Temporal Properties over Contextualized Description Logics

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Abstract. A context-sensitive system is often employed in a dynamic environment due to its adaptivity. To represent temporal properties over an evolving system, we study an extension of Contextualised Description Logics (ConDLs) with linear temporal logic (LTL) operators, where ConDL axioms are used in place of propositional variables. The resulting language is interpreted over an infinite sequence of nested DL-interpretations. With $\mathcal{EL}$, $\mathcal{ALC}$, and $\mathcal{SHOQ}$ in consideration, we show that the formalism is rather well-behaved in the sense that the satisfiability problem in most of the instances have the same complexity with underlying ConDLs. This holds even in the presence of rigidity constraints on the object level.

1 Introduction

Contextual knowledge can represent many aspects of a system, either for describing internal structures or adjusting behaviors with accordance to the outside world. For instance, consider *role-based* paradigm where the contexts are defined by the roles that are currently played by an object. In this setting, an object can adapts its behaviour dynamically instead of getting a fixed behaviour. To represent and reason over context-sensitive knowledge, *description logics* (DLs) of context have been introduced. The notion of context can take many forms in extension of DLs, e.g., as attributes [13,15]. Frequently, they are of the form two-layered DLs, one for the contextual knowledge and another for the domain knowledge [6,11,12].

On the other hand, a role-based system is often employed in a dynamic environment due to its capability to adapt. Objects can adjust their behaviours quickly by taking different roles to accommodate a different context. Temporal logics are often employed to dynamic properties of an evolving system. These properties can be positive, that should be satisfied by the system, or negative, that should not happen in any run of the system. Consider an example of a role-based programming language that span a non-deterministic transition system. This is a standard setting in classical formal verification system where the states are defined over a set of propositional variables that are interpreted true. In our role-based setting, the state will be adjusted with a proper semantics to describe a context-sensitive world. A programmer might want to verify whether a critical role is played by at least one object in any time point.

In this work, we present a family of logic-based language to represent temporal properties over context-sensitive system. We consider the family of *contextualized DLs* (ConDLs) to describe context-sensitive knowledge [6]. While previous approaches are quite expressive due to possibilities of describing direct relation of object domains between two contexts, this leads to undecidability in the admission of rigid names. ConDLs take a different approach by restricting contextual view in top-down manner with a meta-level concept constructor to collect all contexts that satisfy an object-level axiom. The practicability of the language has been shown by utilizing ConDLs to reason over a role-based modeling language [5,14]. For the temporal part, we utilize *linear temporal logic* (LTL) which is interpreted over infinite sequence of states. A similar approach is studied in temporalizing DLs where temporal operators are allowed in front of DL axioms [3,9]. With a similar argument in the complexity perspective, we allow temporal operators only in front of ConDL axioms. The obtained language is a family of three-dimensional description logics where temporal operators are used to express properties over evolving contextual knowledge represented by ConDL axioms.

An example of formula that can be expressed by this logic is

$$(\textsc{Private} \sqsubseteq [\![\mathit{HasMoney}(\mathit{Bob})]\!]) \,\mathbf{U}\, (\textsc{Work} \sqsubseteq \neg[\![\mathit{worksFor}(\mathit{Bob}, \mathit{CompanyX})]\!]).$$

The axiom says that Bob has money in a private context until he is not working for Company X anymore in a work context. However it is not possible to say that in general if *someone* has money in private context, until *that someone* is not working for some companies. We investigate the satisfiability problem in this family of logic. Furthermore, it is very common in multi-dimensional DLs to desire the ability of expressing rigidity. For example, the role human is rigid since if something is a human, then it holds in any context or time. We say that a concept or role is rigid if their interpretation are the same across dimensions of the interpretation. Such constraints often make the reasoning problem harder. In this work, we consider the setting where rigidity constraints occur only on the object domain level and holds across all contexts and time points.

2 Basic Notions

2.1 The Contexualized Description Logics $\mathcal{L}_M[\![\mathcal{L}_O]\!]$

For representing context-dependent knowledge, we consider ConDLs, a family of DLs of context studied in [6]. First, we recall the syntax of considered DLs and assume the reader familiar with their standard semantics. For a thorough introduction to DLs (specifically $\mathcal{ALC}$, $\mathcal{EL}$ and $\mathcal{SHOQ}$), we refer to [1,2,4,10].

