Approximation of parametrically given polyhedral sets

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Abstract
In our paper we consider systems of linear inequalities which linearly depend on multidimensional parameters. A technique for approximating set of parameters for which the considered system is consistent is described. Approximations are constructed by means of convex and concave piece-wise linear functions. An illustrative example is given.

1 Introduction
Parametric polyhedral set $D(p)$ is defined in the following way

$$D(p) = \{ x \in X : g_i(x, p) \leq 0, \ i = 1, \ldots, m \}, \quad (1)$$

where $X \subset \mathbb{R}^n$ is a polytope ($X \neq \emptyset$), $g_i : \mathbb{R}^n \times \mathbb{R}^q, \ i = 1, \ldots, m$ are bilinear functions, i.e. each function is linear in variable $x$ when variable $p$ is fixed and vice versa. Vector $p$ is called a parameter and may vary within a given polytope $P \subset \mathbb{R}^q, \ p \in P$. Define set

$$P^* = \{ p \in P : D(p) \neq \emptyset \}.$$

In general, $P^*$ is a nonconvex, disconnected and implicitly given set. We consider the following problem: find an outer $P_{out}^*$ and an inner $P_{in}^*$ explicit approximations of $P^*$:

$$P_{in}^* \subset P^* \subset P_{out}^*.$$

Similar problem were considered in [BorrelliEtAl03] and [JonesEtAl08] under some additional conditions which allow one to effectively apply the well elaborated linear programming parametric technique. Interval linear optimization [FiedlerEtAl06] is also tightly connected to the topic of our paper. However, here we suggest to use approximations generated by support function-majorants and function minorants [Khamisov99].

2 Approximation technique
Consider the following function

$$w(p) = \min_{x \in X} \max_{1 \leq i \leq m} g_i(x, p).$$

Then set $P^*$ has the equivalent description

$$P^* = \{ p \in P : w(p) \leq 0 \}.$$
Let $\tilde{p} \in P$ be given, find
\[ \tilde{x} \in \text{Argmin} \max_{x \in X} \max_{1 \leq i \leq m} g_i(x, \tilde{p}) \] (2)
and define function
\[ \varphi(p, \tilde{p}) = \max_{1 \leq i \leq m} g_i(\tilde{x}, p). \] (3)
Since functions $g_i$ are liner in $p$ function $\varphi$ is a convex piece-wise linear function which satisfies two conditions
\[ w(\tilde{p}) = \varphi(\tilde{p}, \tilde{p}),\; w(p) \leq \varphi(p, \tilde{p}) \; \forall p \in P. \] (4)
Due to these conditions function $\varphi$ is called a support function-majorant of function $w$. Rewrite now function $w$ in the following way
\[ w(p) = \min_{x \in X} \max_{1 \leq i \leq m} g_i(x, p) = \min_{x \in X} \max_{u \in S_u} \sum_{i=1}^m u_i g_i(x, p) = \max_{u \in S_u} \min_{x \in X} \sum_{i=1}^m u_i g_i(x, p), \] (5)
where $S_u$ is the standard simplex, $S_u = \left\{ u \in \mathbb{R}^m : \sum_{i=1}^m u_i = 1, u_i \geq 0, \; i = 1, \ldots, m \right\}$. Let again $\tilde{p} \in P$ be given. Solve the corresponding max-min problem in (5) and find $\tilde{x}$ and $\tilde{u}$ such that
\[ w(\tilde{p}) = \sum_{i=1}^m \tilde{u}_i g_i(\tilde{x}, \tilde{p}). \] (6)
Define function
\[ \psi(p, \tilde{p}) = \min_{x \in X} \sum_{i=1}^m \tilde{u}_i g_i(x, p). \] (7)
Then, from (5) and (6) we have
\[ w(\tilde{p}) = \psi(\tilde{p}, \tilde{p}),\; w(p) \geq \psi(p, \tilde{p}) \; \forall p \in P. \] (8)
By construction function $\psi(\cdot, \tilde{p})$ is concave and due to the properties (8) is called a support function-minorant of function $w$.
Assume, that $w(\tilde{p}) \leq 0$ and define set
\[ P_{in}(\tilde{p}) = \{ p \in P : \varphi(p, \tilde{p}) \leq 0 \}. \] (9)
From (4) we have $w(p) \leq 0 \; \forall p \in P_{in}(\tilde{p})$. By construction $P_{in}(\tilde{p})$ is a convex polyhedral set. Therefore, $P_{in}(\tilde{p})$ is a convex polyhedral inner approximation of $P_{in}^*$. In the case of $w(\tilde{p}) > 0$ we define set
\[ P_0(\tilde{p}) = \{ p \in P : \psi(p, \tilde{p}) > 0 \}. \] (10)
From (8) we have $w(p) > 0 \; \forall p \in P_0(\tilde{p})$. It follows from the concavity of $\psi(\cdot, \tilde{p})$ that $P_0(\tilde{p})$ is a convex set. Then the set $P_{out}(\tilde{p}) = P \setminus P_0(\tilde{p})$ is an outer approximation of $P_{out}^*$.
It follows from (3) and (9) that set $P_{in}(\tilde{p})$ has explicit description as the polyhedron
\[ P_{in}(\tilde{p}) = \{ p \in P : g_i(\tilde{x}, p) \leq 0, \; i = 1, \ldots, m \}. \] (11)
The set $P_{out}(\tilde{p})$ is the complement of convex set $P_0(\tilde{p})$. Therefore, $P_{out}(\tilde{p})$ is nonconvex, can be disconnected and has a disjunctive structure [Balas18]. Assume, that vertices $v_1, \ldots, v_N$ of $X$ are known, $X = \text{conv}\{v_1, \ldots, v_N\}$. Then, from (7) we have
\[ \psi(p, \tilde{p}) = \min_{1 \leq j \leq N} \sum_{i=1}^m \tilde{u}_i g_i(v^j, p) \] (12)
and
\[ P_0(\tilde{p}) = \{ p \in \mathbb{R}^m : \sum_{i=1}^m \tilde{u}_i g_i(v^j, p) > 0, \; j = 1, \ldots, N \}. \] (13)
Define sets
\[ P_j(\bar{p}) = \{ p \in P : \sum_{i=1}^{m} \bar{u}_i g_i(v^j, p) \leq 0 \}, \quad j = 1, \ldots, N. \]

