Multi-Fuzzy Sets as Aggregation Subjective and Objective Fuzziness

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Abstract. The non-factor fuzziness is considered as one of the forms of uncertainty modeling by detecting hidden knowledge contained in the original data array (cloud of data) and fuzzy set. It is shown that the objective component of the fuzziness consists of 2 parts: 1-subset of ordered pairs, calculated on the basis of tensor decomposition structured as a 2D or 3D tensor of the initial data array; 2 - the immersion of the interval of possible values of the fuzzy set into a special matrix (Toeplitz), followed by a tensor decomposition. The results obtained: it is shown that in the general case analysis of uncertainty must be performed using a multi-fuzzy set; with restrictions on the formation of the membership function it is suggested to use a subset of ordered pairs on the basis of objective fuzziness; the developed algorithm for reducing the multi-fuzzy set to FS-1 type. The following examples illustrate the effectiveness of the proposed methodology.

Keywords: multi-fuzzy set, a subset of ordered sequences, tensor decomposition, special matrix, objective fuzziness.

1 Introduction

The theory of fuzzy sets (TFS) as a method and apparatus for solving problems in uncertain conditions has shown its high efficiency. However, at present, its application has encountered a number of difficulties due, in particular, to the following factors:

- the emergence of fundamentally new problems associated with the complication of technological processes, which limits the possibility of assigning a heuristic membership function (MF); the need to work with data of super large volumes and high measurements, which require consideration of their life cycle;
- philosophical interpretation of the phenomenon of fuzziness, which declares in the composition of this phenomenon the objective and subjective components, involves taking into account its influence on the structure and type of fuzzy set (FS), because the standard FS is constructed only on the basis of the subjective component - FS, that is, the standard type of FS is a special case;

- the initial data set (IDS), on the basis of which the universal set (US) is calculated, is the bearer of hidden knowledge, but at present, research on the influence of the structure of IDS on the form and structure of FS has not been carried out;
- IDS containing fuzziness (or inaccuracy), allows to distinguish several types of formal (objective) fuzziness in the form of subsets of ordered pairs (SOP).

Let's remind that FS $\tilde{A}$ in X is SOP:

$$\tilde{A} \overset{\Delta}{=} \left\{ \left(x, \mu_{\tilde{A}}(x) \right) \mid x \in X, \mu_{\tilde{A}} \rightarrow [0,1] \right\} \quad (1)$$

where $\mu_{\tilde{A}}$ is the function of membership or degree of membership [1] (also the degree of compatibility or truth) *x* in $\tilde{A}$, which reproduce X in the space of membership *M* [2]. When *M* contains only two points 0 and 1, $\tilde{A}$ is not fuzzy and $\mu_{\tilde{A}}$ is identical to the characteristic function of the classical set.

In this regard, it is advisable to turn to the theory of non-factors [3], because the first non-factors were identified and studied within the framework of the problems of TFS, specifically fuzzy mathematics [4]. Four non-factors are considered in the theory of non-factors: inaccuracy, insufficiency, ambiguity and fuzziness due to the fact that their formal representation is associated with the use of a range of values. Note that in the appendix to different categories of data and knowledge, non-factors are treated differently, for example, the fuzziness of data, fuzziness of conclusions, the fuzziness of the problem statement - these are completely different fuzziness. At the same time, one of the principal differences lies precisely in the fact that the considered fuzziness has different proportions of the objective and subjective components.

In this context, only data fuzziness is considered. Separately, pay attention to inaccuracy, in the work an example of modeling of inaccuracy is given by a subset of ordered sequences. It is shown in [5] that inaccuracy is a universal factor inherent in all real parameters. In [6], it has been shown that inaccuracies and insufficiency are associated with the use of the range of values, which simplifies the consideration of their meaningful differences; they do not have the "external resemblance", but they are interrelated, moving to each other at different levels of modeling.

2 Review of the Literature

2.1 Main statements

M. Black [7] first applied the many-valued Lukasiewicz logic to lists as multiplicative of objects and called such sets indeterminate (vague), L. Zade based on the logic of Lukasevich constructed a complete algebraic system [8]. This was the first publication in which the term *fuzzy set* was used and the main idea of fuzziness was formulated. Attempts to give a philosophical interpretation of the actual positions of TFS made in the works [9, 10], in the works [11, 12] considered the problem of fuzziness as a scientific concept. Let's pay attention that interest in the conceptual foundations of TFS, due to the expansion of TFS to new areas, new tasks, solve which using standard concepts is difficult.

Here are the main provisions of work [11]. Fuzziness is a concept that characterizes the non-continuity of the transition from the absence of manifestations of the full discovery of the quality of objects, properties and relations of the real world, which is reflected in the cognitive and intellectual activity of the individual. The fuzziness in the real world is manifested in 2 forms - audio and visual, blur (fuzziness) of the sound signal, fuzziness (blurring) of the image, although the forms of the emergence of the virtual (conceivable abstract) fuzziness are much larger.

The methodological basis for considering the phenomenon of fuzziness and the concept that it expresses is the formalization of intuitive notions of fuzziness. For this purpose, an analysis of the forms of the appearance of fuzzy bones is used to identify the components that make up its contents and the synthesis of the system of interconnections of the fuzziness of the object's property in the process of cognitive activity of the subject. For the formalization of intuitional notions of fuzziness, the method of determination through abstraction is used; the method of an equation is used in the process of understanding the ontological grounds of the phenomenon of fuzziness and the appearance of the relationship between its objective and subjective aspects.

One of the most important conclusions of the work [11] is that the dialectical nature of the relationship between the subjective and objective aspects of fuzziness is determined, this important conclusion is confirmed in the works of Nobel laureates D. Kahneman and A. Tversky [13], which showed that the use of common sense (the main tool for the formation and assessment of subjective fuzziness) is not rational in all cases, the person rational supposed abnormalities of rational behavior. The second important conclusion from the work [11] is that the phenomenon of fuzzy, having objective and subjective aspects, differing in depth of reflection of the essence of processes, must be studied in a complex manner, while simultaneously considering both types.

Since fuzziness is a characteristic, first of all, the cognitive process, then the solution to this issue is the most effective method from the establishment of consistency patterns to the construction of a formal system based on them. In work [11], this approach is realized by revealing intuitive notions of fuzziness, establishing their specific content and further formalization in order to identify their logical structure, and in the further clarification of the relationship between the formal and substantive aspects of the concept of fuzziness. If the fuzziness is understood as a property, then, as noted by Canadian researchers *A.M. Norwich* and *I.B. Turksen*, it is also necessary to distinguish between objective and subjective fuzziness.

