Comparison of numerical integration methods

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***Abstract*—The calculation of the integral is formally based on the calculation of the integral in a given range, i.e. the area between the specified function and the OX axis. Integrals find a variety of applications in the description of certain phenomena, whether objects in industry or information technology. Therefore, the numerical approach is an important element. In this work, we analyze three different approaches to this problem relative to selected, test functions.**

I. Introduction

Numeric integration consists in the elimination of the integral value by means of a specific algorithm. Integration of a specific function will consist in approximating the value of the surface area between this function determined in a given range and the OX axis. Such algorithms determine the value of the definite integral is important in the industry, as well in informatics, which can be seen on the example of the latest research results published around the world.

As an intermediate tool, numerical integration was used in estimating the population abundance [1] or in reconstructing medical images [2] or in physics [3]. Again, in [4], the authors analyze the complexity of this type of algorithms, and in [5], [6] the idea of integrating in many dimensions was considered.

In this paper, we compare classic numerical numeric algorithms on selected sets of test functions to determine the selection of the approximation method.

II. Numerical integration

Integrals are important part in mathematical analysis. There are two types of integral - indefinite and definite integral. The first one is called primitive function and it is reverse of derivative. It is also generalization of definite integral. After using the range $[a, b]$ we get definite integral on this range. It can be understood as the area between the function graph and the OX axis, where for the positive values it takes sign plus and for negative minus. Integrals are very useful not only in mathematics analysis, but in the physics calculations too. Because of this interpretation of the integral, we can use it to calculate values in physics, e.g. work $W = Fs$, which the result is area of the graph.

However many function are not possible to calculate precisely, which means that we can not calculate them with analitical mathematics methods. For this reason the another method was established. It is called numerical integration and its purspose is to find approximation of searched integral. Depending on our needs, we can use methods, which allow us to calculate the result with specific error. There are a few basic methods of numerical integration, differing in a way of approximation:

- midpoint rule,
- trapezoidal rule,
- Simpson's rule.

In all of them we get approximated value of integral, but they are determined with various errors and speed of convergence to the correct result. All of these methods consist on dividing the interval $[a, b]$ on n same subintervals and calculating area of function for each of the subintervals with using specific formulas.

A. Midpoint rule

Le us assume that $f : [a, b] \to \mathbb{R}$ is our function, which we want to integrate. In the case of midpoint rule, we use value $f(\frac{x_i+x_{i+1}}{2})$ as an approximation for the value in the subinterval. We get

$$h = \frac{b-a}{n} \tag{1}$$

$$p_i = f(\frac{x_i + x_{i+1}}{2}) \tag{2}$$

where h is the length of subintervals, x_i is end of $i-th$ subinterval for $i = 0, ..., n-1$, $x_0 = a$ and p_i is approximated value for whole subinterval, which means rectangle $h \times f(\frac{x_i+x_{i+1}}{2})$. Then

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} p_i h \tag{3}$$

Midpoint rule is one of the least accurate methods, however it gives us quite accurate approximation in the case, when function doesn't change a lot in the subintervals. Error of this rule is estimated by the following formula

$$error = \frac{(b-a)^3}{24n^2} M^2 \tag{4}$$

where M is maximum of second derivative of the function f. According to this formula, we can see, that significant impact of the value of error has number n of subintervals – the bigger n, the smaller error.

B. Trapezoidal rule

Trapezoid rule is similar to midpoint rule, but instead of taking rectangles, we use trapezoid. In other words, we approximate by inscribing polygonal chain in the graph of the function, taking separate segment for each subinterval. In this case we calculate following values:

$$h = \frac{b-a}{n} \tag{5}$$

$$p_i = \frac{y_i + y_{i-1}}{2} h \tag{6}$$

where h is a length of subinterval and $y_i = f(x_i)$ for $i = 1, ..., n$. Then p_i means area of the trapezoid for $[x_{i-1}, x_i]$. The whole integral on the range of $[a, b]$ equals

$$\int_a^b f(x)dx = \sum_{i=1}^{n} p_i \tag{7}$$

For trapezoid rule error has following formula:

$$error = \frac{(b-a)^3}{12n^2} M^2 \tag{8}$$

and similar to midpoint rule, M is maximum of the second derivative.

C. Simpson's rule

Simpson's rule is based on polynomial interpolation and uses second degree polynomial. It is the most acurate method from these three. It is similar to trapezoid rule, because like there, here we use h, y_{i-1} and y_i as three sides of the figure, but the fourth side is parabola approximated to graph of the function. Values at the beginning and end of the subinterval are used as the points needed to approximate this parabola.

Value of the area for $[x_{i-1}, x_i]$ is calculated in the following way

$$p_i = h(f(x_i) + 4f(x_i + \frac{1}{2}h) + f(x_i + h)) \tag{9}$$

where $i = 1, ..., n$ and $h = \frac{b-a}{n}$. Then the whole integral can be approximated to the value

$$\int_a^b f(x)dx = \frac{1}{6} \sum_{i=1}^{n} hp_i \tag{10}$$

This method has a following error:

$$error = -\frac{1}{90} \frac{(b-a)^5}{n^5} M \tag{11}$$

where M is maxixmum of the fourth derivative. Usually this error is the smallest one from these three rules.

