Theoretical aspects of applying continuous VaR-criterion in option markets

G.A. Agasandyan

Dorodnicyn Computing Centre, FRC CSC RAS, Vavilova 40, 119333, Moscow, Russia
agasand17@yandex.ru

Abstract The problem of optimal behavior of an investor in high-developed option markets is studied. On a basis of a specific investor's risk-preferences function, the continuous VaR-criterion (CC-VaR) as a continuous generalization of well-known common VaR-criterion is introduced. This criterion suggests the investor to construct a portfolio that ensures maximizing the average income and satisfying continuous set of restrictions on income from bottom. In fact, the investor specifies in advance the distribution function and finds the portfolio, which generates random income that satisfies all these restriction. Newman-Pearson procedure as the essence of methods that construct optimal on CC-VaR portfolios is given and their properties in relation to CC-VaR problems are studied. The theoretical δ-market and the theoretical option market as its equivalent are being investigated. As usual, the investor has own forecast on probability properties of future behavior of option underlier. The optimal portfolio in theoretical one-period option market with the given prices picture is constructed. The method is illustrated by an example with two-sided exponential probability distributions.

Keywords: continuous VaR-criterion (CC-VAR), Newman-Pearson procedure, optimal portfolio.

1 Introduction

Any instrument (portfolio) that an investor acquires in the option market generates some random income. The type of randomness forecasts the investor himself. The result of investment is estimated thru the average yield (or income) and risk – the first has to be maximized, and the second is measured usually by the variation or so-called VaR-criterion. In markets with restricted choice of instruments this can suffice. However in markets with broad toolkits, one can construct portfolios from common calls and puts that generate random income as the function of arbitrary future underlier's price. In this case usual measures of risk can be non-adequate [1-3]).

So, the application of VaR-criterion in the problem of maximizing the average income produces an income that is no more than critical income level of criterion (usually less than investment amount) with the probability near the one (at the dense lattice of strikes). This result hardly suits the rational investor. Also, using the variation, as in the theory of Markowitz [1], leads to significant impoverishment of the space of

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the optimum choice. The theory is founded on probability properties of the second order and hence don't admit to allow some nuance of distributions as, e.g., their heavy tails, which needs allowing for probability properties if only of fourth order such as kurtosis.

So we arrive at an idea of using *continuous VaR-criterion* (CC-VaR), when the priority for the investor is to solve market problems with fulfilling *continuous set* of restrictions, which provides the distribution function of incomes given by the investor in advance. Formally, we introduce the CC-VaR as follows.

The risk interests of an investor are described by monotone increasing and non-negative *risk-preferences function* (r.p.f.) $\phi(\varepsilon)$, $\varepsilon\in[0, 1]$. As a rule, we assume that $\phi(1) < \infty$, and even $\phi(1) = 1$. As a typical example, we can consider the function $\phi(\varepsilon) = \varepsilon^{\lambda}$, $\varepsilon\in[0, 1]$, $\lambda > 0$. The more the value of λ, the more the investor is ready to risk for increasing the average yield [4-7]. The investor must fulfill a handsome system of inequalities. CC-VaR denotes a set of requirements $\mathsf{P}\{q \geq \phi(\varepsilon)\} \geq 1 - \varepsilon$ *jointly* for *all* $\varepsilon\in[0, 1]$, which have to be fulfilled for the random income q generated by the investor's portfolio; here $\mathsf{P}\{M\}$ is the probability of the set M from the point of view of the investor. We adopt that the income is non-negative because of mechanism of margin calls that operates in option markets.

Questions about forming the forecast of the investor remain beyond our consideration. The forecast probability distribution of the future underlier's price as well as investor's risk preferences are subjects of the responsibility of the investor and are set in the model from outside. And optimization algorithm has to work with arbitrary forecast and risk-preferences functions.

It is easy to conceive that a portfolio of, for instance, butterflies, which generates the income rather well approximating arbitrary (non-negative) function of future underlier's price, can be constructed in this market. It is sure that in this case butterflies with mutually rather nearby arranged strikes can be needed.

These considerations lead us to the construction of theoretical market where so called δ-instruments that are marginal instruments for the set of all butterflies form a basis, from which we can receive arbitrary portfolios. The offered construction can be conceived also as marginal for generalized roulette whose positions correspond to future prices of underlier, and probabilities of stopping form arbitrary distributions.