The theory of non-factors proposed by A.S. Narinyany, devotes great attention to the analysis of non-precision, which is a multidimensional characteristic of fuzziness, being one of the most common forms of its expression. The content of the concept of fuzzy contains a consistent series of lower-level abstractions, which in turn, in each case, contain abstractions of an even lower level, both explicit and intuitive; the common thing for them is that they characterize the power of the real objects, phenomena, relations and concepts.

Touching TFS, we note that in the general case in the formation of FS (process of fuzzification) objective fuzziness is to calculate the interval US, subjective - in the choice (heuristic) MF, while it should be borne in mind that the basis of the methodological approach to the analysis of the relationship objective and subjective fuzziness is the rationale for choosing one of two aspects of the concept of fuzziness as defining in relation to another. In this context authors, objective inaccuracy is additionally determined by the artificial blurring of the US with the help of the immersion procedure in a special matrix.

In [14], it is shown that the problem of uncertainty (which, unfortunately, reduces only to one of the non-factors - fuzziness) inevitably arises in the simulation of complex systems, in which a person plays an active role. In order to obtain significant conclusions about the behavior of a complex system, it is necessary, in principle, not only to abandon the high standards of accuracy and severity which are characteristic of relatively simple systems, but to consider alternatives, not only varying with different standard FS, but also changing the approach to computation SOP.

Objective fuzziness is a description of the objects, properties and relations of the real world, not dependent on consciousness; subjective fuzziness, in turn, characterizes the process of knowledge, the human way of thinking and knowledge of objective fuzziness measures reflection of the internal patterns of the object in its external manifestations. Subjective fuzziness is defined as a measure of the reflection of the phenomena of the external world into the inner world of the individual, which allows for the personality of the human psyche to be taken into account. At least at the level of sensation and perception, subjective fuzziness has an objective basis, which, in turn, gives grounds to believe that objective obscurity is decisive in relation to subjective.

Subjective and objective types of fuzziness also differ in depth of reflection of the essence of processes and in relation to these types of basic categories of dialectics. Reflecting the contradictory process of cognition, objective and subjective fuzziness, while in a dialectical contradiction, adequately describe the phenomenon of fuzziness in its totality, indicating their dialectical unity. The study of various interpretations of fuzziness, their analysis and generalization showed that no interpretation completely does not cover the content of the concept under study and does not fully reflect its essence. Therefore, there are prerequisites for considering the fuzziness as a theoretical concept, in which any preliminary interpretation will act as a separate case. From a methodological point of view, the interpretation of fuzziness as a theoretical concept has the essential advantage that such an understanding of fuzziness does not allow to reduce any one interpretation of this concept into a general rank and, at the same time, deny any of them due to its incompleteness.

2.2 List of main symbols and abbreviations

In table 1 is presented main abbreviations, that are used in the article.

Table 1. List of main symbols and abbreviations

Notations	Definitions
X	Matrix (or tensor)
x	vector

X(:,j), **X**(i,:)	j-th column, i-row of matrix
$\| \ \|_F$	Frobenius norm
$\otimes$	Kronecker product
FS	Fuzzy set
MF	Membership functions
US	Universal set
TFS	Theory of fuzzy set
IDS	Initial data set
SOP	Set of ordered pairs
KP	Kronecker product
PCA	Principal component analysis
MFS	Multi-fuzzy set
SVD	Singular value decomposition

3 Problem Statement

The main object of TFS - FS $\tilde{x} \overset{\Delta}{=} \left\{ x / \mu_X \right\}, \mu_X \to [0,1], x \in X = \{x\}$, has the property that for its presence there is sufficient minimum number of statistical characteristics of IDS*: (x^{min}, x^{mean}, x^{max})* $\in X$, obtained on the basis of IDS, and an expert assessment of the type of FS. At the same time, a number of structural characteristics that are endowed with IDS can be obtained based on the discovery of hidden knowledge contained in the Data Mining (IDS), is not used. The presence of additional characteristics in conditions of uncertainty can affect the decision-making - to correct the heuristic adopted FS, or in the most unfavorable case - the impossibility of appointing an FS - to offer as an FS equivalent subset of ordered pairs formed on the basis of IDS, or else.

In TFS, the notion of the nearest (to fuzzy $\tilde{x}$) crisp set is considered, the closeness is understood as the proximity of the norm, in particular, Euclid, Frobenius, or elsewhere, that is $\left\| \tilde{x} - x^0 \right\|_F^2 \to \min$, where $\tilde{x} = (x \ \ \mu_x)_1^n$, $x \in X, \mu_x \to [0,1]$; $x^0 = \left(x \ \ \mu_{x^0} \right)_1^n$, $\mu_{x^0} \in \{0;1\}$, $x^0 \in X$. Note that the F-norm of the matrix **A** with the dimension m×n is defined as $\|A\|_F = \left(trace\left(\mathbf{A} \cdot \mathbf{A^H} \right) \right)^{1/2}$, the sets $\tilde{x} = \left(x \ \ \mu_x \right)_1^n$ and $x^0 = \left(x \ \ \mu_{x^0} \right)_1^n$ must be represented in the matrix form, in particular, in the form of the KP- $x \otimes \left(\mu_x \right)^T$ and $x \otimes \left(\mu_{x^0} \right)^T$ respectively.

The concept of the nearest object in the work is expanded by forming the tasks of determining the subsets of ordered pairs: a) nearest to the given FS; b) nearest to the structured IDS. Let's pay attention to the following. The nearest crisp set approximates the fuzzy, subset of ordered pairs nearest to the standard fuzzy can be used for comparison or estimation or to be more convenient for further processing, since it always has the same form and obtained on the basis of formal transformations, which excludes the participation of an expert. Accordingly, the following tasks are formulated.

An object in uncertainty is represented by an IDS (or array), in the general case, a data cloud, for example, an array of data ℵ obtained when measuring a certain value with devices with limited accuracy in conditions of interference. To work with this object, its representation is used in the form of some statement such as "*approximately equal to* ...", "*close to* ...", etc., which, in turn, can be presented to FS $\tilde{x} \overset{\Delta}{=} \{x / \mu_x\}$, $\mu_x \to [0,1]$, $x \in X = \{x\}$, where $\mu_x \to [0,1]$ is MF, which is appointed by the expert heuristically, as a rule, from the composition of the means of a specific application package, which is used in the work.

