On the Mathematical Model of Nonlinear Vibrations of a Biologically Active Rod with Consideration of the Rheological Factor

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Abstract. Qualitative and numerical methods of researching nonlinear vibration systems are used to study the mathematical model of nonlinear vibrations of a biologically active rod. This model is widely used in biomechanics and medical research for designing new materials with biofactor elements that possess certain preset features. Conditions are established for the existence of a unique solution of the boundary value problem for the beam vibration nonlinear differential equation, in which there is an integral summand with the fourth derivative by the spatial variables. This summand models the rheological factor in the system. The existence of classes of nonlinear rheological vibration systems with dissipation that have blow-up regimes is stated theoretically. The relation between nonlinearity indices in such regimes is obtained. The theoretical possibility of using the Runge-Kutta method for numerical solution of the corresponding boundary value problem is shown. The results are illustrated by a model example. The importance of the obtained theoretical assumptions for the practical modeling, analysis, and synthesis of parameters of technological vibration systems is shown.

Keywords: Mathematical Model, Nonlinear Vibrations, Galerkin Method, Biofactor, Rheological System.

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1 Introduction

Mathematical modeling of both normal physiological and pathological processes is one of the current trends of modern medical research. It is especially important to note that modern medicine is largely an experimental science with a vast empirical experience of affecting different diseases with a variety of means. However, more often than not searching for experimental means of studying different process in biological media has many flaws due to our inability to limit ourselves to experiment only. Therefore, mathematical modeling is often the most effective way of studying processes in living organisms (or their parts).

In medical practice, numerical modeling of biomechanical processes is carried out on the basis of the continuous media mechanics models and numerical methods of solving corresponding systems of partial differential equations.

Mathematical modeling methods can narrow down the search of optimal system parameters significantly. After such parameter optimization, experimental research can be carried out with much more information about the functioning of a biological system. The development of the mathematical modeling framework involves building a closed mechanical-mathematical model of the process that describes the behavior of a biological medium on the basis of equations in partial derivatives and the continuous medium mechanics principles. In addition, mathematical modeling involves calculating constitutive relations between the components of stress tensors and deformation tensors. Correct mathematical formulation of the problem and the presetting of initial and boundary conditions are necessary for effective research. The development and software implementation of numerical algorithms adapted to the specifics of the problem under consideration and the visualization of the obtained numerical results are also important.

During the study of biomedical issues, we may come across processes, for whose mathematical description we use the frameworks of ordinary differential equations, mathematical physics equations, algebraic nonlinear equation systems, difference equations, the theory of bifurcations, chaos and order, etc. Examples of a successful use of such mathematical frameworks are presented in [1] for prognosing disease development, in [2-5] - for solving nonlinear dynamics problems in biology, chemical kinetics, etc. The development of numerical methods for solving problems in biomechanics also allowed solving problems in the physics of plasmas, the mechanics of deformable solids, etc. It is known that certain mathematical methods have evolved under the influence of biomedical problems, for example, the methods of mathematical statistics, Volterra equations, neural networks, methods of solving rigid differential equations, etc.

The problems of researching mathematical models of linear and nonlinear dynamic systems have become widespread in recent decades. We are talking about qualitative approaches [6-11], analytical approaches [12-18], and combinations of such approaches and approximate research methods [19].

The biological, medical, and sport problems that require research and numerical solution of partial equations have been formulated relatively recently. They are presented in [20-22]. Rheological relations for biological continuous media have been developed in [23-24]. The range of tasks considered in this area is quite wide.