2 Continuous δ-market and instruments

For simplicity and elucidating peculiarities of the offered criterion, *ideal theoretical* one-period option market generated by some underlier $\boldsymbol{X}$, e.g. a stock, is considered. We call the market *theoretical* because strikes of options can be arbitrary real numbers, and *ideal* because all bid-offer spreads and all commissions are equal to zero, and then all instruments can be traded in *any* quantities (even though fractional). At the beginning of the period, the underlier's price is known and equal, say, to μ_0, but this value is for our purposes inessential. At the end of period, this price is the variate X, that takes on values x from a continuous set $\mathsf{X}\subset\Re_+$ (or even $\Re$), $\Re$ is the set of all real numbers, $\Re_+$ is that of all positive numbers. The value x is the income from $\boldsymbol{X}$ at the end of period, too. Investor elaborates own forecast for the future price of underlier in the form of *forecast* probability density $p(x)$, $x\in\mathsf{X}$.

In the market, the arbitrary instrument generating income $g(x)$, $x\in\mathsf{X}$, which is a non-negative measurable function, can be traded. We call the function $g(x)$ the *payoff* of instrument and write $g(x) = \pi(x; \boldsymbol{G})$; in particular, $\pi(x; \boldsymbol{X}) = x$. Prices of instruments are formed by the market towards the beginning of period. We denote $|\boldsymbol{G}|$ the price of $\boldsymbol{G}$, and $\|\boldsymbol{G}\|$ its average income.

We consider so-called δ-instruments $\boldsymbol{D}(s)$, $s\in\mathsf{X}$, as options of special kind, and s is its *strike*. The income of the instrument $\boldsymbol{D}(s)$ is singular and defined as δ-function $\delta(x - s)$, s, $x\in\mathsf{X}$, which is equal to zero if $x \neq s$ and infinity if $x = s$, with integral within X in x equal to the one. The price and the average income of $\boldsymbol{D}(s)$ are equal, respectively, to $c(s)$ and $p(s)$, $s\in\mathsf{X}$.

In terms of δ-instruments, we can describe practically any continuous-in-s portfolios:

$$\boldsymbol{G} = \int_{\mathsf{X}} g(s)\boldsymbol{D}(s)ds\big(= g(\boldsymbol{X})\big). \quad (1)$$

Therefore we can consider a collection of δ-instruments $\{\boldsymbol{D}(s),\ s\in\mathsf{X}\}$ as a basis of the market, and we call it δ-market. These instruments can be considered as marginal for such instruments of real markets as simplest common butterflies properly normalized. The price and the average income of δ-instruments $\boldsymbol{D}(s)$ are, respectively,

$$|\boldsymbol{D}(s)| = c(s), \quad \|\boldsymbol{D}(s)\| = p(s), \quad s\in\mathsf{X}.$$

Non-negative function $c(x)$, $x\in\mathsf{X}$, has to be produced by the market. Also,

$$|\boldsymbol{G}| = \int_{\mathsf{X}} g(s)|\boldsymbol{D}(s)|ds = \int_{\mathsf{X}} g(s)c(s)ds, \quad \|\boldsymbol{G}\| = \int_{\mathsf{X}} g(s)\|\boldsymbol{D}(s)\|ds = \int_{\mathsf{X}} g(s)p(s)ds.$$

In theoretical markets, such instruments as *indicators of sets* play significant role. The payoff of the indicator $\boldsymbol{H}\{M\}$ of the set $M\subset\mathsf{X}$ is its characteristic function: $\pi(x; \boldsymbol{H}\{M\}) \equiv \chi_M(x) \equiv \{1,\ x\in M;\ 0,\ x\notin M\}$. There is among indicators also the *riskless* asset of unit size $\boldsymbol{U} = \boldsymbol{H}\{\mathsf{X}\}$. We have the representations

$$\boldsymbol{H}\{M\} = \int_M \boldsymbol{D}(s)ds, \quad \boldsymbol{U} = \boldsymbol{H}\{\mathsf{X}\} = \int_{\mathsf{X}} \boldsymbol{D}(s)ds,$$

$$|\boldsymbol{U}| = \int_{\mathsf{X}} c(x)dx = 1/r, \quad \|\boldsymbol{U}\| = \int_{\mathsf{X}} p(x)dx = 1.$$