Need to show:

- the existence, in addition to the standard FS $\tilde{x}^{(j)} \overset{\Delta}{=} \{x^{(j)} / \mu_{x^{(j)}}\}$, $\mu_{x^{(j)}} \to [0,1]$, $x^{(j)} \in X$ of a collection of SOP that is endowed with properties of FS (having one of the components as a weight function) are analyzed on the basis of formal procedures and represent the object no less effectively than FS; it is important that the specified SOP are obtained by detecting hidden knowledge (intellectual analysis) that are found both in the components of the FS and in the IDS;
- the possibility of using the SOP as an FS, if there are restrictions in the designation of FS;
- the expediency of submitting an object in conditions of the uncertainty of a multi of fuzzy sets containing FS and SOP that characterize this object;
- the possibility of using methods of fuzzy mathematics, developed for type FS-1 type, on a multi-fuzzy set (MFS);
- real examples of the solution of the above tasks.

4 Materials and Methods

Definition 1. Subjective fuzziness is a subset of ordered pairs $\{x_i / \mu^{x_i}\}$, one of its components of which $\mu^{x_i} \to [0,1]$ plays the role of weight function and obtained on the basis of mental activity of a person (in particular, common sense); objective fuzziness $\mu^{x_i} \to [0,1]$ obtained on the basis of formal calculations.

Formation of a subset of ordered pairs based on the blurring of the vector of input data [15]. One of the tasks that were posed by the authors was to use the logical and semantic proximity of tasks that arise in the processing of noisy (blurring) images and procedures of blurring (fuzzification) input values in the formation of FS. As it is known, the restoration of blurry and tired images, for example, magnetic resonance tomography (MRT) is the main problem in digital images. Some elements of the blind deconvolution method used to restore a crisp (sharp) image from blurring and noisy

MRT images can be used to solve problems, in particular, to make decisions under uncertainty, using TFS.

Recall that blind deconvolution is a method of restoring a sharp (crisp) version of a blurred image when the source of blurring is unknown. In some applications, particularly in the physical imaging problems, the blurring process is not known and needs to be restored with the image. A similar situation occurs when using TFS - the process of fuzzification $\tilde{x} \overset{\Delta}{=} \left\{ x_i / \mu^{x_i} \right\}, \mu^{x_i} \to [0,1]$, fuzzifier*(x)*→*x* $\in U_x$ of the initial data array or a separate component of this mathematical object - a scalar, variable, vector, etc., is not coded and entirely depends on the position (knowledge, experience and other qualities) of an expert. Without denying the rationality of this approach, we duplicate the expert by performing formal fuzzification of the input by immersing the input vector into a special matrix and, having obtained a new matrix, we execute a singular decomposition over it, define a rank-1 matrix (row and column) which, according to the Young-Ekkart theorems [16] we shall consider as a subset of ordered pairs possessing properties of informally formed FS.

It is known [17] that if one imagines a degradation (image, signal) in the form of a linear fashion, i.e. $g(x, y) = f(x, y) \otimes h(x, y) + \eta(x, y)$, where ⊗- a symbol of Kroneker's product, $f(x, y)$ is the initial image, $h(x, y)$ is the function of scattering points are unknown functions, then the restoration is reduced to the problem of blind deconvolution [18]. The expression for $g(x, y)$ is also considered in the vector-matrix form: $g = [\mathbf{H}] f + \eta$, where the matrix [**H**] can be constructed from the function h of the discrete point distribution and has the structure of the so-called special matrix.

These can be the Toeplitz matrix, Hankel matrix or a combination of them: for example, a block matrix in Toeplitz form with Hankel-shaped blocks or the like. General view - the method of forming a Toeplitz matrix from a vector (in this case US) - is shown in Fig.1. This matrix-vector form can be useful in analyzing the problem of fuzziness, which is the basis for the formation of a SOP with the properties of the FS.

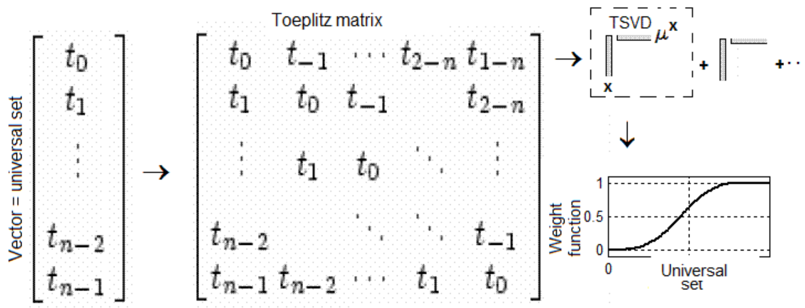

Fig. 1. Formation of the Toeplitz matrix from the vector [$t_0, t_1, ..., t_{n-2}, t_{n-1}$] as the process of US fuzziness - a method of the formal (objective) formation of SOP as an analogue of the MF

It is shown [19] that in most applications where using a blurred image **B** restore its sharp and clear version **X**, the most common model of the blurring process is given as **B** = **A** [**X**] + **N**, where **A** is the blurring process, be it linear or nonlinear process, and **N** denotes some additive noisy process. For digital imaging **X** and **B** are discrete vectors or arrays, and as a result, the linear blur **A** corresponds to a matrix-matrix operator. In the case when the blur operator **A** is invariant with respect to the displacement, the corresponding blurring matrix will be well structured and may be, for example, Toeplitz, circular or Hankel. The algebra of calculations with structured matrices is considered in the paper [20]; in this paper we consider the cut singular decomposition of the matrices because it allows the obtaining of the SOP, and the first component of which is the US, and the second - weight function, that is, SOP structurally and semantically coincides with FS, although calculated on the basis of formal relations.

Tensor models. A tensor is a d-linear form or d-dimensional array: $\mathbf{A} = \left(a_{i_1 i_2 \ldots i_d}\right)$. Tensor has: dimensionality (order) d - number of indices (measurements, axes, directions), size $n_1 \times \ldots \times n_d$ - the number of readings per axis [21]. An example of the separation of variables is singular decomposition $\mathbf{A} = \sum_{a=1.r} \sigma_a v_a u_a$. The canonical tensor decomposition also appears in terms of the Kronecker's product (KP) of the matrices. Case $d \geq 3$ differs substantially from the case $d = 2$: the approximation of a tensor with a decrease in rank is a complex multidimensional optimization task. In [22], the efficiency of using tensor approximations for calculations is shown, and the d-dimensional tensor can be considered as a vector-column with d-dimensional indexation. An extreme case is the transformation of a vector of size $N = 2^d$ into a d-tensor of measurements $2\times2\times \ldots \times2$.

When maintaining the total number of elements of the tensor, the effectiveness of its use (representation of the tensor, the possibility of performing algebra operations, the formation of FS-2 or n-types, etc.) can be significantly increased by increasing the number of measurements and reducing the number of readings for each dimension. One of the specific characteristics of a tensor is its quantization-the introduction of additional (virtual) measurements. While maintaining the total number of elements of the tensor, the effectiveness of its presentation can be substantially increased by increasing the number of measurements and reducing the number of readings for each dimension.