III. Experiments

In the experiments we wanted to compare the results for three rules, suing a few functions:

- $f(x) = sinx$ dla $x \in \left[0, \frac{\pi}{4}\right]$,
- $f(x) = x^3$ for $x \in [0, 2.5]$,
- $f(x) = e^x$ for $x \in [0, 2]$,
- $f(x) = lnx$ for $x \in [1, 3]$,
- $f(x) = x$ for $x \in [0, 2]$,
- $f(x) = x^3 - x$ for $x \in [0, 2]$,

and various number of subintervals. These functions are possible to calculate precise value, which allow us to compare our result. The effect of our calculations was showed in Tab. I, where additionaly was showed precise value of integral.

Also in Tab. II, we computed relative errors for each result. We can draw some conclusions from them:

A. Simple functions

Simple functions, here $f(x) = x$ and $f(x) = x^3 - x$, calculated numericaly have a precise solution very fast. Sometimes even we don't need to divide the interval into many subintervals to find very approximated result and often additionaly the choosen rule covers precisely the graph of the function. For $f(x) = x$ all of these three methods find the precise solution and for $f(x) = x^3 - x$ Simpson's rule finds it very fast, because in specific division graphs in subintervals are covered with Simpson's parabolas.

B. Imperfection of midpoint rule

Midpoint rule is one of the least accurate methods and can lead to very wrong result, for example it can calculate that the integral equals 0. It is visible in $f(x) = x^3 - x$, where for not divided interval, so $n = 1$, the middle of the interval is $x = 1$, for which value of function equals 0. From the formula for the area of the rectangle we can see, that whole integral equals 0. However it changes when we start to divide the intervals and eliminate the $x = 1$ as the midpoint.

C. Simpson's rule

Simpson's rule is the most accurate method and the fastest convergent. The easiest way to see this is the more complicated function, where none of the rules find precise solution. For example for $f(x) = x^3$ or $f(x) = sin(x)$ we can see, that the error for $n = 1$ is big, but it rapidly deacreases for $n = 4$ and $n = 10$.

IV. Conclusions

In this paper we compared three methods of numerical integration – midpoint, trapezoid and Simpson's rules – to find out which one is the most accurate and the fastest. To compare we used a few simple function and calculated values for various number of subintervals and looked at them to find the best one. The Simpson's rule errors decrease fastest, so this method is the best one.