The parameter r denotes the riskless return relative for period, and $r - 1$ is then the riskless yield. Without loss of generality we can set $r \equiv 1$ and consider the price function $c(x)$, $x\in\mathsf{X}$, as probability density. We call it the *cost* density. Usual calls $\boldsymbol{C}_s$ and puts $\boldsymbol{P}_s$ of strikes $s\in\mathsf{X}$ are given, respectively, by relations

$$\boldsymbol{C}_s = \int_{\{x\geq s\}} (x - s)\boldsymbol{D}(x)dx, \quad \boldsymbol{P}_s = \int_{\{x<s\}} (s - x)\boldsymbol{D}(x)dx, \quad x, s \in \mathsf{X}.$$

By analogy, representations of other known option instruments such as spreads, butterflies, condors et cetera can be given.

The methods of constructing optimal portfolios expounded further have to consider divergences between densities $c(\cdot)$ and $p(\cdot)$ (or measures $\mathsf{C}\{\cdot\}$ and $\mathsf{P}\{\cdot\}$). The special significance is therefore attached to the function of return relative $\rho(x) = p(x)/c(x)$, $x\in\mathsf{X}$. Functions of such type are considered in the theory of mathematical statistics; they are called likelihood functions and are at the heart of the so-called *Newman-Pearson procedure* [8].

In the market under consideration, we can state investment problems differently. Some most important settings are as follows:

Problem CO (original). Given investment amount and r.p.f. of the investor $\phi(\varepsilon)$, the portfolio that achieves max $\mathsf{E}q$ (E is the symbol of the expectation) under condition $\mathsf{P}\{q \geq \phi(\varepsilon)\} \geq 1 - \varepsilon$ for *all* $\varepsilon \in [0, 1]$ is searched out.

Problem CB (basic). Investment amount is not given, and $\phi(\varepsilon)$ is investor's r.p.f.. The regular portfolio that achieves min A under condition $\mathsf{P}\{q \geq \phi(\varepsilon)\} \geq 1 - \varepsilon$ for *all* $\varepsilon \in [0, 1]$ is searched out.

Problem CH (homogenious). Given investment amount S (> 0) and r.p.f. in the form $b\phi(\varepsilon)$, where b is a scale parameter, the regular portfolio and the value of parameter b that achieve max Eq under condition $\mathsf{P}\{q \geq \phi(\varepsilon)\} \geq 1 - \varepsilon$ for *all* $\varepsilon \in [0, 1]$ is searched out. Parameter b is defined from equality $S = bA$.

Emphasize that for original *problem CO*, singular solutions are inherent. The *problem CB*, on the contrary, straight requires constructing the regular portfolio, but with minimum investment amount, and hence is a base for solving many other problems. The *problem CH* is an important example of applying *problem CB* to solving *problem CO* in case of r.p.f. that is homogeneous in scale of investment.

3 Fundamental theoretical results

As we have said above, the problem *CB* is of special interest. And all subsequent constructions just aim at solving this problem. If we already know its solution then the solution, particularly, of the *problem CO* is derived if we add to the investment amount received for the problem *CB* all the rest from given original amount investment, which investor directs to acquiring $\boldsymbol{D}(s^*)$ where s^* is an element on X that achieves the maximum of $\rho(s)$. And we receive a singular solution. But this is just the reason why we have refused using VaR-criterion in favor of CC-VaR.

So we suggest that reasonable approach is in adequate setting the problem. For example, the *problem CH* is may be adequate just in fully homogeneous case. And problem *CO* can be transformed into adequate one if investor choices the r.p.f. in a form $\phi(b; \varepsilon)$, where b is a scale parameter, and find its value to nullify the singular component of portfolio.

At the heart of solving problem *CB* is the *Newman-Pearson procedure*, which is broadly used in mathematical statistics [8].