Definition 1 (DLs Syntax). *Let* $\mathsf{N} = (\mathsf{N_C}, \mathsf{N_R}, \mathsf{N_I})$ *be a* signature *of disjoint sets of* concept names*,* role names *and* individual names*, respectively. Let* $A \in \mathsf{N_C}$*,* $r \in \mathsf{N_R}$*,* $a \in \mathsf{N_I}$*, and* $n \geq 0$*. A* concept C *is built according to the following syntax rule*

$$C, D ::= A \mid \neg C \mid C \sqcap D \mid \exists r.C \mid \{a\} \mid \leq_n r.C \mid \top$$

Let C*,* D *be concepts,* $r \in \mathsf{N_R}$ *and* $a, b \in \mathsf{N_I}$*. A* general concept inclusion *(GCI) is of the form* $C \sqsubseteq D$*. An assertion is of the form* $C(a)$ *(concept assertion), or* $r(a, b)$ *(role assertion). A Boolean axiom formula* $\mathcal{B}$ *is a Boolean combination of GCIs and assertions. A* role inclusion axiom *is of the form* $r \sqsubseteq s$ *and a* transitivity *axiom is of the form* $trans(r)$*. Moreover,* r *is a a subrole of* $s \in \mathsf{N_R}$ *(w.r.t* $\mathcal{R}$*) if every model of* $\mathsf{N_R}$ *is a model of* $r \sqsubseteq s$*. An* RBox axiom *is either a role inclusion axiom or a transitivity axiom. An* RBox $\mathcal{R}$ *is a finite set of RBox axioms. A Boolean knowledge base (KB) is a pair* $\mathfrak{B} = (\mathcal{B}, \mathcal{R})$*, where* $\mathcal{B}$ *is a Boolean axiom formula and* $\mathcal{R}$ *is an RBox.*

In the basic DL $\mathcal{ALC}$, these concept constructors are allowed: $\neg C$, $C \sqcap D$ and $\exists r.C$. An expressive DL which allow all possible concept constructors and axioms introduced above is called the DL $\mathcal{SHOQ}$. In the lightweight DL $\mathcal{EL}$, only $C \sqcap D$, $\exists r.C$ and $\top$ are allowed. In $\mathcal{ALC}$ and $\mathcal{EL}$, RBox is empty. Furthermore, we assume at-most restrictions contains only simple roles, i.e., it has no transitive subroles, in $\mathcal{SHQ}$ and all its extensions to maintain the decidability.

The family of logic $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ are *two-sorted* with a *meta level signature* $\mathsf{M} = (\mathsf{M_C}, \mathsf{M_R}, \mathsf{M_I})$ and an *object level signature* $\mathsf{O} = (\mathsf{O_C}, \mathsf{O_R}, \mathsf{O_I})$. We call $\mathsf{M_C}$, $\mathsf{M_R}$ and $\mathsf{M_I}$ the set of *meta concept names*, *role names*, and *individual names* respectively. Analogously, $\mathsf{O_C}$, $\mathsf{O_R}$, $\mathsf{O_I}$ is called the set of *object concept names*, *role names*, and *individual names* respectively. All sets are assumed to be pairwise disjoint.

Definition 2 ($\mathcal{L}_M[\![\mathcal{L}_O]\!]$ Syntax). *Let* α *be an* $\mathcal{L}_O$*-axiom over the object level signature* O *and* $E \in \mathsf{M_C}$*,* $s \in \mathsf{M_R}$ *and* $e \in \mathsf{M_I}$ *meta level names. An* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-meta level concept description G *over* M *and* O *(*m-concept *for short) is the smallest set that contains* E *for all* $E \in \mathsf{M_C}$ *(basic meta concept),* $[\![\alpha]\!]$ *for all* $\mathcal{L}_O$*-axioms* α *(referring meta concept), and all complex concepts that can be built with the concept constructors allowed in* $\mathcal{L}_M$*.*