The main object of the TFS - SOP is to some extent universal, but in a number of cases it can be used in other. forms [23], in particular, when performing operations of fuzzy mathematics and fuzzy logic than is customary in the TFS. FS-1 type $\tilde{a} \overset{\Delta}{=} \left\{a / \mu^a\right\}, a \in U, \mu^a \rightarrow [0,1]$, regardless of the form of MF and the method of its formation, can be represented by means of a KP in the form of a 2D tensor:

$\tilde{a} \to \mathbf{T}^a = a \otimes \left(\mu^a\right)^T$. The FS-tensor $\mathbf{T}^a$ with the help of the singular decomposition procedure can be compared to the SOP:

$$ {}^p\tilde{a} \overset{\Delta}{=} \left\{ a / {}^p\mu^a \right\}, a \in U, {}^p\mu^a \to [0,1] \tag{2} $$

which has the property when the SOP , ${}^p\mu^a$ is nearest (in the sense of the *F*-norm) to the FS $\tilde{a}$, $\left\| \tilde{a} - {}^p\tilde{a} \right\|_F \to min$.

Note that in the TFS, in addition to the most common types of FS-1 type, designed for the simulation of the simplest types of uncertainty, in which the values of the MF from the interval [0,1], proposed higher-level FS, allow for, in particular, the consideration of multidimensional uncertainties. Conceptually, these types of FS are defined in [24]. For example, the type FS-2 type is defined as an FS of a universal set X in which the FN is fuzzy subsets of the interval [0,1], expanded with the FS-*n* type, where *n = 2, 3, ...* is the FS, in which MF is FS -*n*-1 type. The tensor model of the FS-2 type, $\bar{\tilde{a}} \overset{\Delta}{=} \left\{ a / \left\{ a_{\mu^a} / \mu^a \right\} \right\}, a \in U, a_{\mu^a}, \mu^a \to [0,1]$, in particular, has the form of a block tensor $\to \left[a \otimes \mathbf{T}^a \right]$, where $\mathbf{T}^{\mathrm{a}} = a_{\mu^a} \otimes \mu^a$.

The basis of matrix (tensor) decompositions. Since the appearance of matrix decompositions, later extended to tensors, they were treated as a decomposition on a basis and weighted coefficients [25], which are the bases of all decompositions, are most clearly visible in the methodology of Principal component analysis (PCA) and singular value decomposition (SVD) . But if one considers that one of the interpretations of the FS and MF is that the FS is a subset of ordered pairs, one of whose components is a universal set (basis in the notation of the PCA), the second is the heuristic significance of each element of the US in truth, in particular some assertions (weighted coefficients in the notation of the PCA, we can see the identity in the definition and semantics of the FS and the matrix (tensor) decomposition. SVD is a decomposition of a real (complex) matrix in order to bring it to a canonical form.

