Investigation of Mathematical Model of Acoustic Wave Propagation Through Relax Environment in Ultrasound Diagnostics Problems

Zinovii Nytrebych[1[0000-0002-9599-8517]], Volodymyr Il'kiv [1[0000-0002-0359-5025]],

Oksana Malanchuk [2[0000-0001-7518-7824]] and Winfried Auzinger [3[0000-0002-9631-2601]]

[1] Lviv Polytechnic National University, Lviv 79013, Ukraine
znytrebych@gmail.com
[2] Danylo Halytskyi Lviv National Medical University, Lviv 79017, Ukraine
[3] Vienna University of Technology, Karlsplatz 13, 1040 Vienna, Austria
w.auzinger@tuwien.ac.at

Abstract. A mathematical model of the process of an acoustic wave propagation in a relax environment has been investigated. This mathematical model is widely used to describe and determine the basic parameters of the wave process in the problems of ultrasound diagnostics. The model is formulated in the form of the Cauchy problem for hyperbolic equation of third order with the initial data, which are analytical functions. The class of entire functions, which is the class of existence and uniqueness of the Cauchy problem solution for the partial differential equation, which describes this wave, is established. In the selected class of functions, the Cauchy problem solution is constructed using the differential-symbol method. Examples of solving problems with specific initial data are given. The obtained results and the indicated methodology allow us to determine the basic parameters of the process of acoustic wave propagation in the problems of ultrasound diagnostics.

Keywords: Mathematical model, wave process, ultrasound diagnostics, initial problem, differential-symbol method

1 Introduction

Simulation of biomechanical processes in medicine is an extremely important area of scientific researches [1–3]. Such modeling is often based on existing models. For example, models of continuous-environment mechanics (in vibration problems [5–7]) and models of gas-hydrodynamics problems [8, 9] are used particularly in models of biological and medical processes [4]. A characteristic feature of modern models is the using of nonlinear partial differential equations, in addition to the ordinary differential

2019 IDDM Workshops.

equations. The study of such models is quite complicated (see, in particular, [10–12]). Numerical, qualitative, and asymptotic methods are used to research such models [13–15].

In recent years, in modeling complicated biomedical processes of diverse nature, the interest has been increasing not only in traditional partial differential equations of second order, but also in equations of higher order. The wave processes with dispersion and absorption in water dynamics problems [16, 17], viscosity theory [18], and geophysics [19] are described by partial differential equations of third order with respect to time. In particular, such equations include the equations of fourth order in spatial variables which describe the processes of vibration of mechanical systems [20, 21].

The hyperbolic equations of third order in time, which are intensely studied in ultrasound diagnostics, include the equation of the form

$$\tau\left(\frac{\partial^3 u}{\partial t^3}-c_f^2\Delta\frac{\partial u}{\partial t}\right)+\frac{\partial^2 u}{\partial t^2}-c_e^2\Delta u = f(t,x),\quad x=(x_1,x_2,x_3)\in\mathbb{R}^3, \tag{1}$$

in which $u(t,x)$ is dynamic pressure, τ is relaxation time, constants c_e and c_f are limiting phase speeds of sound, $\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$ is three-dimensional Laplace operator.

In research [22], the solution of the Cauchy problem for equation (1) is given by the fundamental solution of equation (1) using modified Bessel functions.

The work is aimed to:

- study of the mathematical model of the acoustic wave propagation process in a relax environment with given initial data, which are entire analytical functions;
- establishing a class of unique solvability of the corresponding Cauchy problem;
- presentation of the analytical method of solving the problem;
- study of the process of acoustic wave propagation for the specific initial data of the problem, development of a method of finding the determining parameters of the wave process.

2 Posing of the problem and main results

Let us consider the Cauchy problem

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2}-\Delta\right)+\frac{\partial^2}{\partial t^2}-\alpha\,\Delta\right]u(t,x)= f(t,x),\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \tag{2}$$

$$u(0,x)=\varphi_0(x),\ \frac{\partial u}{\partial t}(0,x)=\varphi_1(x),\ \frac{\partial^2 u}{\partial t^2}(0,x)=\varphi_2(x),\quad x\in\mathbb{R}^3, \tag{3}$$

where α is the constant which belongs to the interval $(0,1)$, $\mathbb{R}_+=(0,\infty)$.