Then
\[ P_{\text{out}}(\bar{p}) = \bigcup_{j=1}^{N} P_j(\bar{p}), \tag{14} \]
i.e. \( P_{\text{out}}(\bar{p}) \) is a union of polyhedrons. Find scalars \( \bar{\gamma}_j \):
\[ \sum_{i=1}^{m} \bar{u}_i g_i(v^j, p) \leq \bar{\gamma}_j \quad \forall p \in P, \quad j = 1, \ldots, N. \]

It is well-known, that by introducing 0-1 variables \( z_j, \ j = 1, \ldots, N \) the disjunctive structure of \( P_{\text{out}}(\bar{p}) \) in (14) can be described as follows
\[ P_{\text{out}}(\bar{p}) = \left\{ p \in P : \exists z : \sum_{i=1}^{m} \bar{u}_i g_i(v^j, p) \leq \bar{\gamma}_j z_j, \quad z_j = 0 \bigvee 1, \quad j = 1, \ldots, N, \quad \sum_{j=1}^{N} z_j = (N - 1) \right\}. \]

We always can assume that \( X \) is a simplex. Since \( X \) is bounded it has an outer approximation by a simplex and since \( X \) is defined by a system of linear inequalities we can move all inequalities into the list of new functions \( g_i \) in (1). In this case new functions do not have parameter \( p \), however the suggested approach is still correct.

Let us consider how \( \bar{x} \) and \( \bar{u} \) corresponding to a given \( \bar{p} \) can be obtained. Rewrite problem in (2) in the following way
\[ \min_{x, \xi} \{ \xi : g_i(x, \bar{p}) \leq \xi, \quad i = 1, \ldots, m, \quad x \in X \}, \tag{15} \]
where \( \xi \) is an auxiliary unbounded scalar variable. Problem (15) is a linear programming problem which always has a finite solution. Let \( (\bar{x}, \bar{\xi}) \) be a solution. Then, obviously, \( \bar{x} \) satisfies inclusion (2) and \( w(\bar{p}) = \xi \). Write down the Lagrange function
\[ L(\xi, x, u) = \xi + \sum_{i=1}^{m} u_i (g_i(x, \bar{p}) - \xi). \]