Let G *and* H *be m-concepts. An* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-Boolean meta level formula $\mathcal{C}$ *over* M *and* O *(*m-formula *for short) is built according to the following syntax rule*

$$\mathcal{C} ::= G \sqsubseteq H \mid G(e) \mid s(e, f) \mid \mathcal{C} \wedge \mathcal{C}' \mid \neg\mathcal{C}.$$

An m-assertion is either of the form $G(e)$ *(m-concept assertion) or* $s(e, f)$ *(m-role assertion). Furthermore we call an m-formula of the form* $G \sqsubseteq H$ *an m-GCI. An m-axiom is either an m-GCI, an m-assertion, an RBox axiom over* M *or an RBox axiom over* O*. A Boolean knowledge base over* M *and* O *(m-KB) is a tuple* $\mathfrak{C} = (\mathcal{C}, \mathcal{R}_M, \mathcal{R}_O)$ *where* $\mathcal{C}$ *is an m-formula,* $\mathcal{R}_M$ *is an RBox over* M *and* $\mathcal{R}_O$ *is an RBox over* O*.*

The semantics of $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ is defined in terms of *nested interpretations*. The structure consists of a single meta level interpretation over M (called context) where each meta domain element is associated with an object level interpretation over O. Moreover, all object level interpretations have the same domain.

An example of m-axiom that says Bob works for Company X in a work context is $\textsc{Work} \sqsubseteq [\![\mathit{worksFor}(\mathit{Bob}, \mathit{CompanyX})]\!]$. A referring meta concept encompasses contexts in which the object-level axiom inside it holds. In the example, $[\![\mathit{worksFor}(\mathit{Bob}, \mathit{CompanyX})]\!]$ is interpreted as the set of all contexts that satisfies $\mathit{worksFor}(\mathit{Bob}, \mathit{CompanyX})$.

Definition 3 (Nested Interpretation). *A* nested interpretation $\mathfrak{I}$ *(over* M *and* O*) is a tuple of the form* $\mathfrak{I} := (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c\in\mathbb{C}})$*, where* $(\mathbb{C}, \cdot^{\mathfrak{I}})$ *is an* M*-interpretation, and* $\mathcal{I}_c := (\Delta, \cdot^{\mathcal{I}_c})$ *is an* O*-interpretation for each* $c \in \mathbb{C}$*. Furthermore, we have that* $a^{\mathcal{I}_c} = a^{\mathcal{I}_d}$ *for all* $c, d \in \mathbb{C}$ *(rigid object individual assumption).*

Definition 4 ($\mathcal{L}_M[\![\mathcal{L}_O]\!]$ Semantics). *Let* $\mathfrak{I} = (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c\in\mathbb{C}})$ *be a nested interpretation. The* extension of the mapping $\cdot^{\mathfrak{I}}$ to complex m-concepts *is extended to referring meta concepts as follows :* $[\![\alpha]\!]^{\mathfrak{I}} := \{c \in \mathbb{C} \mid \mathcal{I}_c \models \alpha\}$.

Let $\mathcal{C}$ *be an m-formula.* Satisfaction of $\mathcal{C}$ in $\mathfrak{I}$*, written as* $\mathfrak{I} \models \mathcal{C}$ *(*$\mathfrak{I}$ is a model of $\mathcal{C}$*), is defined by induction on the structure of* $\mathcal{C}$ *as follows:*

$$\mathfrak{I} \models s(e,f) \text{ iff } (e^{\mathfrak{I}}, f^{\mathfrak{I}}) \in s^{\mathfrak{I}} \qquad \mathfrak{I} \models \neg\psi \text{ iff } \mathfrak{I} \not\models \psi \qquad \mathfrak{I} \models G(e) \text{ iff } e^{\mathfrak{I}} \in G^{\mathfrak{I}}$$
$$\mathfrak{I} \models G \sqsubseteq H \text{ iff } G^{\mathfrak{I}} \subseteq H^{\mathfrak{I}} \qquad \mathfrak{I} \models \psi_1 \wedge \psi_2 \text{ iff } \mathfrak{I} \models \psi_1 \text{ and } \mathfrak{I} \models \psi_2$$

