathrm{E}_2\right)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)), \quad (30)$$

$$\Pi_{vl(M)}\left(t-\xi\right)=\frac{1}{2\mathrm{E}_2\tau^{\beta}\left(1-2\nu_0\right)}\left(t-\xi\right)^{\beta-1}(E_{\beta,\beta}\left(-\frac{\mathrm{E}_{11}}{2\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)-$$
$$-2\nu_0 E_{\beta,\beta}\left(-\frac{\nu_2\mathrm{E}_{11}}{2\mathrm{E}_2\tau^{\beta}\left(1-\nu_1\nu_2\right)}\left(t-\xi\right)^{\beta}\right)). \quad (31)$$

The use of the iterative method, which is based on the method of least squares and coordinate descent, involves two stages. At the first stage, based on the a priori knowledge about structural parameters of rheological models, the identification of such parameters is made using the creep law for a specified model in its classical interpretation. The structural parameters of the models having been identified, the next stage is to find the values of fractional-differential parameters which can be obtained by minimizing explicit expressions that describe deformation functions for

Voigt's, Kelvin's, and Maxwell's models. Maxwell's one-dimensional fractal model can be presented as follows:

$$\varepsilon_M(t) = \frac{\sigma_0}{\mathrm{E}\tau^{\beta}\beta\Gamma(\beta)}\left[2t^{\beta} - (t-t_1)^{\beta}h(t-t_1)\right] + \\ + \frac{\sigma_0}{\mathrm{E}\tau^{\beta-\alpha}(\beta-\alpha)\Gamma(\beta-\alpha)}\left[2t^{\beta-\alpha} - (t-t_1)^{\beta-\alpha}h(t-t_1)\right] \quad (32)$$

For both stages of the iterative method of objective function:

$$\sum_{i=1}^{n}\left(\varepsilon_i - \varepsilon_{F,K,M}^{kl}\left(t_i, \tau, \sigma_0, \mathrm{E}\right)\right)^2 \Rightarrow \min, \quad (33)$$

$$\sum_{i=1}^{n}\left(\varepsilon_i - \varepsilon_{F,K,M}\left(t_i, \alpha, \beta\right)\right)^2 \Rightarrow \min. \quad (34)$$

The refinement of the identified parameters is carried out by means of the coordinate descent method. The results of the identification and their comparison with the experimental data [9] for Maxwell's fractal model at moisture content $(W = 30\%)$ and with the elastic modulus $\left(\mathrm{E}_M = 15.3 \cdot 10^3 \mathrm{M\Pi}a,\right)$ are presented in Fig.1. Taking into account the statistical criterion that is based on the correlation coefficient, an appropriate estimation of the difference of results for the model is found - $\Delta_M = 26.15, \rho_M = 0.992$.

Taking into consideration rheological and thermophysical characteristics of for biophysical material with different conditional densities and their dependence on temperature and moisture content, a numerical experiment was carried out to study the change in temperature, stress and deformation components with respect to time, taking into account the fractal structure and without its consideration.

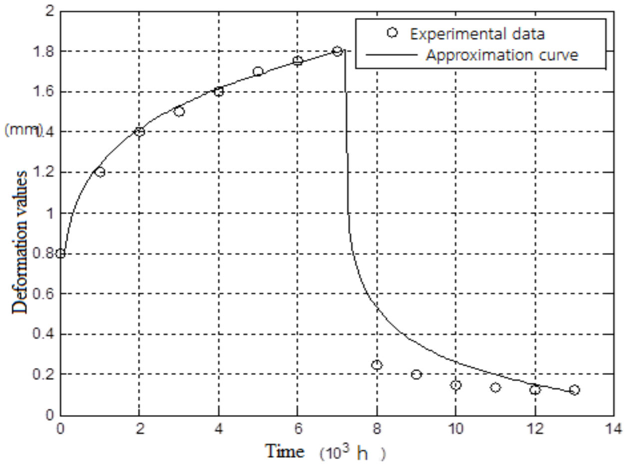

Fig. 1. Identification of the fractal parameters for Maxwell's model

After analyzing the temperature change T for samples of biomaterials with different densities (ρ_1=550 kg/m^3, ρ_2=500 kg/m^3, ρ_3=400 kg/m^3) in the center and at the end of the sample depending on the time t (Fig. 2) with selected fractal model parameters - $\alpha = 0.3, \beta = 1.9, \gamma = 0,1$, the following conclusions can be drawn: at any geometric point, the biomaterial samples with ρ=500 kg/m^3 heats up faster than other one; at the ends of the biomaterial samples the temperature rises faster than in the center of both specimens; the temperature rises to a certain level and becomes almost constant.