Newman-Pearson procedure. The set family $\mathbf{Z} = \{Z(\tau), \tau \geq 0\}$ is constructed by the rule: $x \in Z(\tau)$ if and only if $\rho(x) \leq \tau$, where $\rho(x)$ is defined above and $\tau \geq 0$. The function $\mathrm{f}_{\mathsf{P}}(\tau) = \mathsf{P}\{Z(\tau)\}$, $\tau \in [\tau', \tau'']$, $\tau' = \min_x \rho(x)$, $\tau'' = \max_x \rho(x)$, is introduced. The family $\mathbf{Z}$ non-decreases in τ, hence $\mathrm{f}_{\mathsf{P}}(\tau)$ is a non-decreasing function. Moreover, $0 \leq \mathrm{f}_{\mathsf{P}}(\tau) \leq 1$ and $\mathrm{f}_{\mathsf{P}}(\infty) = 1$. For each $\varepsilon \in [0, 1]$, the set $X_\varepsilon \in \mathrm{Z}$ is defined from condition $\mathsf{P}\{X_\varepsilon\} = \varepsilon$. The function introduced just ascertains a connection between τ and ε: $\varepsilon = \mathrm{f}_{\mathsf{P}}(\tau)$ (also $\tau = \mathrm{f}_{\mathsf{P}}^{\leftarrow}(\varepsilon)$). Additionally, the function $\mathrm{f}_{\mathsf{C}}(\tau) = \mathsf{C}\{Z(\tau)\}$, $\tau \in [\tau', \tau'']$, is introduced. Functions $\mathrm{f}_{\mathsf{P}}(\tau)$, $\mathrm{f}_{\mathsf{C}}(\tau)$, $\tau \in [\tau', \tau'']$, $\gamma(\varepsilon)$, $\varepsilon \in [0, 1]$, are called *forecast function*, *cost function*, *dissonant*, respectively. For regular pair $(\mathsf{C}\{\cdot\}, \mathsf{P}\{\cdot\})$, the set X_ε exists for each

$\varepsilon\in[0, 1]$ and is unique, and the price $\gamma(\varepsilon) = \mathsf{C}\{X_\varepsilon\}$ is maximized among all $\mathsf{C}\{Y\}$, $\mathsf{P}\{Y\} = \varepsilon$, $Y\subset$X.

The next assertions can also be proved:

• The functions $f_{\mathsf{P}}(\tau)$ and $f_{\mathsf{C}}(\tau)$ are distribution functions of ρ(X) for measures $\mathsf{P}\{\cdot\}$ and $\mathsf{C}\{\cdot\}$, respectively, where X is the random future price of the underlier.

• The dissonant $\gamma(\varepsilon)$, $\varepsilon\in[0, 1]$, is the concave function, and is equal to the superposition of functions $f_{\mathsf{C}}(\tau)$ and $f_{\mathsf{P}}^{\leftarrow}(\varepsilon)$.

• The ordering function $w(x)$ defined by equivalence rule

$$w(x)=\varepsilon\left(=\mathsf{f}_{\mathsf{P}}\left(\rho(x)\right)\right) \quad\Leftrightarrow\quad x\in\Gamma_\varepsilon=\lim_{\varepsilon'\uparrow\varepsilon}\left(X_\varepsilon-X_{\varepsilon'}\right),\ \varepsilon\in[0, 1],\ x\in\mathsf{X}.$$

• The variate w(X) is uniform distributed over the interval [0, 1], and does not depend on $\phi(\varepsilon)$.

• The distribution function of the random income $q = g(\mathrm{X}) = \phi(w(\mathrm{X}))$, is equal to the function $\phi^{\leftarrow}(z)$, $z\in[\phi(0), \phi(1))$, and the identity $\mathsf{P}\{q \geq \phi(\varepsilon)\} \equiv 1 - \varepsilon$ holds for all $\varepsilon\in[0, 1]$.

We submit the method connected with the discretization of the problem and the passage to the limit continuous scheme. In discrete case, the discretization parameter n and levels ε_i, γ_i, ϕ_i, $i\in I = \{1, 2, \ldots, n\}$, of continuous parameters ε, γ, ϕ are given. Applying the Newman-Pearson procedure is readily seen to give the next representations of the optimal portfolio at $b = 1$ and its characteristic:

$$\boldsymbol{G}^{(n)}=\sum_{i=1}^{n}\left(\phi_i-\phi_{i-1}\right)\boldsymbol{H}\left\{\mathsf{X}-X_{\varepsilon_i}\right\},$$

$$A^{(n)}=\sum_{i=1}^{n}\left(\phi_i-\phi_{i-1}\right)\left(1-\gamma_i\right)=\sum_{i=1}^{n}\phi_i\left(\gamma_i-\gamma_{i-1}\right),\ \phi_0=0,\gamma_0=0,$$

$$R^{(n)}=\sum_{i=1}^{n}\left(\phi_i-\phi_{i-1}\right)\left(1-\varepsilon_i\right)=\sum_{i=1}^{n}\phi_i\left(\varepsilon_i-\varepsilon_{i-1}\right),\ \phi_0=0,\varepsilon_0=0,$$

$$\mathsf{P}\left\{q\geq\phi_i\right\}\geq 1-\varepsilon_i \quad \text{for } all\ i\in I.$$