Moreover, $\mathfrak{I}$ *is a model of* $\mathcal{R}_{\mathsf{M}}$ *(written* $\mathfrak{I} \models \mathcal{R}_{\mathsf{M}}$*) if* $(\mathbb{C}, \cdot^{\mathfrak{I}})$ *is a model of* $\mathcal{R}_{\mathsf{M}}$*, and* $\mathfrak{I}$ *is a model of* $\mathcal{R}_{\mathsf{O}}$ *(written* $\mathfrak{I} \models \mathcal{R}_{\mathsf{O}}$*) if for all* $c \in \mathbb{C}$*,* $\mathcal{I}_c$ *is a model of* $\mathcal{R}_{\mathsf{O}}$*. Finally,* $\mathfrak{I}$ *is a model of m-KB* $\mathfrak{C} = (\mathcal{C}, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ *(written* $\mathfrak{I} \models \mathfrak{C}$*) if* $\mathfrak{I}$ *is a model of* $\mathcal{C}$*,* $\mathcal{R}_{\mathsf{M}}$ *and* $\mathcal{R}_{\mathsf{O}}$*. We say* $\mathfrak{C}$ *is consistent if it has a model.*

2.2 Temporalizing Contextualized Description Logics

We introduce a family of temporalized DLs of context, denoted as $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL. It is clear that an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL formula is an LTL formula where propositional variables are replaced by $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ m-axioms.

Definition 5 ($\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL Syntax). *Let* $\mathcal{R}_{\mathsf{M}}$ *be an RBox over* M *and* $\mathcal{R}_{\mathsf{O}}$ *be an RBox over* O*. The set of* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *formulae w.r.t.* $\mathcal{R}_{\mathsf{M}}$ *and* $\mathcal{R}_{\mathsf{O}}$ *is the smallest set satisfying:*

- *if* γ *is an m-axiom w.r.t.* $\mathcal{R}_{\mathsf{M}}$ *and* $\mathcal{R}_{\mathsf{O}}$ *then* γ *is an* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *formula w.r.t.* $\mathcal{R}_{\mathsf{M}}$ *and* $\mathcal{R}_{\mathsf{O}}$*;*
- *if* ϕ *and* ψ *are* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *formulae, then* $\phi \wedge \psi$*,* $\phi \vee \psi$*,* $\neg\phi$*,* $\phi\, \mathbf{U}\, \psi$*, and* $\mathbf{X}\, \phi$ *are* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *formulae w.r.t.* $\mathcal{R}_{\mathsf{M}}$ *and* $\mathcal{R}_{\mathsf{O}}$*.*

A temporal context knowledge base (t-KB) is a tuple $\mathfrak{D} = (\phi, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ *where* $\mathcal{R}_{\mathsf{M}}$ *is an RBox over* M*,* $\mathcal{R}_{\mathsf{O}}$ *is an RBox over* O*, and* ϕ *is an* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *formula w.r.t.* $\mathcal{R}_{\mathsf{M}}$ *and* $\mathcal{R}_{\mathsf{O}}$*.*

The semantics of $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL is based on $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structures, sequences of nested interpretations. Constant domain assumption is respected on both meta and object level. We also consider rigid object concept and role names which are interpreted as the same across time-points and contexts.

Definition 6 ($\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL Semantics). *An $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure is a sequence $\mathfrak{T} = (\mathfrak{I}^{(i)})_{i\geq 0}$ of nested-interpretations $\mathfrak{I}^{(i)} = (\mathbb{C}, \cdot^{\mathfrak{I}^{(i)}}, \Delta, \{\mathcal{I}_c^{(i)}\}_{c\in\mathbb{C}})$ obeying the rigid individual assumption such that*

- $c^{\mathfrak{I}^{(i)}} = c^{\mathfrak{I}^{(j)}}$ *for all meta level individual names c and all $i, j \geq 0$, and*
- $a^{\mathcal{I}_c^{(i)}} = a^{\mathcal{I}_d^{(j)}}$ *for all object level individual names a, $i, j \geq 0$, and $c, d \in \mathbb{C}$.*

Given an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL formula ϕ, an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure $\mathfrak{T} = (\mathfrak{I}^{(i)})_{i\geq 0}$ and a time-point $i \geq 0$, validity of ϕ in $\mathfrak{T}$ at time i, written $\mathfrak{T}, i \models \phi$ is defined inductively:

$\mathfrak{T}, i \models \gamma$	*iff*	$\mathfrak{I}^{(i)} \models \gamma$ *for an m-axiom* γ
$\mathfrak{T}, i \models \phi \wedge \psi$	*iff*	$\mathfrak{T}, i \models \phi$ *and* $\mathfrak{T}, i \models \psi$
$\mathfrak{T}, i \models \neg\phi$	*iff*	$\mathfrak{T}, i \not\models \phi$
$\mathfrak{T}, i \models \mathbf{X}\,\phi$	*iff*	$\mathfrak{T}, i+1 \models \phi$
$\mathfrak{T}, i \models \phi\,\mathbf{U}\,\psi$	*iff*	*there is* $k \geq i$ *such that*
		$\mathfrak{T}, k \models \psi$ *and* $\mathfrak{T}, j \models \phi$ *for all* $j, i \leq j < k$

Furthermore if (1) $\mathfrak{I}_i \models \mathcal{R}_\mathsf{M}$ for every $i \geq 0$ (written $\mathfrak{T} \models \mathcal{R}_\mathsf{M}$), (2) $\mathcal{I}_c^{(i)} \models \mathcal{R}_\mathsf{O}$ for every $i \geq 0$ and $c \in \mathbb{C}$ (written $\mathfrak{T} \models \mathcal{R}_\mathsf{O}$), and (3) $\mathfrak{T}, 0 \models \phi$, then we call $\mathfrak{T}$ is a model of ϕ w.r.t. $\mathcal{R}_\mathsf{M}$ and $\mathcal{R}_\mathsf{O}$. We say that $\mathfrak{T}$ is a model of $\mathfrak{D} = (\phi, \mathcal{R}_\mathsf{M}, \mathcal{R}_\mathsf{O})$ (written $\mathfrak{T} \models \mathfrak{D}$) if $\mathfrak{T}$ is a model of ϕ w.r.t. $\mathcal{R}_\mathsf{M}$ and $\mathcal{R}_\mathsf{O}$. Finally we say $\mathfrak{D}$ is consistent if it has a model.

We say that an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure $\mathfrak{T} = (\mathfrak{I}^{(i)})_{i\geq 0}$ respects rigid object concept names (object role names) iff for any rigid object concept name A (object role name r), we have that $A^{\mathcal{I}_c^{(i)}} = A^{\mathcal{I}_d^{(j)}}$ ($r^{\mathcal{I}_c^{(i)}} = r^{\mathcal{I}_d^{(j)}}$) holds for all $i, j \in \{0, 1, ...\}$ and all $c, d \in \mathbb{C}$. We denote the set of all rigid object concept and role names with O_RC and O_RR, respectively. We say that the $\mathsf{O}_\mathsf{C} \setminus \mathsf{O}_\mathsf{RC}$ are flexible (object) concept names and $\mathsf{O}_\mathsf{R} \setminus \mathsf{O}_\mathsf{RR}$ are flexible (object) role names.

Let $\mathfrak{D} = (\phi, \mathcal{R}_\mathsf{M}, \mathcal{R}_\mathsf{O})$ be an t-KB: we say $\mathfrak{D}$ is satisfiable w.r.t. rigid names iff there is an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure $\mathfrak{T}$ respecting rigid object role names s.t. $\mathfrak{T}$ is a model of $\mathfrak{D}$. Analogously satisfiable w.r.t. rigid concepts for respecting rigid object concept names, and simply satisfiable without rigid names in consideration. Notice that rigid concepts can be simulated by rigid roles, therefore there are only three cases.

3 Deciding Satisfiability in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL

Table 1 summarizes the result of our investigation of the complexity of the satisfiability problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL. We use $\mathcal{L}_{\mathcal{ALC}}^{\mathcal{SHOQ}}$ to denote DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$.