Note that equation (2) is obtained from (1) by introducing dimensionless variables $\overline{x}_i = x_i / (c_f \tau)$ i $\alpha = c_f / c_e$.

In work [23], the solution of problem (2), (3) is based on the fundamental solution of equation (2), but it has a very complicated structure and needs simplification.

In this research, we recommend another approach to solving problem (2), (3). We use the differential-symbol method, which was effectively applied to solving the Cauchy problem [24] and two-point in time problems [25–27].

Let us write ordinary differential equation

$$\left[\frac{d^3}{dt^3} + \frac{d^2}{dt^2} - \beta\frac{d}{dt} - \alpha\beta\right] V(t,\alpha,\beta) = 0\,, \tag{4}$$

in which β is the Laplace operator symbol, that is $\beta = |\nu|^2 \equiv \nu_1^2 + \nu_2^2 + \nu_3^2$, $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$.

Let $\lambda_1 = \lambda_1(\alpha,\beta)$, $\lambda_2 = \lambda_2(\alpha,\beta)$, $\lambda_3 = \lambda_3(\alpha,\beta)$ is the roots of the algebraic equation

$$\lambda^3 + \lambda^2 - \beta\lambda - \alpha\beta = 0\,. \tag{5}$$

They belong to the set

$$\left\{-\frac{1}{3} + \frac{\sqrt[3]{2}(1+3\beta)}{3A} + \frac{A}{3\sqrt[3]{2}}, -\frac{1}{3} - \frac{(1\pm i\sqrt{3})(1+3\beta)}{3\sqrt[3]{4}A} - \frac{(1\mp i\sqrt{3})A}{6\sqrt[3]{2}}\right\},$$

where $A = \sqrt[3]{B + \sqrt{-4(1+3\beta)^3 + B^2}}$, $B = -2 - 9\beta + 27\alpha\beta$, $i^2 = -1$.

Remark 1. If $\beta = 0$, then $B = -2$, $A = -\sqrt[3]{2}$, roots λ_1, λ_2 and λ_3 of equation (5) are independent of the parameter α, in particular, $\lambda_1(\alpha,0) = -1$, $\lambda_2(\alpha,0) = 0$, $\lambda_3(\alpha,0) = 0$.

Remark 2. If $\beta = -\dfrac{1}{3}$, $\alpha = \dfrac{1}{9}$, then $\lambda_1 = \lambda_2 = \lambda_3 = -\dfrac{1}{3}$. For the other pairs (α,β) at least the two roots are different.

In the case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, the elements of normal fundamental system of solutions of equation (4) have the form

$$\begin{aligned}
V_0(t,\alpha,\beta) &= \frac{\lambda_3\lambda_2(\lambda_3-\lambda_2)e^{\lambda_1 t} - \lambda_3\lambda_1(\lambda_3-\lambda_1)e^{\lambda_2 t} + \lambda_2\lambda_1(\lambda_2-\lambda_1)e^{\lambda_3 t}}{(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)},\\
V_1(t,\alpha,\beta) &= \frac{(\lambda_3^2-\lambda_2^2)e^{\lambda_1 t} - (\lambda_3^2-\lambda_1^2)e^{\lambda_2 t} + (\lambda_2^2-\lambda_1^2)e^{\lambda_3 t}}{(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)},\\
V_2(t,\alpha,\beta) &= \frac{(\lambda_3-\lambda_2)e^{\lambda_1 t} - (\lambda_3-\lambda_1)e^{\lambda_2 t} + (\lambda_2-\lambda_1)e^{\lambda_3 t}}{(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}.
\end{aligned} \tag{6}$$

They are entire functions of variable $\beta = \nu_1^2 + \nu_2^2 + \nu_3^2$ and vector-parameter $\nu = (\nu_1, \nu_2, \nu_3)$ according to the Poincare's theorem [28].