In the writings of authors it is shown that SVD can be used as a procedure for determining subsets of ordered pairs similar to FS. This allows us to expand the possibilities of singular expansion - from solving standard tasks - approaching the least-squares method, compression of images, etc. - to tasks in conditions of uncertainty. An example, when the matrix 3×3 is represented by a subset of ordered pairs (Tabl.2) in such way:

```
(abs(u(:,1)*s(1,1,),abs(v(:,1)))
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the KP of a component of which allows a new matrix (initial approximation) with an accuracy of not less than 1%, is given below. Note that the absolute values are left and the right singular vectors have the meaning of weight functions, in addition,

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sum (v (:, 1)²)=sum (u (:, 1)² ) = 1.
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Table 2. Example SVD and its properties.

initial matrix	singular decompositions: [u s v] = svd(x)		
x = 3.82 3.71 3.28 4.79 4.63 3.41 3.12 3.02 3.40	u= -0.56 0.06 -0.83 -0.67 -0.62 0.41 -0.49 0.78 0.39	s = 11.16 0 0 0 0.85 0 0 0 0.01	v = -0.61 -0.38 0.69 -0.60 -0.35 -0.72 -0.52 0.86 0.01
norm(x,'fro')= = 11.20	x1=kron(u(:,1)*s(1,1),v(:,1)^T) $\rightarrow$ norm(x1,'fro')= 11 .16 norm(x1,'fro')= norm(x,'fro') , x1≅x, sum(v(:,1)^2= sum(u(:,1)^2=1		

In [26], the SVD model is proposed for the *N*-th order of tensors.

Theorem [26] (*N*-order SVD). Each complex ($I_1 \times I_2 \times \bullet \bullet \bullet \times I_N$) -the tensor **A** an be written as a product $\mathbf{A} = S \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \bullet \bullet \bullet \times_N \mathbf{U}^{(N)}$, in which:

1. $\mathbf{U}^{(n)} = \left(\mathbf{U}_{I_1}^{(n)} \mathbf{U}2_{I_2}^{(n)} ... \mathbf{U}_{I_n}^{(n)}\right)$ - unitary ($I_n \times I_n$) - matrix,
2. S is a complex ($I_1 \times I_2 \times \bullet \bullet \times I_N$), the tensor for which the sub-tensors obtained by fixation of the n-th index in α have the properties of ordering: for all possible values of n.

Decomposition of the 3rd order tensor in Fig. 2. A comparison of matrix and tensor theorems shows a clear analogy between two cases, in particular, the left and right singular vectors of the matrix are generalized as *n*-modal singular vectors

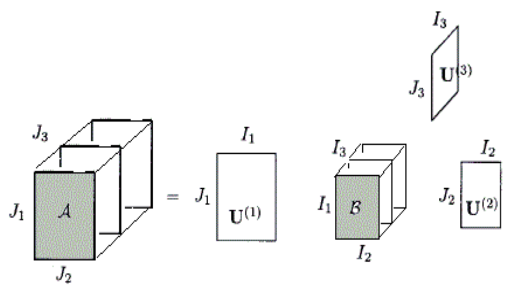

Fig. 2. Representation of the 3-rd order tensor $A \in C^{J_1 \times J_2 \times J_3}$ as a product of the 3-rd order tensor $B \in C^{J_1 \times J_2 \times J_3}$ and matrix $\mathbf{U}^{(1)} \in R^{J_1 \times I_1}$, $\mathbf{U}^{(2)} \in R^{J_2 \times I_2}$, $\mathbf{U}^{(3)} \in R^{J_3 \times I_3}$.

Links between high-order (HOSVD) and matrix SVD. Let HOSVD for A be given by Theorem 2. Then there $\mathbf{A}_{(n)} = \mathbf{U}^{(n)} \cdot \sum^{(n)} \cdot \mathbf{V}^{(n)^H}$ is SVD for $\mathbf{A}_{(n)}$, where the diagonal matrix $\sum^{(n)} \in \mathrm{R}^{I_n \times I_n}$ and the column wise extended orthonormal matrix $\mathbf{V}^{(n)} \in C^{I_{n+1}I_{n+2}I_N I_1 I_2 I_{n-1} \times I_n}$ are defined in accordance

$$\sum{}^{(n)} \overset{\Delta}{=} diag\left(\sigma_1^{(n)}, \sigma_2^{(n)}, ...\sigma_{I_n}^{(n)}\right), \tag{3}$$

$$\mathbf{V}^{(n)H} \overset{\Delta}{=} \tilde{S}_{(n)} \cdot \left(\mathbf{U}^{(n+1)} \otimes \mathbf{U}^{(n+2)} ... \mathbf{U}^{(N+1)} \otimes \mathbf{U}^{(1)} \otimes \mathbf{U}^{(2)} \otimes ... \otimes \mathbf{U}^{(n-1)}\right)^T, \tag{4}$$

in which $\tilde{S}_{(n)}$ - a normalized version $S_{(n)}$ with rows, varied according to the scale to one-dimensional length $S_{(n)} = \sum^{(n)} \cdot \tilde{S}_{(n)}$.

Multi-fuzzy sets as a way of storing, analyzing and processing data in conditions of uncertainty. Given that in the general case, as a model of uncertainty (fuzziness), has an objective and subjective component, it should be recognized that the MFS will be more adequate to the model than the application of the FS. In addition, MFS is a unique way of representing, analyzing and processing large data, a subset of ordered sequences, the most natural form of representation of uncertainty. The theory of multiple fuzzy sets in terms of multidimensional membership functions is an extension of a series of related theories: fuzzy sets, L-fuzzy sets, intuitionistic fuzzy sets, which are discussed in [27], from which the basic definitions are taken.

Definition 2. Let X be a non-empty set and $\{L_i: i \in P\}$ be the family of complete lattices. Multi FS **A** in *X* is a set of ordered sequences:

$\tilde{A} = \{\langle x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ...\rangle : x \in X\}$, where $\mu_i \in L_i^X$, for $i \in P$.

We recall that a complete lattice is a partially ordered set in which any non-empty subset has an exact upper and lower bound, usually called the union and intersection of the elements of a subset.

Remark 1. If the sequences of membership functions have only *k*-terms (the end-of-number of terms), k is called dimensional $\tilde{A}$. If $L_i = [0, 1]$ (for $i = 1, 2, ..., k$), then the set of all the multi-fuzzy sets in *X* of dimension k are denoted by $\mathbf{M^kFS}$ (*X*). The function of multi membership (plural membership) μ_A is a function of *X* in I_k such that for all *x* in *X*, $\mu_A(x) = \langle \mu_1(x), \mu_2(x), ..., \mu_k(x) \rangle$. For simplicity, we denote the multi-FS $\tilde{A} = \{\langle x, \mu_1(x), \mu_2(x), ..., \mu_k(x) \rangle : x \in X\}$ as = $(\mu_1, \mu_2, ..., \mu_k)$. In this work, $L_i = [0, 1]$ (for $i = 1, 2, ..., k$), and some properties of the multi-fuzzy sets of dimension k are given.