The most important area in traumatology is the problem of mathematical modeling of human leg movement while walking in order to build orthopedic prostheses that imitate this movement. To model the distribution of dynamic loads and deformations at the time of movement of the entire foot, it is necessary to use the framework of partial differential equations, in particular the system of equa-tions of the mechanics of deformable solid body. Creating such models for the needs of traumatology and orthopedics is a new and relevant task for computational biolo-gy and medicine. Computer-assisted implementation of virtual surgeries and predic-tion of their consequences is another prospective area. This is a very complex area of research that is just beginning to emerge. The formulation of certain mathematical models and methods of their research are not totally clear. However, the implementation of some virtual surgeries is a real task. Thus, in [25], numerical modeling of lithotripsy surgeries (fragmentation of renal stones with acoustic waves initiated by a spark discharge or a laser pulse) is presented. The purpose of such studies is to find lithotripter operating modes (pulse duration and intensity, number of pulses), at which fragments of destroyed stone would be small enough for natural excretion. For this purpose, the picture of acoustic pulse propagation in the body and the stone was investigated numerically, and the problem of its destruction was solved.

The problems of biomechanics, as well as the tasks of controlling and regulating vibration processes in structural systems, are largely related to the problem of contact interactions with the medium, whose response to external influence depends on the prehistory or the history of load. In other words, external influence turns into the response of the coupling medium. This feature of the medium is called self-regulation. Models of self-regulatory systems in biomechanics are models of bioactive materials, or materials with biofactor. Similar models have been developed, for example, in [26-28]. A model of a self-regulatory medium, whose response to force impact is described by a hereditary-type biofactor model [27], is used in this case. The solution of the corresponding mixed problem for the fifth-order equation is built and the impact of the biofactor and material viscosity on the vibration process is investigated.

The aim of the presented studies is to develop qualitative approaches and on their basis to theoretically substantiate the possibility of creating proper computational methods for solving problems in biomechanics. These tasks arise in the process of creating new orthopedic materials, as well as the modeling, synthesis and optimization of parameters of corresponding orthopedic systems.

2 Investigation of the mathematical model of a nonlinear vibration system that generalizes the rheological vibration model with consideration of the effect of the biofactor

2.1 Problem statement. The main result

Let us denote $Q_T = (0,l)\times(0,\tau)$, $\tau \in \left(0,T\right]$, $0 < l < \infty$, $T < \infty$. In the domain Q_T, we consider the first mixed problem for the nonlinear equation with real coefficients

$$U_{tt} + \left(a_2(x)U_{xxt}\right)_{xx} + \left(b_2(x)U_{xx}\right)_{xx} + \left(b_1(x)\left|U_{xx}\right|^{q-2}U_{xx}\right)_{xx} +$$

$$+\int_0^t g(t-\theta)\left(d(x)U_{xx}(x,\theta)\right)_{xx}d\theta = c_0(x)\left|U\right|^{p-2}U + f(x,t) \qquad (1)$$

with the initial conditions

$$U(x,0) = U_0(x), \qquad (2)$$

$$U_t(x,0) = U_1(x) \qquad (3)$$

and the boundary conditions

$$U(0,t) = U_{xx}(0,t) = 0, \qquad U(l,t) = U_{xx}(l,t) = 0. \qquad (4)$$

The mixed problem for the fifth-order nonlinear evolution equation considered here describes the vibrations of an elastic bioactive rod with consideration of the "memory" effect. The aim of this article is to conduct a qualitative study of the solution to the problem (1) - (4) in a limited range and obtain sufficient conditions for the existence of a generalized solution of the mixed problem in Sobolev spaces for the fifth-order differential equation (1), in which there is an integral summand with the fourth derivative according to the spatial variable that models the effect of "memory" in the vibration system. The obtained results will make it possible to apply adequate computational methods and computer simulation to the above problem for the optimal synthesis of the parameters of a vibration system whose mathematical model is the problem (1)-(4). Let us assume the following conditions are true:

(1) functions $a_2(x)$, $\left(a_2(x)\right)_{xx}$ are bounded on $(0,l)$; $a_2(x) \ge A_2$, $\left(a_2(x)\right)_{xx} \ge A_2$, $A_2 > 0$;