By replacing differences in these formulae by differentials in ε, γ and ϕ, and sums by integrals, we obtain in continuous case

$$A=\int_0^1\left(1-\gamma(\varepsilon)\right)d\phi(\varepsilon)=\int_0^1\phi(\varepsilon)\,d\gamma(\varepsilon),\quad R=\int_0^1(1-\varepsilon)\,d\phi(\varepsilon)=\int_0^1\phi(\varepsilon)\,d\varepsilon.$$

The next assertion can be proved:

Theorem 1. The family of sets $\mathbf{Z} = \{X_\varepsilon, \varepsilon\in[0, 1]\}$ constructed by the Newman-Pearson procedure achieves the minimum of the investment amount A.

Theorem 2. The average income R and the investment amount A for the optimal theoretical portfolio at $b = 1$ are defined by formulae

$$R=\int_0^1\phi(\varepsilon)\,d\varepsilon,\quad A=\int_0^1\phi(\varepsilon)\,d\gamma(\varepsilon)=\int_0^1\phi(\varepsilon)\gamma'(\varepsilon)\,d\varepsilon.$$

4 The illustrative examples

The second Laplace distribution (also called two-sided exponential distribution), which we denote Exp(μ, α), has the density

$$\tfrac{1}{2\alpha}\exp\left(-|x-\mu|/\alpha\right),\quad x\in\Re,$$

with expectation $\mathsf{E}\mathrm{X} = \mu$, variation $\mathsf{D}\mathrm{X} = 2\alpha^2$ and kurtosis equal to 6 (excess = 3). Let's set $p(x) \sim \mathrm{Exp}(\mu, \alpha)$, $c(x) \sim \mathrm{Exp}(\mu, \beta)$ и $\beta \neq \alpha$. We may set $\mu = 0$. So

$$p(x)=\tfrac{1}{2\alpha}\exp\left(-\tfrac{|x|}{\alpha}\right),\quad c(x)=\tfrac{1}{2\alpha}\exp\left(-\tfrac{|x|}{\beta}\right),\quad \rho(x)=\tfrac{\beta}{\alpha}\exp\left(\tfrac{\alpha-\beta}{\alpha\beta}|x|\right),\quad x\in\Re .$$

Let's set $\kappa = \alpha/\beta$. If $\kappa < 1$ the function $\rho(x)$ increases at $x < 0$, decreases at $x > 0$, and at $x = 0$ is maximized; the investor «sells volatility». If $\kappa > 1$ the $\rho(x)$ decreases at $x < 0$, increases at $x > 0$, and at $x = 0$ is minimized; the investor «buys volatility». We set $\phi(\varepsilon) = \varepsilon^{\lambda}$, $\varepsilon\in[0, 1]$, $\lambda > 0$.

Volatility selling. At $\kappa < 1$ the investor supposes the market to be less volatile than option prices speak about it. Algorithm generates the next symmetric contractions

$$Z(\tau)=\left\{x\middle||x|\ge\rho^{\leftarrow}(\tau)=\tfrac{\alpha\beta}{\beta-\alpha}\ln\left(\tfrac{\beta}{\alpha\tau}\right)\right\},\quad X_\varepsilon=\left\{x\middle||x|\ge x_\varepsilon\right\},$$

$$-x'_\varepsilon=x''_\varepsilon=x_\varepsilon=\rho^{\leftarrow}(\tau)\ge 0,\ \tau=\rho(x'_\varepsilon)=\rho(x''_\varepsilon)=\rho(x_\varepsilon),$$

$$\mathsf{P}\{X_\varepsilon\}=2\int_{x_\varepsilon}^{\infty}p(x)\,dx=\alpha^{-1}\int_{x_\varepsilon}^{\infty}e^{-x/\alpha}dx=\exp(-x_\varepsilon/\alpha)=\varepsilon ,$$

$x_\varepsilon=-\alpha\ln\varepsilon$ is $(1-\varepsilon/2)$-quantile of distribution Exp(0, α),

$\Gamma_\varepsilon=\{x'_\varepsilon,x''_\varepsilon\}=\{-x_\varepsilon,x_\varepsilon\}$ is the set of all (here, two) limit points of X_ε.