Definition 3. Let *k* be a positive integer and let it go μ, ν in $\mathbf{M^kFS}$ (*X*) such, that $\mu = (\mu_1, ..., \mu_k) = \{\langle x, \mu_1(x), \mu_2(x), ..., \mu_k(x)\rangle : x \in X\}$ and $\nu = (\nu_1, ..., \nu_k) = \{\langle x, \nu_1(x), \nu_2(x), ..., \nu_k(x)\rangle : x \in X\}$, then we have the following relations and operations:

1. $\mu \subseteq \nu$ if and only if $\mu_i \le \nu_i$ to all i=1,2,…,k;
2. $\mu = \nu$ if and only if $\mu_i = \nu_i$ to all i=1,2,…,k;
3. $\mu \cup \nu = = (\mu_1 \cup \nu_1, \ldots, \mu_k \cup \nu_k) = \{\langle x, \max(\mu_1(x),\nu_1(x)), \ldots, \max(\mu_k(x),\nu_k(x))\rangle : x \in X\}$;
4. $\mu \cap \nu = (\mu_1 \cap \nu_1, \ldots, \mu_k \cap \nu_k) = \{\langle x, \min(\mu_1(x),\nu_1(x)), \ldots, \min(\mu_k(x),\nu_k(x))\rangle : x \in X\}$;
5. $\mu + \nu = = (\mu_1+\nu_1, \ldots, \mu_k+\nu_k) = \{\langle x, \mu_1(x)+\nu_1(x) - \mu_1(x) \cdot \nu_1(x), \ldots, \mu_k(x)+\nu_k(x)-\mu_k(x) \cdot \nu_{1k} x)\rangle : x \in X\}$;

Remark 2. Let $\mu = (\mu_1, \mu_2)$ be the fuzzy set of measure 2 and let $\mu_1(x)$, $\mu_2(x)$ be the gradation of the membership and non- membership of the quantity x in μ, respectively. If $\mu_1(x) + \mu_2(x) \le 1$, then μ is an intuitionist FS. Consequently, every intuitionist FS in X is a multi-fuzzy set in the X dimensionality of 2, and each intuitionist fuzzy operation is a multi-fuzzy mapping on the multi fuzzy sets. But multi-FS does not necessarily have to be an intuitionistic fuzzy set, for example, multi-FS $\mu = \{(x, \mu_1(x), \mu_2(x)): \mu_1(x) = .9, \mu_2(x) = .8, x \in X\}$ is not an intuitionistic FS.

Arithmetic on multi-fuzzy sets:

$A+B=\{(x, \mu_{A+B}(x), \nu_{A+B}(x)) | x \in X\}$, where $\mu_{A+B}(x)=\mu_A(x)+\mu_B(x)-\mu_A(x) \cdot \mu_B(x)$, $\nu_{A+B}(x)= \nu_A(x) \cdot \nu_B(x)$,

$A \cdot B=\{(x, \mu_{A \cdot B}(x), \nu_{A \cdot B}(x)) | x \in X\}$, where $\mu_{A \cdot B}(x)=\mu_A(x) \cdot \mu_B(x)$, $\nu_{A \cdot B}(x)= \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)$.

We note that between the subjective and the objective fuzziness, which are characterized, above all, by the proximity of the F-norm and the value of the defuzzified value of the FS, there are certain interconnections, and there are cases when these values are close or coincide.

5 Experiment

Assume that an object is an inaccurately measured set (array) of values, which is structured as a matrix of 8×8

A=

[7.91	8.03	7.77	8.25	4.71	5.60	7.70	5.49;
6.96	6.41	8.74	7.42	5.82	4.36	7.09	4.83;
5.05	5.22	6.14	3.82	3.39	6.48	5.77	8.25;
4.74	7.22	8.28	3.07	8.93	7.56	6.41	3.09;
5.05	6.28	4.04	8.36	6.50	6.18	7.77	7.61;
6.20	5.67	8.88	4.19	5.54	6.84	3.36	8.83;
7.36	7.17	4.63	4.79	6.09	4.25	6.62	8.94;
4.86	6.73	4.51	6.97	5.00	5.28	3.30	7.73]

The object obtained by the above-mentioned method (under uncertainty) is formalized: a) in the form of an FS with a heuristic designated MF using standard library MF; b) SOP №1, which is formed on the basis of immersion of the interval US in a special matrix; c) SOP №2, which is formed on the basis of IDS (Fig. 3). Recall that in the general case we have 3 channels of formalization of uncertainty:

- standard (for TFS) - FS $\tilde{a} \stackrel{\Delta}{=} \{a / \mu_A\}, \mu_A \to [0,1], a \in A$ is formed heuristic, MF μ_A is selected by an expert from the existing set of MF, which is placed in the working package of applications, US A = {a} is formed on the basis of IDS; additionally formed 2D tensor T^A: $\tilde{a} \stackrel{\Delta}{=} \{a / \mu_A\} \to \mathbf{T}^A = a \otimes (\mu_A)^T$;
- the formation of an objectively blurred object, the method of immersion of the vector (the interval of values of US) into a special matrix (Toeplitz, Hankel, etc.) $T^{A_1} = toeplitz(a)$, the singular decomposition of which allows calculating the SOP $\tilde{a}^{(1)} \stackrel{\Delta}{=} \{a / \mu_A^{(1)}\}, \mu_A^{(1)} \to [0,1], a \in A^{(1)} \subseteq A$;
- structuring of IDS A in the form of a 2D tensor (a matrix of dimension $m \times m$), $\mathrm{T}^{A_2} = reshape(A, m, m)$, a singular decomposition of which allows the calculation of the SOP $\tilde{a}^{(2)} \stackrel{\Delta}{=} \{a / \mu_A^{(2)}\}, \mu_A^{(2)} \to [0,1], a \in A^{(2)} \subseteq A$.

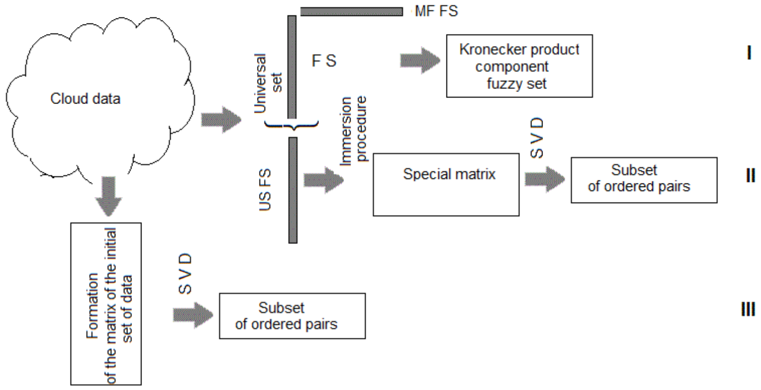

Fig. 3. Formation of multi-fuzzy sets as the structuring of 3 matrices: Kronecker product component FS, IDS and fuzziness US fuzzy set.

Difficulties in the formation of MFS from three objects: FS $\tilde{a}$, SOP $\tilde{a}^{(1)}$ and $\tilde{a}^{(2)}$ consist in the fact that they are constructed on different US $A^{(1)} \subseteq A, A^{(2)} \subseteq A, A^{(1)} \neq A^{(2)} \neq A$, therefore the following steps of the algorithm are as follows: 1-st way - consider 2D tensors as the frontal slices of the 3D tensor ℵ with the following the high-order tensor decomposition, that is, the implementation of the procedure [U1, U2, U3, S] = svd3 (rank, ℵ), a fragment of the algorithm and an example showing its performance is given below:

1. .An unstructured array of initial data is specified: $\mathbf{X}=\{x_i\}$, i=1,n;

2. .Formation FS: $\tilde{x} \overset{\Delta}{=} \{x / \mu^x\}$, $\mu^x \to [0,1]$, $x \in X = \{x_j\}$, $j = 1, m$; m – number of α-levels in FS $\tilde{x}$, $m^2 \le n$;
3. Formation of 3D tensor **x** based on frontal slices: {**x**(:,:,1), **x**(:,:,2),**x**(:,:,3)} -fig.4: **x**(:,:,1)= $x \otimes (\mu^x)^T$, where ⊗ - symbol of a tensor product; **x**(:,:,2)=reshape(**X**, *m*, *m*) – structuring of IDS in the form of a 2D matrix; **x**(:,:,3)= toeplitz([x1,…,xm] – fuzziness US;
4. High-order singular decomposition of the 3D tensor: [**U1,U2,U3,S**]=svd3(rank, **X**) [28]; reconstruction of a complete tensor:

```
F = tmul( tmul( tmul( S, U1, 1), U2, 2), U3, 3).
```

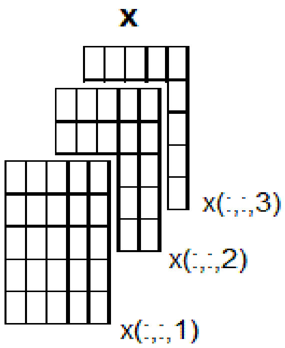

Fig. 4. Representation of the object in conditions of uncertainty in the form of a 3D tensor

5. Formation of a subset of ordered sequences – MFS:

```
m=max(abs(U2(:,1)).* abs(U3(:,1)))
sort([abs(U1(:,1)*S(1))*m,abs(U2(:,1))/max(abs(U2(:,1))),
abs(U3(:,1))/max(abs(U3(:,1)))]),
```

where m is the norm factor, introduced for compatibility of the results in the 2D-tensor. In the MFS notation we have:

```
x=abs(U1(:,1)*S(1))*m,
μ(1) =abs(U2(:,1))/max(abs(U2(:,1))),
μ(2)=abs(U3(:,1))/max(abs(U3(:,1))),
```

that is $\bar{\tilde{x}} \overset{\Delta}{=} \{x / < \mu^{(1)}, \mu^{(2)} >\}$.

Due to the fact that the solution under uncertainty conditions is most often adopted on the basis of the standard FS-1 type {x / μ}, $x \in X$, $\mu \to [0,1]$, it is expedient to bring the MFS $\bar{\tilde{x}} \overset{\Delta}{=} \{x / < \mu^{(1)}, \mu^{(2)} >\}$ to form {x / μ}. This can be realized in two ways: the first is formation of FS on the basis of singular vectors of the 3D tensor (fig.5).

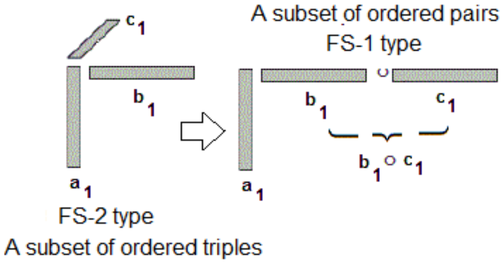

Fig. 5. Formation of FS on the basis of singular vectors of the 3D tensor.

The second method is to immerse the set of frontal slices **x** (:,:, 1), **x** (:,:, 2) and **x** (:,:, 3) in the Toeplitz matrix: ℵ 1 = toeplitz ([**x** (:,: , 1), **x** (:,:, 2), **x** (:,:, 3)]) we have ℵ 1 = [**x** (:,:, 1) **x** (:,:, 2) **x** (:,:, 3); **x** (:,:, 2) **x** (:,:, 1) **x** (:,:, 2); **x** (:,:, 2) **x** (:,:, 1)]. The singular decomposing of matrix ℵ 1 allows one to obtain a SOP $\tilde{x} \overset{\Delta}{=} \{x / \mu\}$ which represents the integral characteristic of the object under uncertainty conditions.

Example. An object in conditions of uncertainty (inaccuracy of measurements, the effect of interference, etc.) is described in the form of a statement <more than 3 and not more than 9 ⟩ and represents one of the realizations of the expression X = 3 + rand (9) * 6; for its formalization by an expert, the proposed FS $\tilde{x} = (x / \mu_x)$ with a triangular MF of the form: μ_{trimf} = trimf (x, [3.00 6.00 9.00]), $x \in X$ = [3.06 3.76 4.47 5.17 5.88 6.59 7.29 8.00 8.70], defuzzified value of FS, calculated by the center of gravity (CG) method and the F-norm are given below: def ($\tilde{x}$) = 5.98, $\|\tilde{x}\|_F$ =norm ($\tilde{x}$, 'fro') = 18.54. In addition to heuristic FS, this object can be characterized by a further number of subsets of ordered pairs, endowed with properties of FS, which is accepted as a standard, the results of calculations are given in Table 3.

Table 3. Comparative characteristics FS and SOP

FS $\tilde{x}$	Subsets of ordered pairs			
	Structuring and decomposition IDS	Immersion of US into a special (Toeplitz) matrix	Truncated models columns 1 and 3	Integral model of uncertainty
1	2	3	4	5
3.17 0	5.64 0.84	5.38 0.80	3.17 0	5.13 0.66
3.87 0.25	6.04 0.86	5.46 0.82	4.56 0.50	5.27 0.68
4.56 0.50	6.07 0.87	5.46 0.82	5.96 1.00	5.50 0.74
5.26 0.75	6.32 0.88	5.70 0.85	7.35 0.50	5.59 0.77
5.96 1.00	6.72 0.90	5.70 0.85	8.74 0	5.67 0.78
6.65 0.75	6.80 0.90	6.10 0.91	5.38 0.80	6.00 0.81
7.35 0.50	6.94 0.91	6.10 0.91	5.46 0.82	6.18 0.86
8.04 0.25	7.35 0.94	6.69 1.00	5.70 0.85	6.29 0.88
8.74 0	7.69 1.00	6.69 1.00	6.10 0.91	6.40 0.94
			6.69 1.00	

$\|\tilde{x}\|_F$ ...$def(\tilde{x})$	$\|\tilde{x}\|_F$ ...$def(\tilde{x})$	$\|\tilde{x}\|_F$ ...$def(\tilde{x})$	$\|\tilde{x}\|_F$ ...$def(\tilde{x})$	$\|\tilde{x}\|_F$ ...$def(\tilde{x})$
18.74 5.96	20.13 6.65	18.01 5.96	14.08 5.96 13.30 5.90	17.55 5.83

6 Results and Discussion

Thus, in conditions of uncertainty, under which in this case we will understand the situation when a semantically uniquely identified object (for example, the result of measuring a certain value in the conditions of interference), the known initial data array, which allows structuring and determine the interval of values, which is accepted as a US for the formation of FS, an object in conditions of uncertainty can be represented by the totality of FS and SOP.

In this case, the standard FS $\tilde{x} \overset{\Delta}{=} \{x/\mu_x\}, \mu_x \to [0,1], x \in X$ intended heuristic;

SOP1 $\tilde{x}_1 = \{x/\mu_{x_1}\}, \mu_{x_1} \to [0,1], x \in X_1$ is formed on the basis of formal procedures for simulating the fuzziness process by immersion of the US interval into a special matrix (Toeplitz) and subsequent singular decomposition;

SOP2 $\tilde{x}_2 = \{x/\mu_{x_2}\}, \mu_{x_2} \to [0,1], x \in X_2$ is formed on the basis of structuring the IDS in the form of a matrix and the subsequent singular decomposition.

All SOP and FS have the properties: proximity to F-norm, defuzzified values- $def(\tilde{x}) \approx def(\tilde{x}_1) \approx def(\tilde{x}_2)$ takes place inclusion - X1∈X, X2 ∈X. It should be noted separately that the integral model of uncertainty containing the of the Kronecker product components of FS, IDS, the matrix of the Toeplitz model of fuzziness, which is considered as a block vector, immersed in a special matrix, has the value $\|\tilde{x}\|_F$ and def ($\tilde{x}$) that is practically the same as the FS, but the length of the interval US , on which the SOPs are defined, is much shorter than the corresponding magnitude for FS (1.27 versus 5.5). Failure to take into account this factor can have a significant effect on decision making as a result of the use of fuzzy mathematics. FS by its nature gives guaranteed partial overvaluation, it can be effectively used as a control parameter in decision-making.

Operations of fuzzy mathematics in the system of subsets of ordered pairs calculated on the basis of the singular decomposition of the Toeplitz matrices formed by immersing the real component of the FS into a special matrix.

Let 2 fuzzy number (or FS) be given: $\tilde{a} = \{a/\mu_a\}$, $\mu_a \to [0,1]$, $a \in X$; $\tilde{b} = \{b/\mu_b\}$; $\mu_b \to [0,1]$, $b \in X$; the principle of fuzzy expansion the result of a mathematical (arithmetic) operation on them allows you to represent in the form $\tilde{c} = \tilde{a} *_f \tilde{b} = \{c/\mu_c\}$ where $*_f \in \{+, -, *, /\}$, c = a * b, $\mu_c = \min(\mu_a, \mu_b), \mu_c \to [0,1], c \in X$.