(2) functions $b_2(x)$, $\left(b_2(x)\right)_{xx}$ are bounded on $(0,l)$; $b_2(x) \ge B_2$, $\left(b_2(x)\right)_{xx} \ge B_2$, $B_2 > 0$;

(3) functions $b_1(x)$, $\left(b_1(x)\right)_x$ are bounded on $(0,l)$; $b_1(x) \ge b_0 > 0$;

(4) function $c_0(x)$ is bounded on $(0,l)$;

(5) $g(t) \ge 0$, $g'(t) \le 0$ for all $t \in \left[0,+\infty\right)$, $0 \le \int_0^{+\infty} g(t)dt = G < +\infty$;

(6) function $d(x)$ is bounded on $(0,l)$, $d(x) \ge d_2 \ge 0$;

(7) $p > 2$, $q > 2$;

(8) functions $f(x,t)$, $f_t(x,t)$ are integrable with square according to Lebesgue in the domain Q_{τ_0} for any $\tau_0 > 0$;

(9) the initial deviation has the following features: $U_0(x)$ is a function integrable with power $2p-2$ on $(0,l)$, the second derivative $U_0(x)$ is a function integrable with power q on $(0,l)$, the fourth derivative $U_0(x)$ is a function integrable with square on $(0,l)$, $|U_{xx}|^{q-3}(U_{xxx})^2$, $|(U_0)_{xx}|^{q-3}((U_0)_{xxx})^2$ are the functions integrable with square on $(0,l)$, while $U_0(x)$ satisfies the conditions (4);
(10) the initial deviation has the following features: the second and the fourth derivatives of $U_1(x)$ are functions integrable with square on $(0,l)$, while $U_1(x)$ satisfies the conditions (4).

The function $U:(0,l)\times[0,T)\to\mathbb{R}$ (T is a positive number or $+\infty$) is called the generalized solution to the problem (1)-(4) in the domain Q_T if it satisfies the initial conditions (2) and the integral equality

$$\int_0^l \Big[U_{tt}V + a_2(x)U_{xxt}V_{xx} + b_2(x)U_{xx}V_{xx} + b_1(x)|U_{xx}|^{q-2}U_{xx}V_{xx} +$$

$$+\int_0^t g(t-\theta)d(x)U_{xx}(x,\theta)V_{xx}(x)d\theta - c_0(x)|U|^{p-2}UV - f(x,t)V\Big]dxdt = 0 \qquad (5)$$

for almost all $t\in(0,T)$ and for all testing functions V, for which the equality (5) is correct.

The solution $U(x,t)$ has the following features:

- the functions U, U_t are continuous on $[0,T_0]$ according to the time variable, the second derivative U_{tt} is bounded on $[0,T_0]$ according to the time variable for an arbitrary number T_0 from the interval $(0,T)$;

- the function U is integrable according to the spatial variable with power q on $(0,l)$; the function U_t is integrable with square according to the spatial variable on $(0,l)$; the function U_{tt} is integrable with square together with the second derivative according to the spatial variable on $(0,l)$.

The main result. Let the conditions **(1), (2), (3), (4), (5), (6), (7), (8), (9), (10)** be satisfied. Then the finite time T, which depends on the coefficients, the right-hand side of the equation, and the initial data, can be specified, at which a generalized solution U of the problem (1) - (4) exists in the domain Q_T.

2.2 Galerkin method

Because the space $\hat{V}(0,l)=W^{2,r}(0,l)\cap H^4(0,l)\cap L^{2p-2}(0,l)$ with $r=\max\{q,2q-4\}$ is a separable Banach, there is a countable set in it $\{\omega^k\}_{k\in\mathbb{N}}$, where any finite number of elements is linearly independent and the closure of its linear shell in $\hat{V}(0,l)$ coincides with $\hat{V}(0,l)$. Let us note that $\{\omega^k\}_{k\in\mathbb{N}}$ can be selected orthonormal in the space $L^2(0,l)$. Let's consider the functions $U^N(x,t)=\sum_{k=1}^{N}c_k^N(t)\omega^k(x)$, $N=1,2,\dots$, where c_1^N, c_2^N, …, c_N^N are solutions of the corresponding Cauchy problems