The ordering function with two symmetric branches, the weight function and the dissonant with its derivative have views

$$u_{1,2}(\varepsilon)=\{x'_\varepsilon,x''_\varepsilon\}=\pm x_\varepsilon=\mp\alpha\ln\varepsilon ,$$

$$w(x)=\left\{u_1^{\leftarrow}(x),\,x<0;\ u_2^{\leftarrow}(x),\,x>0\right\}=\exp\left(-|x|/\alpha\right),$$

$$g(x)=\phi(w(x))=e^{-\theta|x|},\quad g'(x)=\mp\theta e^{-\theta|x|},\quad g''(x)=\theta^2e^{-\theta|x|},$$

$$\gamma(\varepsilon)=\mathsf{C}\{X_\varepsilon\}=\beta^{-1}\int_{-\alpha\ln\varepsilon}^{\infty}e^{-x/\beta}dx=\varepsilon^{\kappa},\quad \gamma'(\varepsilon)=\kappa\varepsilon^{\kappa-1}.$$

The main characteristics of investment

$$A=\int_0^1\phi(\varepsilon)\gamma'(\varepsilon)\,d\varepsilon==\kappa(\kappa+\lambda)^{-1},\quad R=\int_0^1\phi(\varepsilon)\,d\varepsilon=(1+\lambda)^{-1},\quad y=\tfrac{\lambda(1-\kappa)}{\kappa(1+\lambda)}>0 .$$

The optimal portfolios both in put-spreads and call-spreads and both in puts and calls themselves are given by mixed representations, respectively,

$$\boldsymbol{G}=\left(\boldsymbol{U}-\theta\int_{-\infty}^{0}e^{\theta x}d\boldsymbol{P}(x)+\theta\int_0^{\infty}e^{-\theta x}d\boldsymbol{C}(x)\right),$$

$$\boldsymbol{G}=\boldsymbol{U}-\theta\left(\boldsymbol{P}(0)+\boldsymbol{C}(0)\right)+\theta^2\int_{-\infty}^{0}e^{\theta x}\boldsymbol{P}(x)\,dx+\theta^2\int_0^{\infty}e^{-\theta x}\boldsymbol{C}(x)\,dx .$$

The payoffs of these portfolios are defined uniquely by parameter θ. Hence the less α and the greater λ the more acute is the peak of the function $g(x)$ at $x = 0$ and the greater is the yield y. We can consider this portfolio as some continuous expansion of the long butterfly, which realizes in real markets the same volatility selling.

Volatility buying. At $\kappa > 1$ the investor supposes the market to be more volatile than option prices speak about it. The analogical computations in this case give sequentially

$$Z(\tau)=\left\{x\middle||x|\le\rho^{\leftarrow}(\tau)=\tfrac{\alpha\beta}{\beta-\alpha}\ln\left(\tfrac{\beta}{\alpha\tau}\right)\right\},\quad X_\varepsilon=\left\{x\middle||x|\le x_\varepsilon\right\},$$

$$-x'_\varepsilon=x''_\varepsilon=x_\varepsilon=\rho^{\leftarrow}(\tau),\quad \tau=\rho(x'_\varepsilon)=\rho(x''_\varepsilon)=\rho(x_\varepsilon),$$

$$\mathsf{P}\{X_\varepsilon\} = 2\int_0^{x_\varepsilon} p(x)\,dx = \alpha^{-1}\int_0^{x_\varepsilon} e^{-x/\alpha}dx = 1-\exp(-x_\varepsilon/\alpha) = \varepsilon\,,$$

$x_\varepsilon = -\alpha\ln(1-\varepsilon)$ – $(1+\varepsilon)/2$-quantile of distribution Exp(0, α),

$\Gamma_\varepsilon = \{x'_\varepsilon, x''_\varepsilon\} = \{-x_\varepsilon, x_\varepsilon\}$ is the set of all limit points of X_ε.