Suppose that the FS $\tilde{a}$ and $\tilde{b}$ are represented in the form of FS with a triangular or trapezoidal MF, that is, trimf (x, [a, b, c]) or trapmf (x, [a,b,c,d]), the choice of these FS is due to the fact that they explicitly contain the interval of values for trimf x=$\mathbf{I}^{ac}$, for trapmf x = $\mathbf{I}^{ad}$. If the number of α-levels is n, then the FS as the SOP has the form of matrices with the dimension $2\times n$, the 1-st vector-column $(a_1 a_2 ... a_n)^T$ or $(b_1 b_2 ... b_n)^T$ used to form the Toeplitz matrix simulating the process of blurring (fuzzines) the interval.

$$\tilde{a} = \begin{pmatrix} a_1 \mu_{a_1} \\ a_2 \mu_{a_2} \\ . \\ . \\ . \\ a_n \mu_{a_n} \end{pmatrix}, \tag{5}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{pmatrix} \rightarrow toeplitz(a) = \mathbf{T}^a = \begin{pmatrix} a_1 & a_2 & ... & a_n \\ a_2 & a_1 & ... & a_{n-1} \\ & & . & \\ & & . & \\ & & . & \\ a_n & a_{n-1} & ... & a_1 \end{pmatrix} \rightarrow svd(\mathbf{T}^a) \rightarrow \tilde{a}^{(n)} = \left(\begin{array}{cc:} a_1^{(T)} & \mu_{a_1}^{(T)} \\ a_2^{(T)} & \mu_{a_2}^{(T)} \\ . & \\ . & \\ . & \\ a_n^{(T)} & \mu_{a_n}^{(T)} \end{array}\right) \tag{6}$$

The procedure for $\tilde{b}$ is performed similarly. The singular decomposition of the Toeplitz matrixes Ta and Tb allows us to calculate the SOP $\tilde{a}^{(n)}$ and $\tilde{b}^{(n)}$, which are structurally and functionally similar to $\tilde{a}$ and $\tilde{b}$, because $\| \tilde{a}^{(n)} \|_F \approx \| \tilde{a} \|_F$, $def\left(\tilde{a}^{(n)}\right) \approx def\left(\tilde{a}\right)$, $\| \tilde{b}^{(n)} \|_F \approx \| \tilde{b} \|$, $def\left(\tilde{b}^{(n)}\right) \approx def\left(\tilde{b}\right)$.

The result of the operation $\tilde{a} *_{\mathrm{f}} \tilde{b}$ is nearest to the F-norm and the defuzzification value of the result $\tilde{a}^{(n)} *_{\mathrm{f}} \tilde{b}^{(n)}$.

Table 4 shows the results of an arithmetic operation, which is performed on a FS (with expertly assigned MF) - a subjectivistic fuzziness, and over a SOP calculated on the basis of formal interval blurring models by immersion of interval (US) into the Toeplitz matrix.

Table 4. Fuzzy arithmetic in the SOP system, formed on the basis of immersion the interval of possible values of FS into a special (Toeplitz) matrix

Standart FS x =[4:4/8:8] ; a=trimf(x,[4 6 8]); x1=[7:4/8:11] ;b=trapmf(x1,[7 8 10 11])			Subsets of ordered pairs, calculated on the basis of blurring (fuzziness) of the universal set by immersion into the Toeplitz matrix		
a	b	c=a+b	a^n	b^n	$c^n = a^n + b^n$
4.00 0 4.50 0.25 5.00 0.50 5.50 0.75 6.00 1.00 6.50 0.75 7.00 0.50 7.50 0.25 8.00 0	7.00 0 7.50 0.50 8.00 1.00 8.50 1.00 9.00 1.00 9.50 1.00 10.00 1.00 10.50 0.50 11.00 0	11.00 0 12.00 0.25 13.00 0.50 14.00 0.75 15.00 1.00 16.00 0.75 17.00 0.50 18.00 0.25 19.00 0	5.61 0.85 5.64 0.85 5.70 0.86 5.78 0.87 5.89 0.89 6.02 0.91 6.19 0.94 6.39 0.97 6.62 1.00	8.63 0.90 8.65 0.90 8.71 0.91 8.79 0.91 8.90 0.93 9.03 0.94 9.20 0.96 9.39 0.98 9.61 1.00	14.24 0.85 14.29 0.85 14.40 0.86 14.56 0.87 14.78 0.89 15.06 0.91 15.39 0.94 15.78 0.97 16.22 1.00
F-norms and defuzzification values					
18.49 6.00	27.13 9.00	45.69 15.00	18.18 6.00	27.13 9.00	45.03 15.01

Remark 3. Specificity of the Toeplitz matrix is such that it must be formed on the basis of a vector of US with a dimension of $1\times m$, $m = 2n$, whereas the FS is based on a US measure of $1\times n$. The SOP with the dimension of $2\times m$, which is calculated as a result of the singular decomposition of the Toeplitz matrix, has the same content in two adjacent lines. In order to be able to compare on the basis of calculated F-norms and defuzzified values of FS and SOP, measuring $2\times n$ and $2\times m$, respectively, SOP are cut to a dimension of $2\times n$, using only odd lines.

7 Conclusions

1.The complication of tasks in conditions of uncertainty, due, above all, the complication of technological processes requiring automation, the need to take into account the data of super-large volumes and measurements, which are represented by rank-1 tensors, leads to the need to take into account the situation when the MF, cannot be fundamentally formed or have insurmountable difficulties in their formation. Another reason that determines the limited use of TFS for solving problems under uncertainty is that under real conditions, the object is described not only by the standard MF, which is designated heuristically, but also by a number of SOP, caused by the discovery of latent knowledge that takes into account the structural features of the initial data array and the value interval - US, on which the MF and SOP are formed.

2. The technology of blurring of the interval on which the FS is formed is proposed as a procedure for immersing this interval (vector) into a special matrix (Toeplitz), the

singular decomposition of which allows one to calculate the SOP, which is one of the characteristics of uncertainty. It is shown that this SOP is the nearest (in the sense of the F-norm) to the heuristic FS intended for this range and can be to some extent a verification test for the expediency of using the adopted MF.

3.The necessity (in a number of cases) of the simultaneous consideration of the subjective component of the fuzziness in the form of the KP of the components of the FS (2D tensor $m \times m$), structured IDS and the blurred interval of FS (objective fuzziness), characterizing the uncertainty, by creating a block of the Toeplitz Matrix, the singular decomposition of which allows for a generalized SOP, which integrally characterizes the uncertainty. Over SOP, obtained by means of singular decompositions (2D and 3D tensors), it is possible to apply without limitation all processing methods developed for standard FS.

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