$$\int_0^l\Big[U_{tt}^N\omega^k+a_2(x)U_{xxt}^N\omega_{xx}^k+b_2(x)U_{xx}^N\omega_{xx}^k+b_1(x)\left|U_{xx}^N\right|^{q-2}U_{xx}^N\omega_{xx}^k+$$

$$+\int_0^t g(t-\theta)d(x)U_{xx}^N(x,\theta)\omega_{xx}^k d\theta-c_0(x)\left|U^N\right|^{p-2}U^N\omega^k-f(x,t)\omega^k\Big]dxdt=0\,, \qquad (6)$$

$$c_k^N(0)=U_{0,k}^N\,,\quad \left(c_k^N\right)_t(0)=U_{1,k}^N\,, \qquad (7)$$

where $$U_0^N(x)=\sum_{k=1}^{N}U_{0,k}^N(x)\omega^k\,,\quad U_1^N(x)=\sum_{k=1}^{N}U_{1,k}^N(x)\omega^k\,,$$

$$\left\|U_0^N-U_0\right\|_{\hat{V}(0,l)}\to 0\,,\quad \left\|U_1^N-U_1\right\|_{H_0^2(0,l)\cap H^4(0,l)}\to 0\,,$$

$N\to\infty$. On the basis of the Karatheodori theorem [29] there exists an absolutely continuous solution to the problem (6), (7), determined in a certain interval $[0,t_0)$. From the evaluations obtained below, it follows that $t_0=T$, while number T will be determined later.

Let us multiply (6) by $\left(c_k^N\right)_t$, sum it up by k from 1 to N and integrate it by t from 0 to $\tau\le T$. We will obtain

$$\frac{1}{2}\int_0^l\left(U_t^N(x,\tau)\right)^2dx+\int_{Q_\tau}\Big[a_2(x)\left(U_{xxt}^N\right)^2+b_2(x)U_{xx}^NU_{xxt}^N+b_1(x)\left|U_{xx}^N\right|^{q-2}U_{xx}^NU_{xxt}^N+$$

$$+\int_0^t g(t-\theta)d(x)U_{xx}^N(x,\theta)U_{xxt}^N d\theta-c_0(x)\left|U^N\right|^{p-2}U^NU_t^N-f(x,t)U_t^N\Big]dxdt=\frac{1}{2}\int_0^l\left(U_1^N\right)^2dx\,. \quad (8)$$

Let us evaluate the summands of the equality (8). Based on condition **(1)**

$$I_1=\int_{Q_\tau}a_2(x)\left(U_{xxt}^N\right)^2dxdt\ge A_2\int_{Q_\tau}\left(U_{xxt}^N\right)^2dxdt\,.$$

According to condition **(2)**

$$I_2=\int_{Q_\tau}b_2(x)U_{xx}^NU_{xxt}^Ndxdt\ge\frac{B_2}{2}\int_0^l\left(U_{xx}^N(x,\tau)\right)^2dx-\frac{B^2}{2}\int_0^l\left(\left(U_0^N\right)_{xx}(x,\tau)\right)^2dx,\quad B^2=\sup_{x\in(0,l)}\left|b_2(x)\right|^2\,.$$

Using condition **(3)**, we will obtain

$$I_3 = \int_Q b_1(x)\left|U_{xx}^N\right|^{q-2} U_{xx}^N U_{xxt}^N \ge \frac{b_0 C_1}{q}\int_0^l \left(U_{xx}^N(x,\tau)\right)^2 dx - \frac{C_2}{q}\int_0^l \left(\left(U_0^N(x,\tau)\right)_{xx}\right)^q dx\,,$$

at that $C_1 > 0$, the positive constant C_2 depends on $b^0 = \sup\limits_{x\in(0,l)}\left|b_1(x)\right|$.