Provided $\lambda_1 \neq \lambda_2 = \lambda_3$ functions (6) have the form

$$V_0 = \frac{\lambda_2^2 e^{\lambda_1 t} - \lambda_1(2\lambda_2 - \lambda_1)e^{\lambda_2 t} + \lambda_2\lambda_1(\lambda_2 - \lambda_1)\, t\, e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)^2},$$
$$V_1 = \frac{-2\lambda_2 e^{\lambda_1 t} + 2\lambda_2 e^{\lambda_2 t} - (\lambda_2^2 - \lambda_1^2)\, t\, e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)^2},$$
$$V_2 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t} + (\lambda_2 - \lambda_1)\, t\, e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)^2}.$$

If $\lambda_1 = \lambda_2 = \lambda_3$, then the normal fundamental system of solutions of equation (4) is the following:

$$V_0 = \left(1 - \lambda_1 t + \frac{1}{2} t^2 \lambda_1^2\right) e^{\lambda_1 t},$$
$$V_1 = t\left(1 - \lambda_1 t\right) e^{\lambda_1 t},$$
$$V_2 = \frac{1}{2} t^2 e^{\lambda_1 t}.$$

Remark 3. According to Remark 1, if $\beta = 0$ the condition $\lambda_1 \neq \lambda_2 = \lambda_3$ is fulfilled and functions (6) have the following form:

$$V_0(t,\alpha,0) = 1,\ V_1(t,\alpha,0) = t,\ V_2(t,\alpha,0) = e^{-t} - 1 + t.$$

Remark 4. If $\beta = -\frac{1}{3}$ and $\alpha = \frac{1}{9}$, then according to Remark 2 all roots of equation (5) are identical and equal to $-\frac{1}{3}$. Functions (6) have such form

$$V_0 = \left(1 + \frac{1}{3} t + \frac{1}{18} t^2\right) e^{-\frac{1}{3}t},$$
$$V_1 = t\left(1 + \frac{1}{3} t\right) e^{-\frac{1}{3}t}, \tag{7}$$
$$V_2 = \frac{1}{2} t^2 e^{-\frac{1}{3}t}.$$

Let the initial functions $\varphi_0(x), \varphi_1(x), \varphi_2(x)$ and right-hand side $f(t,x)$ of equation (2) are arbitrary entire functions. Then there is only one solution of problem (2), (3) in the class of entire functions. This solution can be presented in the form

$$u(t,x) = \sum_{k=0}^{2} \varphi_k\left(\frac{\partial}{\partial \nu}\right)\left\{ V_k(t,\alpha,\beta) e^{\nu \cdot x} \right\}\Bigg|_{\nu = O}$$
$$+ f\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\right)\left\{ \frac{e^{\lambda t} - \sum_{k=0}^{2} \lambda^k V_k(t,\alpha,\beta)}{\lambda^3 + \lambda^2 - \lambda\beta - \alpha\beta} e^{\nu \cdot x} \right\}\Bigg|_{\lambda = 0, \nu = O}, \tag{8}$$

where $\nu \cdot x = \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3$, $O = (0,0,0)$, the differential expressions $\varphi_0\left(\frac{\partial}{\partial \nu}\right)$, $\varphi_1\left(\frac{\partial}{\partial \nu}\right)$, $\varphi_2\left(\frac{\partial}{\partial \nu}\right)$ and $f\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\right)$ are obtained from the functions $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$ and $f(t,x)$ with the change x to $\frac{\partial}{\partial \nu}$ and t to $\frac{\partial}{\partial \lambda}$.

The differential expressions $\varphi_0\left(\frac{\partial}{\partial \nu}\right)$, $\varphi_1\left(\frac{\partial}{\partial \nu}\right)$, $\varphi_2\left(\frac{\partial}{\partial \nu}\right)$ and $f\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\right)$ for entire data of problem can be defined as corresponding Maclaurin series. The actions of these expressions on functions in curly brackets of formula (8) are correct because the functions in brackets are entire functions of first order with respect to the vector ν and the variable λ in the last brackets.

Function (8) satisfies equation (2). It follows from the commutativity of differentiation operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial \nu}$ and $\frac{\partial}{\partial \lambda}$, and from the fact that functions (6) satisfy equation (4). Also, the function $u(t,x)$ of the form (8) satisfies initial conditions (3), since these differentiation operators are commutative, and corresponding initial conditions are satisfied for functions (6).

The fact that the found solution of the Cauchy problem in the class of entire functions is unique, can be proved by method of contradiction (see, for example, [25]).

The main result. The process of an acoustic wave propagation in a relax environment with data at the initial (zero) moment of time is described by the Cauchy problem (2), (3) for hyperbolic equation of third order; equation (3) is important in the problems of ultrasound diagnostics; a class of entire functions as a class of uniqueness solvability of problem is established; formula (8) for constructing the solution of the problem is proposed.