The Lagrange dual problem is
\[ \max_{u \geq 0, x \in X, \xi \in \mathbb{R}} L(\xi, x, u) = \max_{u \geq 0, x \in X, \xi \in \mathbb{R}} \min_{u_i \geq 0} \left\{ \xi \left( 1 - \sum_{i=1}^{m} u_i \right) + \sum_{i=1}^{m} u_i g_i(x, \bar{p}) \right\} = \max_{u \in S_u, x \in X} \min_{i=1}^{m} u_i g_i(x, \bar{p}), \]
i.e. \( \bar{u} \) introduced in (6) is a dual solution of (15). Note also, that (15) is a linear programming problem and hence can be easily solved for any given \( \bar{p} \in P \).

Let us make some intermediate conclusions. For any arbitrary given \( \bar{p} \in P \) we solve linear programming problem (15) obtaining primal solution \( (\bar{x}, \bar{\xi}) \) and dual solution \( \bar{u} \). If \( \xi \leq 0 \) then we know that system (1) is consistent not only for \( p = \bar{p} \) but also \( \forall p \in P_\text{in}(\bar{p}) \), where \( P_\text{in}(\bar{p}) \) is given in (11). If \( \xi > 0 \) then system (1) is inconsistent not only for \( p = \bar{p} \) but also \( \forall p \in P_\text{in}(\bar{p}) \) in (10). The main result here consists in the following: checking a given parameter for feasibility we obtain a set with the same property.

**Example.** Sets \( P = [-0.2, 1.3], X = \{(x_1, x_2) : -5 \leq x_j \leq 5, j = 1, 2\} \). System (1) is defined by the following inequalities
\[ g_1(x_1, x_2, p) = 5p x_1 + 10x_2 + 2p - 10 \leq 0, \]
\[ g_2(x_1, x_2, p) = -2x_1 - 3px_2 + 10p + 10.5 \leq 0. \]

Set \( P^* \) is union of three intervals, \( P^* = [-0.2, -0.05] \cup [0.075, 0.94] \cup [1.235, 1.3] \). Function \( w \) and set \( P^* \) are shown in Fig. 1. In this example we check three values of parameter for feasibility and construct the corresponding sets.

First parameter \( p^1 = 0.01 \). Solving problem (15) with \( \bar{p} = p^1 \) we obtain the corresponding primal solution \( x^1 = (5, 1.015), \xi^1 = 0.4196, \) and dual solution \( u^1 = (0.003, 0.997) \). Since \( w(p^1) = \xi^1 > 0 \) we have \( D(p^1) = 0 \). Set
X has four vertices $v^1 = (-5, -5)$, $v^2 = (-5, 5)$, $v^3 = (5, -5)$, $v^4 = (5, 5)$. Support function-minorant of $w$ has the following form (see (12))
\[
\psi(p,p^1) = \min\{9.901p + 20.2585, -20.009p + 20.5585, 10.051p + 0.3185, -19.859p + 0.6185\} = \\
= \min\{10.051p + 0.3185, -19.859p + 0.6185\} \forall p \in [-0.2, 1.3].
\]
Set $P_\emptyset(p^1) = \{p : \psi(p,p^1) > 0\}$ is open interval \((-0.0317, 0.0311)\). Therefore, $D(p) = \emptyset \forall p \in (-0.0317, 0.0311)$. Hence $P_{out}(p^1) = P\backslash P_\emptyset(p^1) = [-0.2, -0.0317] \cup [0.0311, 1.3]$. See Fig 1 for geometrical interpretation of function $\psi(., p^1)$ and set $P_\emptyset(p^1)$.

Take now the second value of the parameter, $p^2 = 0.6$. The corresponding problem (15) ($\bar{p} = p^2$) has solutions $x^2 = (5, -0.737)$, $\xi^2 = -1.1729$, $u^2 = (0.153, 0.847)$. Since $\xi^2 < 0$ set $D(p^2) \neq \emptyset$. From (3) we obtain
\[
\varphi(p, p^2) = \max\{27p - 17.37, -2.789p + 0.5\},
\]
$P_{in}(p^2) = \{p : \varphi(p, p^2) \leq 0\} = [0.179, 0.643]$ and $D(p) \neq \emptyset \forall p \in [0.179, 0.643]$.