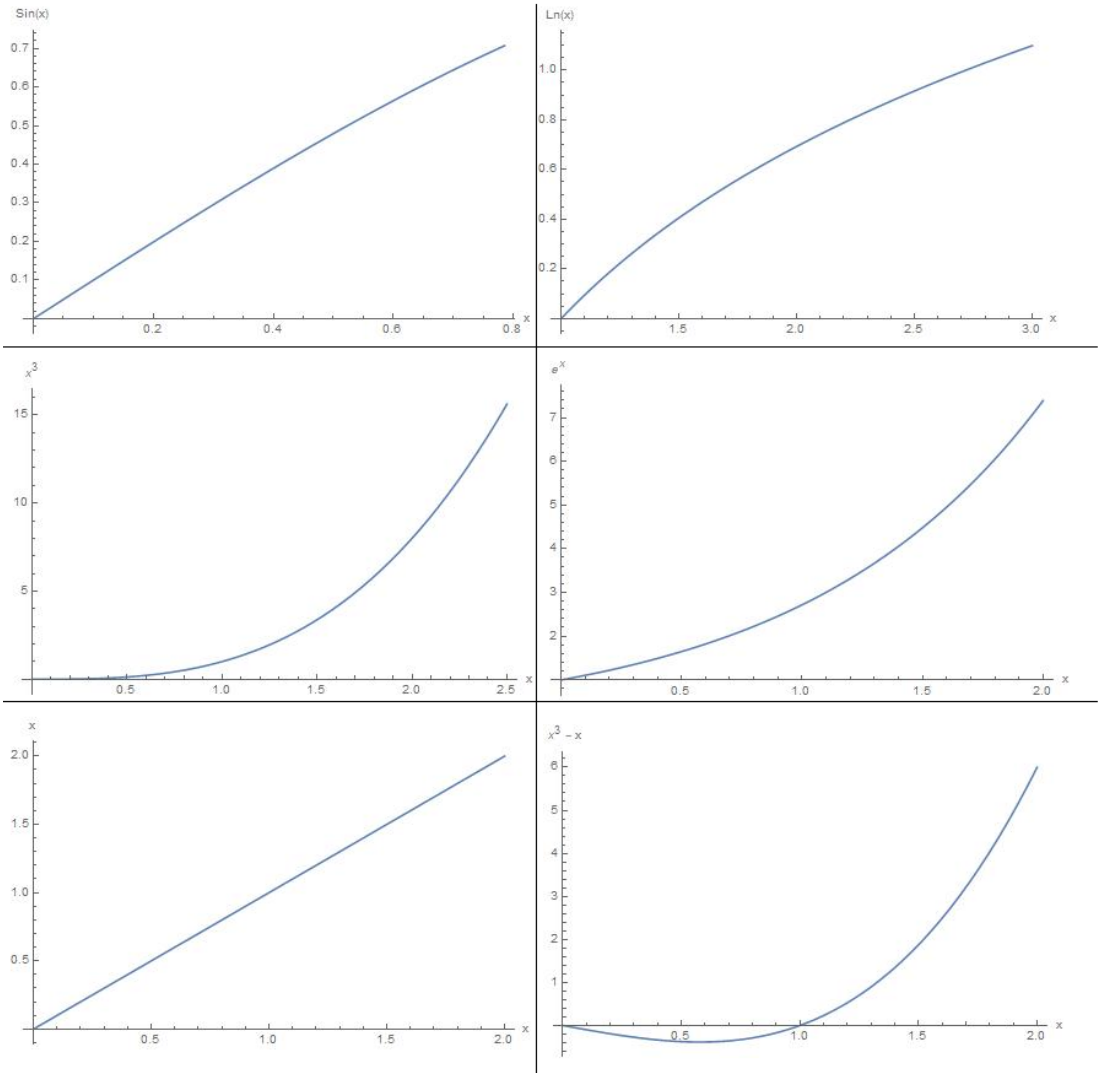

Figure 1: Charts of the analyzed functions.

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	a	b	n	The exact value	Simpson's rule	Trapezoid rule	Midpoint rule
$sin(x)$	0	0.77	1	0.28208933000000003	0.178675	0.26801206700000002	0.28918053500000002
	0	0.77	4	0.28208933000000003	0.28209099999999998	0.28121769400000002	0.28252535000000001
	0	0.77	10	0.28208933000000003	0.28208899999999998	0.28194994099999998	0.28215902999999998
x^3	0	2.5	1	9.765625	13.020799999999999	19.53125	4.8828125
	0	2.5	4	9.765625	9.7656299999999998	10.37597656	9.4604492189999991
	0	2.5	10	9.765625	9.7656299999999998	9.86328125	9.716796875
e^x	0	2	1	6.3890560990000003	6.4207278040000002	8.3890560989999994	5.4365636569999998
	0	2	4	6.3890560990000003	6.3891937250000002	6.5216101100000001	6.3229855329999998
	0	2	10	6.3890560990000003	6.3890596439999996	6.4103387679999999	6.3784200819999999
$log(x)$	1	3	1	1.2958368659999999	1.2904003369999999	1.0986122890000001	1.2969442799999999
	1	3	4	1.295836867	1.2957983500000001	1.2821045820000001	1.3026452340000001
	1	3	10	1.2958368680000001	1.2958358109999999	1.2936188740000001	1.2969442799999999
x	0	2	1	2	2	2	2
	0	2	4	2	2	2	2
	0	2	10	2	2	2	2
$x^3 - x$	0	2	1	2	4	6	0
	0	2	4	2	2	2.25	1.875
	0	2	10	2	2	2.04	1.9799924799999999

Table I: Comparison results for selected methods.

	a	b	n	Error for Simpson's rule	Error for trapezoid rule	Error for midpoint rule
$sin(x)$	0	0.77	1	0.3666013537144141	4.9903565900870923E-2	2.5138153565741946E-2
	0	0.77	4	5.9187314800328924E-6	3.0899296609271398E-3	1.5456808262322172E-3
	0	0.77	10	1.1712211858265657E-6	4.9413208506619789E-4	2.4708439425509267E-4
x^3	0	2.5	1	0.33332991999999995	1	0.5
	0	2.5	4	5.1199999998061685E-7	6.25E-2	3.125E-2
	0	2.5	10	5.1199999998061685E-7	0.01	5.0000000000000001E-3
e^x	0	2	1	4.9571806194195298E-3	0.31303528549466747	0.1490818717605904
	0	2	4	2.1540908057286479E-5	2.0747041271432749E-2	1.0341209174399794E-2
	0	2	10	5.5486599967827115E-7	3.3311132255614338E-3	1.6647242702880168E-3
$log(x)$	1	3	1	4.195380717004561E-3	0.1521986155624622	8.5459355281236604E-4
	1	3	4	2.9723648848697756E-5	1.0597232838259647E-2	5.2540306157380376E-3
	1	3	10	8.1568909350790468E-7	1.7116305723136754E-3	8.5459200808910485E-4
x	0	2	1	0	0	0
	0	2	4	0	0	0
	0	2	10	0	0	0
$x^3 - x$	0	2	1	1	2	1
	0	2	4	0	0.125	6.25E-2
	0	2	10	0	2.0000000000000018E-2	1.0003759765624953E-2

Table II: The results of the comparison of the error value.