3 The examples of application of the developed method and constructing the solution of problems with specific initial data

Let us investigate the process of acoustic wave propagation for specifically given initial functions and the right-hand side of the equation, which are integer functions. We use the method of constructing the solution of problem from the previous section.

Example 1. Let the initial functions in problem (2), (3) and the right-hand side of equation (2) are polynomials, such as $\varphi_0(x) = x_1 + x_2 - x_3$, $\varphi_1(x) = 2x_2$, $\varphi_2(x) = x_1^2 x_3$, $f(t,x) = 3$. Then the solution of problem exists in the class of entire functions. It is unique and can be found by formula (8) due to Remark 1 and 3:

$$u(t,x) = \left(\frac{\partial}{\partial \nu_1} + \frac{\partial}{\partial \nu_2} - \frac{\partial}{\partial \nu_3}\right)\left\{V_0(t,\alpha,\beta)\, e^{\nu \cdot x}\right\}\Big|_{\nu=O}$$

$$+2\frac{\partial}{\partial\nu_2}\left\{V_1(t,\alpha,\beta)e^{\nu\cdot x}\right\}\Big|_{\nu=O}+\frac{\partial^3}{\partial\nu_1^2\partial\nu_3}\left\{V_2(t,\alpha,\beta)e^{\nu\cdot x}\right\}\Big|_{\nu=O}$$

$$+3\frac{e^{\lambda t}-\sum_{k=0}^{2}\lambda^k V_k(t,\alpha,\beta)}{\lambda^3+\lambda^2-\lambda\beta-\alpha\beta}e^{\nu\cdot x}\Bigg|_{\lambda=0,\nu=O}$$

$$=\frac{\partial V_0(t,\alpha,\beta)}{\partial\nu_1}\Big|_{\nu=O}+\frac{\partial V_0(t,\alpha,\beta)}{\partial\nu_2}\Big|_{\nu=O}-\frac{\partial V_0(t,\alpha,\beta)}{\partial\nu_3}\Big|_{\nu=O}$$

$$+V_0(t,\alpha,0)(x_1+x_2-x_3)+2\frac{\partial V_1(t,\alpha,\beta)}{\partial\nu_2}\Big|_{\nu=O}$$

$$+V_1(t,\alpha,0)\cdot 2x_2+\frac{\partial^3}{\partial\nu_1^2\partial\nu_3}\left\{V_2(t,\alpha,\beta)e^{\nu\cdot x}\right\}\Big|_{\nu=O}$$

$$+3\frac{e^{\lambda t}-V_0(t,\alpha,0)-\lambda V_1(t,\alpha,0)-\lambda^2 V_2(t,\alpha,0)}{\lambda^3+\lambda^2}\Bigg|_{\lambda=0}$$

$$=0+0+0+1\cdot(x_1+x_2-x_3)+0+t\cdot 2x_2$$

$$+x_3\left\{(6\alpha-4)t+6-8\alpha+(8\alpha-6)e^{-t}+2t(\alpha-1)e^{-t}\right.$$

$$\left.+\frac{1}{3}\alpha t^3+(1-2\alpha)t^2+x_1^2(e^{-t}-1+t)\right\}+\frac{3}{2}\left(t^2-2(e^{-t}-1+t)\right).$$

So,

$$u(t,x)=x_1+x_2-x_3+2tx_2$$

$$+x_3\left\{(6\alpha-4)t+6-8\alpha+(8\alpha-6)e^{-t}+2t(\alpha-1)e^{-t}\right.$$

$$\left.+\frac{1}{3}\alpha t^3+(1-2\alpha)t^2+x_1^2(e^{-t}-1+t)\right\}+\frac{3}{2}\left(t^2-2t+2-2e^{-t}\right).$$

Since the initial data were polynomials, the obtained solution of the problem is also a polynomial. Only operations of differentiation were used to construct the solution.

We note that the solution of problem linearly depends on parameter α.

Example 2. Let us describe the process of an acoustic wave propagation with zero initial conditions for $\alpha=\frac{1}{9}$ under the influence of external force $f(t,x)=e^{-t}\sin\frac{x_1}{\sqrt{3}}$.