Based on conditions **(5)**, **(6)**,

$$I_4 = \int_{Q_\tau}\int_0^t g(t-\theta)d(x)U_{xx}^N(x,\theta)U_{xxt}^N d\theta dxdt \le \frac{C_3 G\delta_1}{2}\int_{Q_\tau}\left(U_{xxt}^N\right)^2 dxdt + \frac{C_4}{2\delta_1}\int_{Q_\tau}\left(U_{xx}^N\right)^2 dxdt\,,$$

where $\delta_1 > 0$ is an arbitrary constant, while positive constants C_3, C_4 depend on $d^0 = \sup\limits_{x\in(0,l)}\left|d(x)\right|$, T. According to condition **(4)**,

$$I_5 = \int_{Q_\tau} c_0(x)\left|U^N\right|^{p-2} U^N U_t^N dxdt \le C^0 C_5\int_{Q_\tau}\left|U^N\right|^p dxdt + C_6\int_{Q_\tau}\left|U_t^N\right|^p dxdt =$$

$$= C^0 C_5\int_{Q_\tau}\left|U^N(x,0) + \int_0^t U_t^N(x,\tau)\right|^p dxdt + C_6\int_{Q_\tau}\left|U_t^N\right|^p dxdt \le C_7\int_0^l\left|U_0^N\right|^p dx +$$

$$+ C_8\int_{Q_\tau}\left|U_t^N\right|^p dxdt \le C_7\int_0^l\left|U_0^N\right|^p dx + C_9\int_0^\tau\left|\int_0^l\left(U_t^N(x,\tau)\right)^2 dx\right|^{\frac{p}{2}} dt,\ C^0 = \sup_{x\in(0,l)}\left|c_0(x)\right|,$$

positive constants $C_5 - C_9$ are independent from N.

Using condition **(8)**, one can get

$$I_6 = \int_{Q_\tau} f(x,t)U^N dxdt \le \frac{1}{2}\left[\int_{Q_\tau} f^2(x,t) + \left(U^N\right)^2\right]dxdt\,.$$

Taking into account the evaluation of integrals $I_1 - I_6$, after proper choice of a sufficiently small constant δ_1 the next inequality is true:

$$\frac{1}{2}\int_0^l\left[\left(U_t^N(x,\tau)\right)^2 + \left|U_{xx}^N(x,\tau)\right|^q + \left(U_{xx}^N(x,\tau)\right)^2\right]dx + \int_{Q_\tau}\left(U_{xxt}^N(x,\tau)\right)^2 dxdt \le$$

$$\le C_{10}\int_{Q_\tau}\left[\left(U_t^N\right)^2 + \left(U_{xx}^N\right)^2\right]dxdt + C_{11}\int_0^l\left[\left|U_0^N\right|^p + \left(U_1^N\right)^2 + \left(U_0^N\right)_{xx} + \left|\left(U_0^N\right)_{xx}\right|^q\right]dx +$$

$$+ C_{12}\int_{Q_\tau}\left(f(x,t)\right)^2 dxdt + C_{13}\int_0^\tau\left|\int_0^l\left(U_t^N(x,\tau)\right)^2 dx\right|^{\frac{p}{2}} dt\,,\ \tau\in\left(0,T\right), \tag{9}$$

where $C_{10} - C_{13}$ are positive constants independent on N. Using the Grönwall-Bellman inequality, from (9) we obtain

$$\frac{1}{2}\int_0^l\left[\left(U_t^N(x,\tau)\right)^2 + \left|U_{xx}^N(x,\tau)\right|^q + \left(U_{xx}^N(x,\tau)\right)^2\right]dx + \int_{Q_\tau}\left(U_{xxt}^N(x,\tau)\right)^2 dxdt \le$$

$$\le M_1 + M_2\int_0^\tau\left|\int_0^l\left(U_t^N(x,\tau)\right)^2 dx\right|^{\frac{p}{2}} dt\,,\ \tau\in\left(0,T\right), \tag{10}$$

while positive constants M_1, M_2 depend on the coefficients, the right-hand side of the equation, and the initial data and are independent of N.