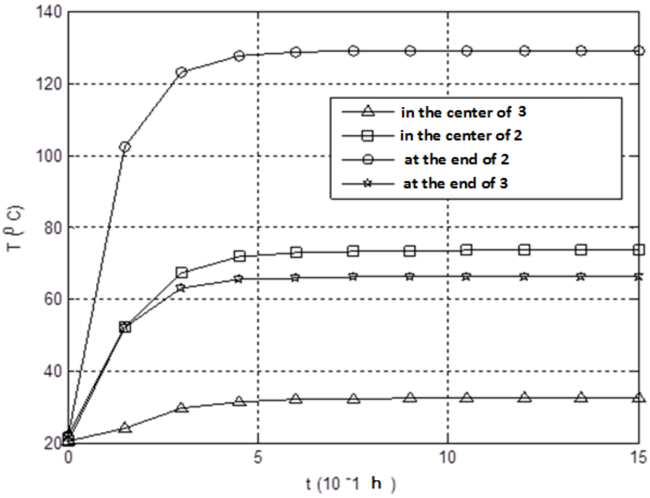

Fig. 2. Temperature change at different geometric points of biomaterial samples with different densities, taking into account fractal parameters.

Investigated is the influence of fractal parameters on the dynamics of stress and deformation components in the radial-tangent anisotropy direction for the rheological Voigt's model. The differences in the use of fractional and integer differentiations are given.

From the graphic dependencies (Figures 3, 4) it can be noted that with increasing time, deformation and stress somewhat decrease, in particular, the greatest values of deformation and stress are reached by the specimen with ρ=400 kg/m^3, and the smallest - another.

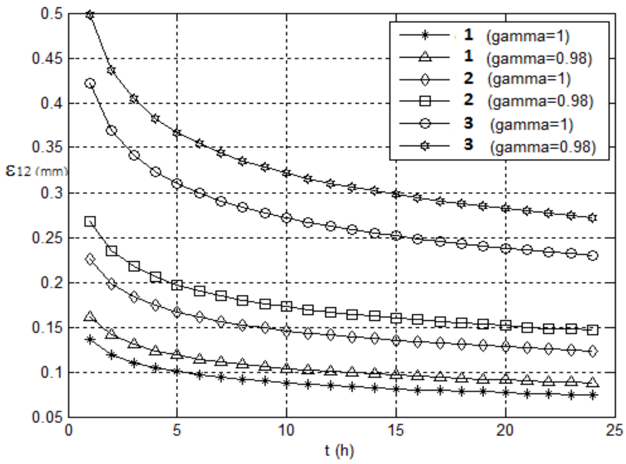

Fig. 3. Change in the deformation component ε_{12} depending on the type of biomaterial

Thus, it can be concluded that the fractal structure of the material has a greater impact on the species with a lower density than on the biomaterials with a higher density.

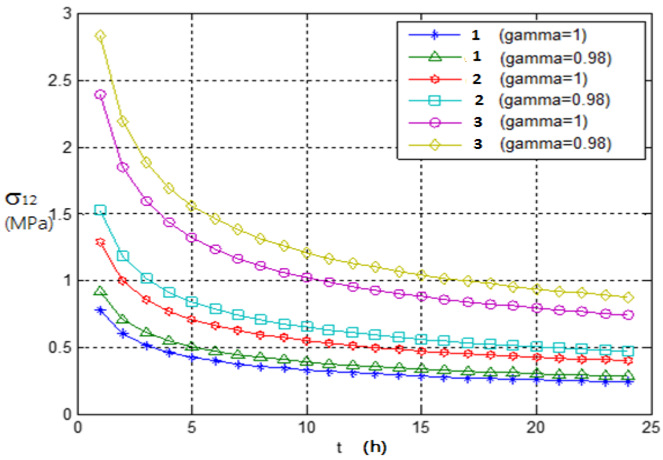

Fig. 4. Change in the σ_{12} stress component depending on the type of biomaterial

5 Conclusions

A mathematical model of nonisothermic moisture transfer using a fractional integro-differential apparatus was constructed, which makes it possible to take into account the thermophysical characteristics of biophysical materials as anisotropic material and, unlike the known ones, extends the set of its realizations by taking into account the fractality not only in time α ($0<\alpha\leq 1$) , but also for spatial coordinates, the order of which in the mathematical model is β ($1<\beta\leq 2$) and in boundary conditions of the third kind – γ ($0<\gamma\leq 1$). Obtained are one - and two-dimensional models of visco-elastic deformation processes in biophysical materials with fractal structure. The results of splitting two-dimensional creep kernels for fractional-differential rheological models are presented, which allows obtaining the function of rate of volumetric and shear creeps. The difference schemes are developed to obtain the numerical results of the study on the processes of heat-and-mass transfer and visco-elastic deformation in the two-dimensional region, taking into account the fractal structure of the material; the stability conditions of the explicit difference schemes are identified. The algorithmic aspects of identification are given and fractional-differential parameters for the Maxwell model are determined, which makes it possible to compare the obtained results with experimental data and to find an explicit expression that describes the fractional-exponential creep kernel.

Identified are the patterns of fractal parameters influence on the dynamics of temperature changes, the components of deformation and stress, according to which it is possible to draw appropriate conclusions about the impact of the fractal structure of the material on different types of biomaterials.

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