Therefore, we find the solution of problem (2), (3), for $\varphi_0(x)=\varphi_1(x)=\varphi_2(x)=0$ and $f(t,x)=e^{-t}\sin\frac{x_1}{\sqrt{3}}$. Then according to Remark 4 by formula (8), we get

$$u(t,x)=e^{-\frac{\partial}{\partial\lambda}}\sin\left[\frac{1}{\sqrt{3}}\frac{\partial}{\partial\nu_1}\right]H(\lambda,\nu,t,x)\Big|_{\lambda=0,\nu=O},$$

where $H(\lambda,\nu,t,x)=\left(\lambda-\frac{1}{3}\right)^{-3}\left(e^{\lambda t}-V_0-\lambda V_1-\lambda^2 V_2\right)e^{\nu\cdot x}$, and V_0,V_1,V_2 are functions (7).

We have

$$u(t,x)=e^{-\frac{\partial}{\partial\lambda}}\sin\left[\frac{1}{\sqrt{3}}\frac{\partial}{\partial\nu_1}\right]H(\lambda,\nu,t,x)\Big|_{\lambda=0,\nu=O}$$

$$=\sin\left[\frac{1}{\sqrt{3}}\frac{\partial}{\partial\nu_1}\right]H(\lambda,\nu,t,x)\Big|_{\lambda=-1,\nu=O}$$

$$=\frac{H\left(-1,\frac{i}{\sqrt{3}},0,0,t,x\right)-H\left(-1,-\frac{i}{\sqrt{3}},0,0,t,x\right)}{2i}$$

$$=\frac{\frac{e^{-t}-V_0+V_1-V_2}{(-2/3)^3}e^{\frac{i}{\sqrt{3}}x_1}-\frac{e^{-t}-V_0+V_1-V_2}{(-2/3)^3}e^{-\frac{i}{\sqrt{3}}x_1}}{2i}$$

$$=-\frac{27}{8}\left(e^{-t}-V_0+V_1-V_2\right)\sin\frac{x_1}{\sqrt{3}}$$

$$=-\frac{27}{8}\left(e^{-t}-\left(1-\frac{2}{3}t+\frac{2}{9}t^2\right)e^{-\frac{1}{3}t}\right)\sin\frac{x_1}{\sqrt{3}}.$$

Therefore,

$$u(t,x)=-\frac{27}{8}\left(e^{-t}-\left(1-\frac{2}{3}t+\frac{2}{9}t^2\right)e^{-\frac{1}{3}t}\right)\sin\frac{x_1}{\sqrt{3}}.$$

The obtained solution of the problem $u(t,x)$ does not depend on the coordinates x_2 and x_3. If $t\to\infty$ it goes to zero.

The function $u(t,x)$ describes periodic oscillations of an acoustic wave with a period $T=2\pi\sqrt{3}$ for coordinate x_1 (see fig. 1). The amplitude of these oscillations is determined by the formula

$$A(t)=-\frac{27}{8}\left(e^{-t}-\left(1-\frac{2}{3}t+\frac{2}{9}t^2\right)e^{-\frac{1}{3}t}\right).$$

Graphic time dependence of amplitude $A(t)$ and $u(t,x)$ for $x_1=\frac{\sqrt{3}}{6}\pi$ and $x_2=\frac{\sqrt{3}}{4}\pi$ is depicted on Fig. 2 with solid and dashed lines.

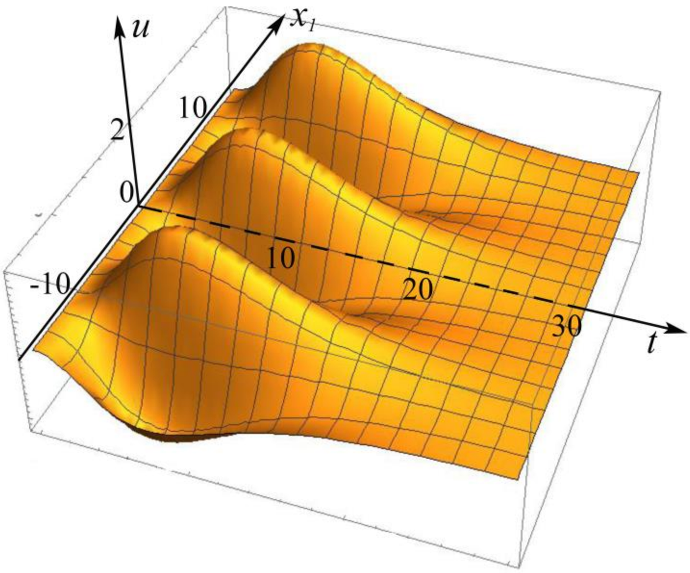

Fig. 1. The dependence of the value of the solution on the time variable and the spatial coordinate x_1