The Bihari lemma can be applied to inequality (10) [30, p. 110].

$$\frac{1}{2}\int_0^l\left[\left(U_t^N(x,\tau)\right)^2+\left|U_{xx}^N(x,\tau)\right|^q+\left(U_{xx}^N(x,\tau)\right)^2\right]dx+\int_{Q_\tau}\left(U_{xxt}^N(x,\tau)\right)^2dxdt\le$$
$$\le\frac{2M_1}{\left[2-(p-2)M_1^{(p-2)/2}M_2T\right]^{2/(p-2)}} \tag{11}$$

at $T<\dfrac{2}{(p-2)M_1^{(p-2)/2}M_2}$. Therefore, from (11) it follows

$$\begin{gathered}\left\|U^N\right\|_{L^\infty\left((0,T_1);W_0^{2,q}(0,l)\right)}\le M_3\\ \left\|U_t^N\right\|_{L^2\left((0,T_1);H_0^2(0,l)\right)\cap L^\infty\left((0,T_1);L^2(0,l)\right)}\le M_3\end{gathered}, \tag{12}$$

where positive constant M_3 is independent on N, $T_1\in(0,T)$.

Let us further differentiate (6) according to variable t, multiply the obtained equality by $\left(c_k^N\right)_{tt}$, sum up all the equations according to k from 1 to N and integrate the result according to the variable t from 0 to τ, $\tau\in(0,T_1]$. Let us evaluate the summands of the obtained equality using conditions **(1)-(10)** just as the previous evaluations were obtained. Based on the above evaluations, on can get

$$\int_0^l\left[\left(U_{tt}^N(x,\tau)\right)^2+\left|U_{xxt}^N(x,\tau)\right|^2\right]dx+\int_{Q_\tau}\left(U_{xxtt}^N(x,\tau)\right)^2dxdt\le\frac{2M_4}{\left[2-(2q-2)M_4^{q-1}M_5T\right]^{1/(q-1)}} \tag{13}$$

at $T<\dfrac{1}{(q-1)M_4^{q-1}M_5}$. From inequality (13) we conclude that

$$\begin{gathered}\left\|U_t^N\right\|_{L^\infty\left((0,T_2);H_0^2(0,l)\right)}\le M_6\\ \left\|U_{tt}^N\right\|_{L^2\left((0,T_2);H_0^2(0,l)\right)\cap L^\infty\left((0,T_2);L^2(0,l)\right)}\le M_6\end{gathered},$$

where the positive constant M_6 is independent on N, $T_2\in(0,T)$. Let $T=\min\left\{\dfrac{2}{(p-2)M_1^{(p-2)/2}M_2},\dfrac{1}{(q-1)M_4^{q-1}M_5}\right\}$. After performing additional a priori evaluations and conclusions, for the arbitrary $T_0\in(0,T)$ one can obtain

$$\int_{Q_{T_0}}\left[(x)U_{tt}U+a_2(x)U_{xx}U_{xxt}+b_2(x)U_{xx}U_{xx}+\int_0^t g(t-\theta)d(x)U_{xx}(x,\theta)d\theta U_{xx}U_{xx}(x,t)-\right.$$
$$\left.-c_0(x)|U|^p-f(x,t)U\right]dxdt+\int_{Q_{T_0}}b_1(x)|U_{xx}|^q\,dxdt=0. \tag{14}$$

Given the arbitrariness of T_0, it follows from (14) that U satisfies equation (1) in terms of distributions. Taking into account the smoothness of the obtained function, we conclude: U is a generalized solution of the problem (1)-(4) in Q_T .