Third parameter $p^3 = 1.1$. The corresponding primal and dual solutions $x^3 = (5, -1.857)$, $\xi^3 = 1.1286$, $u^3 = (0.248, 0.752)$. In this case $\xi^3 > 0$ and $D(p^3) = \emptyset$. Support function-minorant
\[
\psi(p,p^3) = \min\{14.216p - 14.504, -8.344p + 10.296, -20.744p + 25.336\}.
\]
Set $P_\emptyset(p^3) = \{p : \psi(p,p^3) > 0\}$ is interval $(1.02, 1.22)$.

![Figure 1: Geometrical interpretation of the Example.](image)

### 3 An approximation procedure

Parameter $p \in P$ such that $D(p) \neq \emptyset$ is called feasible parameter and infeasible otherwise. Below we describe a procedure which is based on generating finite number of random parameters in $P$ and checking feasibility property of them. Since every parameter generates a set containing either only other feasible parameters or only infeasible parameters such procedure is a covering type procedure. In general, we finish with a collection of sets with feasible parameters and collection of sets with infeasible parameters.

It is assumed that vertices $v^1, \ldots, v^N$ of set $X$ are known. We also fix maximum number of iterations $\bar{k} > 1$ in advance. The procedure has the following description.
Step 0. Set $\mathcal{P}_\text{in} = \emptyset$, $\mathcal{P} = \emptyset$, $k = 1$, $k_\text{in} = 0$, $k_\emptyset = 0$;

Step 1. Choose randomly parameter $p^k \in P$;

Step 2. If $p^k \in \bigcup_{P_i \in \mathcal{P}_\text{in}} P_i$ then goto step 7;

Step 3. If $p^k \in \bigcup_{P_i \in \mathcal{P}} P_i$ then goto step 7;

Step 4. Solve problem (15) with $\tilde{p} = p^k$. Let $(x^k, \xi^k)$ and $\tilde{u}^k$ be primal and dual solutions.

Step 5. If $\xi^k > 0$ then define $P^k_\emptyset = P_\emptyset(\tilde{p})$ in (13) for $\tilde{p} = p^k$ and $\tilde{u} = u^k$, and set $\mathcal{P}_\emptyset = \mathcal{P}_\emptyset \cup P^k_\emptyset$, $k_\emptyset = k_\emptyset + 1$;

Step 6. If $\xi^k \leq 0$ then define $P^k_\text{in} = P_\text{in}(\tilde{p})$ in (11) for $\tilde{p} = p^k$ and $\tilde{x} = x^k$, and set $\mathcal{P}_\text{in} = \mathcal{P}_\text{in} \cup P^k_\text{in}$, $k_\text{in} = k_\text{in} + 1$;

Step 7. Set $k = k + 1$;

Step 8. If $k > k_\emptyset$ then stop, otherwise goto step 1.

When the procedure stops we have a collection $\mathcal{P}_\text{in} = \{P^1_\text{in}, \ldots, P^{k_\text{in}}_\text{in}\}$ of feasible parameters sets and a collection $\mathcal{P}_\emptyset = \{P^1_\emptyset, \ldots, P^{k_\emptyset}_\emptyset\}$ of infeasible parameters sets. Therefore, we can define $P^*_\text{in}$ and $P^*_\text{out}$ in the following way

$$P^*_\text{in} = \bigcup_{i=1}^{k_\text{in}} P^*_i, \quad P^*_\text{out} = P \setminus \left( \bigcup_{i=1}^{k_\emptyset} P^*_i \right).$$

In the example considered above we have $k = 3$, $k_\text{in} = 1$, $k_\emptyset = 2$ and

$$\mathcal{P}_\text{in} = \{[0.179, 0.643]\}, \quad \mathcal{P}_\emptyset = \{(-0.0317, 0.0311), (1.02, 1.22)\}.$$

Therefore,

$$P^*_\text{in} = [0.179, 0.643], \quad P^*_\text{out} = [-0.2, -0.0317] \bigcup [0.0311, 1.02] \bigcup [1.22, 1.3].$$

Theoretically the suggested procedure is finite since $P$ is a compact set, hence can be covered by a finite number of convex sets. However, in order to increase the efficiency other covering techniques, e.g. borrowed from global optimization technology [EvtushenkoEtAl17] can be used.

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**References**