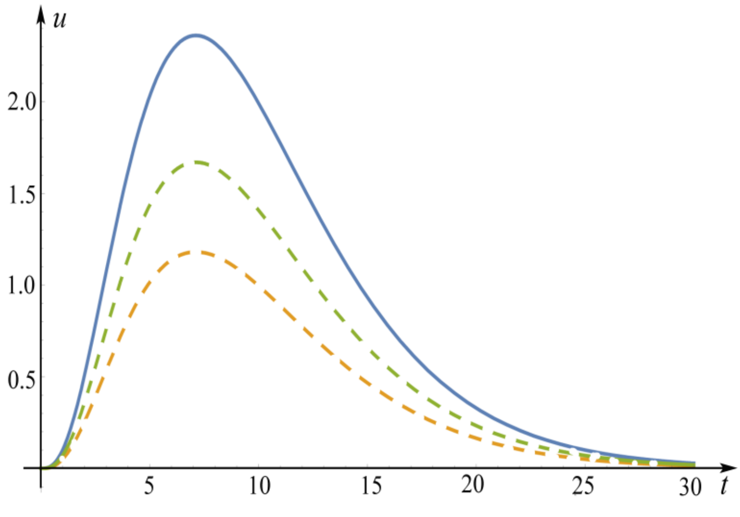

Fig. 2. Graphs of amplitude (solid line) and profiles of sound wave for $x_1 = \frac{\sqrt{3}}{6}\pi$ and $x_2 = \frac{\sqrt{3}}{4}\pi$ (dashed line)

Only values of function $H(\lambda, \nu, t, x)$ in the points $(-1, -i, 0, 0, t, x)$ and $(-1, i, 0, 0, t, x)$ were used to solve the problem. The equalities

$$e^{i\frac{\partial}{\partial\nu_1}}H(\lambda,\nu,t,x)=H(\lambda,\nu_1+i,\nu_2,\nu_3,t,x,),$$
$$e^{-i\frac{\partial}{\partial\nu_1}}H(\lambda,\nu,t,x)=H(\lambda,\nu_1-i,\nu_2,\nu_3,t,x),$$
$$e^{-\frac{\partial}{\partial\lambda}}H(\lambda,\nu,t,x)=H(\lambda-1,\nu,t,x),$$

that are correct for an arbitrary integer function $H(\lambda,\nu,t,x)$ with variables ν_1, ν_2, ν_3 and λ were used.

4 Conclusions

The class of existence and uniqueness of the Cauchy problem solution for the hyperbolic equation of third order has been established. The method of constructing the solution of Cauchy problem for arbitrary entire initial functions and an arbitrary entire right-hand side of the equation is given. In the case if the data of problem has a quasi-polynomial form, according to the proposed method, the solution of Cauchy problem can be found only with operations of differentiation. In particular, it is illustrated by Examples 1 and 2.

The results of the researches can be used in the problems of ultrasound diagnostics. The obtained results and the developed method constitute an important theoretical basis for the mathematical modeling of the acoustic wave propagation process in a relax environment.

The main conclusion of the application of the presented results in medical practice is the possibility to find exactly the determining parameters of the wave process.

References

1. Kapur, N.J.: Mathematical models in biology and medicine. New Delhi: Affiliated East-West Press (1985).
2. Begun, P.I., Afonin, P.N.: Modeling in biomechanics. Moscow: Vysshaya shkola (2004) [in Russian].
3. Horn, M.A., Simonett, G., Webb, G.F.: Mathematical models in medical and health science. Vanderbilt University Press (1998).
4. Belotserkovsky, O.M. (ed.): Computer and brain. New technologies. Nauka, Moscow (2005) [in Russian].
5. Pukach, P.Ya., Kuzio, I.V.: Resonance phenomena in quasi-zero stiffness vibration isolation systems. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu, 3, 62–67 (2015).
6. Pukach, P.Ya., Kuzio, I.V.: Nonlinear transverse vibrations of semiinfinite cable with consideration paid to resistance. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu, 3, 82–86 (2013) [in Ukrainian].
7. Pukach, P.Ya.: Investigation of bending vibrations in Voigt-Kelvin bars with regard for nonlinear resistance forces. Journal of Mathematical Sciences, 215(1), 71–78 (2016).
8. Samarskii, A.A.: The theory of difference schemes. CRC Press (2001).