3 Model example. Results of numerical integration

The following equation can serve as the simplest model example (1)

$$U_{tt} + aU_{xxxxt} + bU_{xxxx} + a_0U_t + b_0U = c_0 |U|^{p-2} U + f(x,t), \ p > 2. \qquad (15)$$

In equation (15), the function $U(x,t)$ is transverse movement of beam cross-section with the coordinate x at any given time t; $a > 0$, $b > 0$, $b_0 > 0$ are constants that are expressed through geometric and physical-mechanical parameters of the beam, constant $a_0 > 0$ characterizes the effect of resistance forces in the vibration system (linear case), constant c_0 describes nonlinearly elastic forces affecting the system, $f(x,t)$ is external driving force. Boundary conditions (4) correspond to the model of the beam with fixed pivot bearings at the ends $x = 0$ and $x = l$. In case of the mixed problem (15), (2)-(4) can be obtained using the above considerations, the value of the critical time T_0, at which the vibration system functions in a regime without blow-up at $t < T_0$, and goes into the blow-up regime at $t \ge T_0$. It is easy to show that value T_0 satisfies the condition

$$T_0 < \frac{2}{(p-2)\left(\tilde{M}\right)^{(p-2)/2} \hat{M}},$$

while M^*, M^{**} are some generalized parameters of the vibration system which depend on the constant of equation (15) and the initial data.

Fig. 1 shows the dependence of the critical value of T_0 on generalized parameters of the vibration system M^*, M^{**} at nonlinearity index $p = 3$ which characterizes nonlinearly elastic features of the environment.

The qualitative results obtained in the previous section make it possible to investigate with the help of numerical methods the dynamic regimes of vibrations for equation (15) in case of the problem with initial deviation $U_0(x) = \begin{cases} 2x/l, & 0 \le x \le l/2 \\ 2 - 2x/l, & l/2 \le x \le l \end{cases}$ and zero initial velocity of deflection of the pivot points and zero boundary conditions. The problem set describes natural transverse vibrations of the rod, which at the initial moment of time is loaded by concentrated force at the point with coordinate $x = l/2$. The above problem is a problem of the same form as (15), (2)-(4). As shown above, there is a single generalized solution to this problem. Therefore, for numerical integration of motion equations, the choice of method is important only from the computational point of view. Numerical solution of the problem is carried out using the fourth-order Runge-Kutta method. Figure 2 presents the law of time deviation of the rod midpoint, depending on the correlation between the frequencies of natural and forced vibrations under the following conditions: $l=1$, $a=-0{,}001$, $b=-1$, $a_0=b_0=0$, $c_0=100$, $p=5$, $f(x,t)=300\sin 9{,}48\pi t$.

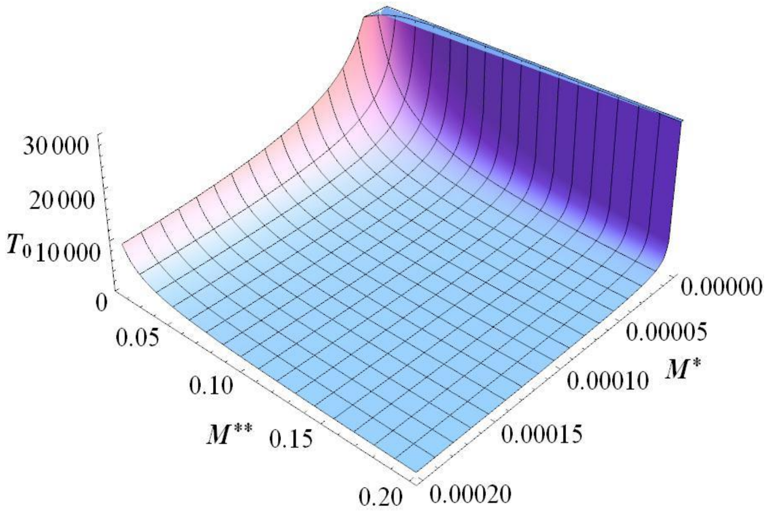

Fig. 1. Dependence of value T_0 on the generalized parameters of the vibration system at $p=3$.