The ordering function with two symmetric branches, the weight function and the dissonant with its derivative have views

$$u_{1,2}(\varepsilon) = \{x'_\varepsilon, x''_\varepsilon\} = \pm x_\varepsilon = \mp\alpha\ln(1-\varepsilon)\,,$$

$$w(x) = \{u_1^{\leftarrow}(x), x<0;\ u_2^{\leftarrow}(x), x>0\} = 1-\exp(-|x|/\alpha)\,,$$

$$g(x) = \phi(w(x)) = \left(1-\exp(-|x|/\alpha)\right)^{\lambda},$$

$$g'(x) = \pm\tfrac{\lambda}{\alpha}e^{-|x|/\alpha}\left(1-e^{-|x|/\alpha}\right)^{\lambda-1}, \quad g'(\pm 0) = 0, \pm\alpha^{-1}, \pm\infty \ \text{ at } \ \lambda>1, =1, <1,$$

$$g''(x) = \tfrac{\lambda^2}{\alpha^2}e^{-|x|/\alpha}\left(1-e^{-|x|/\alpha}\right)^{\lambda-2}\left(e^{-|x|/\alpha}-\tfrac{1}{\lambda}\right),$$

$$\gamma(\varepsilon) = \mathsf{C}\{X_\varepsilon\} = \beta^{-1}\int_0^{-\alpha\ln(1-\varepsilon)} e^{-x/\beta}dx = 1-(1-\varepsilon)^{\kappa}, \quad \gamma'(\varepsilon) = \kappa(1-\varepsilon)^{\kappa-1}.$$

The main characteristics of investment

$$A = \int_0^1 \phi(\varepsilon)\gamma'(\varepsilon)\,d\varepsilon = \tfrac{\Gamma(1+\alpha/\beta)\Gamma(1+\lambda)}{\Gamma(1+\alpha/\beta+\lambda)}\,, \quad R = \int_0^1 \phi(\varepsilon)\,d\varepsilon = (1+\lambda)^{-1},$$

$$y = R/A - 1 = \tfrac{\Gamma(1+\alpha/\beta+\lambda)}{\Gamma(1+\alpha/\beta)\Gamma(2+\lambda)} - 1 > 0\,, \quad \Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}dx\,.$$

The optimal portfolio both in put-spreads and call-spreads is given by mixed representation

$$\boldsymbol{G} = \int_{-\infty}^0 |g'(x)|\,d\boldsymbol{P}(x) - \int_0^\infty g'(x)\,d\boldsymbol{C}(x)\,.$$

The character of the optimal portfolio in puts and calls themselves depends on the parameter λ, and corresponds to the behavior of the function $g(x)$ near zero. We have at $\lambda > 1$, $\lambda = 1$, $\lambda < 1$, respectively,

$$\boldsymbol{G} = \int_{-\infty}^0 g''(x)\boldsymbol{P}(x)\,dx + \int_0^\infty g''(x)\boldsymbol{C}(x)\,dx\,,$$

$$\boldsymbol{G} = \tfrac{1}{\alpha}\left(\boldsymbol{P}(0)+\boldsymbol{C}(0)\right) - \tfrac{1}{\alpha^2}\int_{-\infty}^0 e^{-|x|/\alpha}\boldsymbol{P}(x)\,dx - \tfrac{1}{\alpha^2}\int_0^\infty e^{-|x|/\alpha}\boldsymbol{C}(x)\,dx\,,$$

$$\boldsymbol{G} = \int_{-\infty}^0 g''(x)\left(\boldsymbol{P}(x)-\boldsymbol{P}(0)\right)dx + \int_0^\infty g''(x)\left(\boldsymbol{C}(x)-\boldsymbol{C}(0)\right)dx\,.$$

These portfolios are some continuous expansions of the short butterflies or condors in dependency on the value of λ, which realize in real markets the same volatility buying. At $\lambda < 1$ the function $g(x)$ has at $x = 0$ the singularity of the type "spike", and the less λ, the more the instrument looks like the riskless asset. At $\lambda = 1$ the function $g(x)$ has at $x = 0$ a fracture and reminds the payoff of the short butterfly. At $\lambda > 1$ the function $g(x)$ at $x = 0$ is smooth with $g'(0) = 0$, and reminds the payoff of the condor; here qualitative analogy with the real markets can be readily traced, too.

5 Conclusion

The consideration of illustrative examples demonstrates the rationality of the optimization problem setting with the continuous VaR-criterion and efficiency of theoretical algorithm. However, very few problems can be solved by such analytical way. Hence the method of the solution has to be expanded on discrete models, which could be considered as approximation to the continuous ones. All the more so that it is better to interpret real markets as discrete. Indeed, they are such in many aspects. So, to more adequate formalize the optimization problems, we ought to introduce the so called scenario markets (see, e.g., [6, 9]).

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