9. Samarskii, A.A., Mikhailov, A.P.: Principles of mathematical modelling: ideas, methods, examples. CRC Press (2013).
10. Lavrenyuk, S.P., Pukach, P.Ya.: Mixed problem for a nonlinear hyperbolic equation in a domain unbounded with respect to space variables. Ukrainian Mathematical Journal, 59(11), 1708–1718 (2007).
11. Pukach, P.Y.: On the unboundedness of a solution of the mixed problem for a nonlinear evolution equation at a finite time. Nonlinear Oscillations, 14(3), 369–378 (2012).
12. Pukach, P.Ya.: On the problem without initial conditions for a nonlinear degenerating parabolic system. Ukrainian Mathematical Journal, 46(4), 484–487 (1994).
13. Pukach, P.Ya.: Qualitative Methods for the Investigation of a Mathematical Model of Nonlinear Vibrations of a Conveyer Belt. Journal of Mathematical Sciences, 198(1), 31–38 (2014).
14. Agapov, P.I., Vasyukov, A.V., Petrov, I.B.: Computer simulation of wave processes in the integument of the brain during traumatic brain injury. Processes and methods of information processing, M .: MFTI, 154–163 (2006) [in Russian].
15. Pukach, Petro, Il’kiv, V., Nytrebych, Z., Vovk, M., Pukach, Pavlo: On the Asymptotic Methods of the Mathematical Models of Strongly Nonlinear Physical Systems. Advances in Intelligent Systems and Computing, 689, 421–433 (2018).
16. Clarke, J.F., McChesney M.: Dynamics of Relaxing Gases, Butterworth’s, London (1976).
17. Lick, W.: Wave propagation in real gases. Adv. in Appl. Mech., 10, 1–72 (1967).
18. Morrison, J.A.: Wave propagation in rods of Voigt material and visco-elastic materials with three-parameter models. Quart. Appl. Math., 14, 153–169 (1956).
19. Rudenko, O.V., Soluyan S.I.: Theoretical Foundations of Nonlinear Acoustics. Nauka, Moscow (1975) [in Russian].
20. Pukach, P.Ya., Kuzio, I.V., Nytrebych, Z.M., Ilkiv, V.S.: Analytical methods for determining the effect of the dynamic process on the nonlinear flexural vibrations and the strength of compressed shaft. Nauk. Visnyk Natsion. Hirn. Univer., 5, 69–76 (2017).
21. Pukach, P.Ya., Kuzio, I.V., Nytrebych, Z.M., Ilkiv, V.S.: Asymptotic method for investigating resonant regimes of non-linear bending vibrations of elastic shaft. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu, 1, 68–73 (2018).
22. Renno, P.: The fundamental solution of a hyperbolic operator in three-dimensional thermochemistry. Rend. Accad. Naz. Sci. XI. Mem. Mat. 4, 43–62 (1979/1980) [in Italian].
23. Varlamov, V.: Time estimates for the Cauchy problem for a third-order hyperbolic equation. Intern. Journ. of Mathematics and Mathematical Sciences, 17, 1073–1081 (2003).
24. Nitrebich, Z.M.: An operator method of solving the Cauchy problem for a homogeneous system of partial differential equations. Journal of Math. Sciences, 81(6), 3034–3038 (1996).
25. Nytrebych, Z.M., Malanchuk, O.M., Il'kiv, V.S., Pukach, P. Ya.: Homogeneous problem with two-point conditions in time for some equations of mathematical physics. Azerb. J. of Math., 7(2), 180–196 (2017).
26. Nytrebych, Z., Il'kiv, V., Pukach, P., Malanchuk, O.: On nontrivial solutions of homogeneous Dirichlet problem for partial differential equation in a layer. Krag. J. of Math., 42(2), 193–207 (2018).
27. Nytrebych, Z.M., Malanchuk, O.M., Il'kiv, V.S., Pukach, P. Ya.: On the solvability of two-point in time problem for PDE. Italian Journ. of Pure and Appl. Math., 38, 715–726 (2017).
28. Tikhonov, A.N., Vasil'eva, A.B., Sveshnikov, A.G.: Differential Equations, Springer-Verlag (1985).