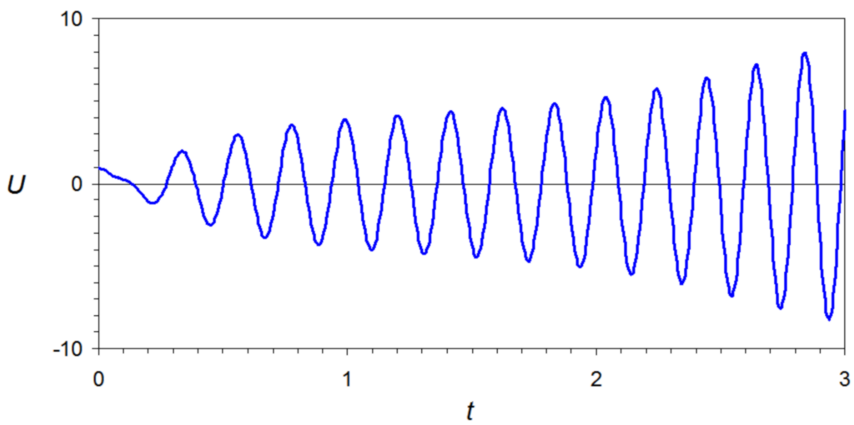

Fig. 2. The law of rod midpoint deviation time change (resonant regime)

Figure 3 shows the same law provided $f(x,t)=300\sin 9{,}48\pi t+2$.

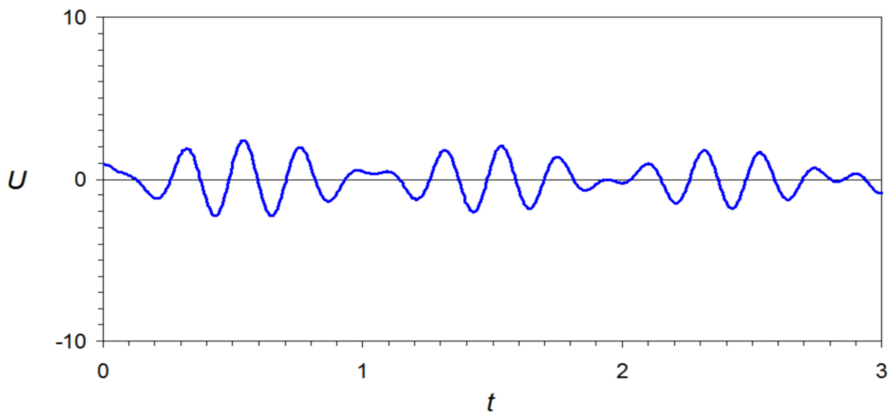

Fig. 3. The law of rod midpoint deviation time change (non-resonant regime)

4 Conclusions

The mathematical model of nonlinear vibrations of a bioactive rod was investigated using combined qualitative and numerical approaches with consideration of the self-regulation phenomenon. This mathematical model is used in biomechanical studies of new materials and to synthesize vibration system parameters. This, in turn, is an important issue in current medical research. The mathematical model of a vibration system is presented as a mixed problem for a fifth-order equation with memory. Subcritical and critical system operation regimes were evaluated. Analytical correlations that characterize the moment of process transition to the blow-up regime were established.

The qualitative and numerical results are the next:

- physical and mechanical parameters of a vibration system determine the critical value of the time parameter, up to which the system is in the blow-up-free regime;
- the attenuation rate does not depend much on the degree of nonlinearity of the resistance force, while the effect of the resistance force on the vibration period at small values of a, p is minor;
- depending on the correlation of frequencies of natural and induced vibrations in the system, there will be a time increase of vibration amplitude (resonance) or vibration beating.

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