Table 1. Complexity of the satisfiability problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL (all are tight).

	without rigid names		w.r.t. rigid concepts		w.r.t. rigid names	
$\mathcal{L}_M \backslash \mathcal{L}_O$	$\mathcal{EL}$	$\mathcal{L}_{\mathcal{ALC}}^{\mathcal{SHOQ}}$	$\mathcal{EL}$	$\mathcal{L}_{\mathcal{ALC}}^{\mathcal{SHOQ}}$	$\mathcal{EL}$	$\mathcal{L}_{\mathcal{ALC}}^{\mathcal{SHOQ}}$
$\mathcal{EL}$	PSpace	ExpTime	NExpTime	NExpTime	NExpTime	2-ExpTime
$\mathcal{L}_{\mathcal{ALC}}^{\mathcal{SHOQ}}$	ExpTime	ExpTime	NExpTime	NExpTime	NExpTime	2-ExpTime

We follow a similar idea of checking satisfiability of DL-LTL formulae and ConDL knowledge base. However, a naive approach to check temporal satisfiability and the contextual admissibility, i.e., collective satisfiability of guessed set of possible m-axiom combinations yields a triple exponential time algorithm. We show a double exponential time algorithm for this problem, and hence the same complexity with satisfiability problem in both $\mathcal{SHOQ}$-LTL and $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ w.r.t. rigid names. The idea is as follows: we guess the combinations of possible m-axioms that are satisfiable with appropriate sets of O-axioms for each m-axiom combination. Then, we check the satisfiability of temporal abstraction with respect of guessed m-axioms. Then, we check the admissibility of the meta-level by checking the existence of M-interpretations with an appropriate referring meta concepts abstraction that are guessed. Finally, we have to check guessed O-axiom combinations if they are also admissible, i.e., there are O-interpretations that satisfy them.

3.1 General Procedure

In the following, let $\mathfrak{D} = (\phi, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ be an t-KB to be tested. Let $Ax_{\mathsf{m}}(\phi)$ be the set of all m-axioms occuring in ϕ. Let $\mathbf{p}$ be a bijection mapping every occuring m-axiom γ in ϕ to a propositional variable p_γ. We assume that p_γ does not occur in ϕ and we define $\mathcal{P}_\phi := \{p_\gamma \mid \gamma \in Ax_{\mathsf{m}}(\phi)\}$. Let $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$ which represents consistent m-axioms combinations.

Definition 7 (Temporal Abstraction). *Let $\mathcal{R}_{\mathsf{M}}$ be an RBox over M, $\mathcal{R}_{\mathsf{O}}$ be an RBox over O, ϕ be an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL formula w.r.t. $\mathcal{R}$, and function $\mathbf{p}$: $Ax_{\mathsf{m}}(\phi) \to \mathcal{P}_\phi$ be a bijection where $\mathbf{p}(\gamma) = p_\gamma$.*

- *The propositional LTL-formula $\phi^{\mathbf{p}}$ is obtained from ϕ by replacing every occurrence of an m-axiom γ by p_γ. We call $\phi^{\mathbf{p}}$ the propositional abstraction of ϕ w.r.t. $\mathbf{p}$.*
- *Given an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure $\mathfrak{I} = (\mathfrak{J}^{(i)})_{i\geq 0}$, we obtain the LTL-structure $\mathfrak{I}^{\mathbf{p}} = (w^{(i)})_{i\geq 0}$ where $w^{(i)} := \{p_\gamma \mid \gamma \in Ax_{\mathsf{m}}(\phi)$ and $\mathfrak{J}^{(i)} \models \gamma\}$ for every $i \geq 0$. We call $\mathfrak{I}^{\mathbf{p}}$ the propositional abstraction of $\mathfrak{I}$ w.r.t. $\mathbf{p}$.*

Lemma 1. *Let $\mathfrak{I}$ be an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$. Then, $\mathfrak{I}$ is a model of ϕ w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$ iff $\mathfrak{I}^{\mathbf{p}}$ is a model of $\phi^{\mathbf{p}}$.*

However, we have to check the existence of each m-axiom combination which induces w_i since it is possible that there is no model for such combination. We ensure each world represents one of possible combinations of the guessed set $\mathcal{S}_{\mathcal{X}}$. Given a set $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$ and a propositional LTL formula $\phi^{\mathbf{p}}$, we define

$$\phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}} := \phi^{\mathbf{p}} \wedge \mathsf{G}\Big(\bigvee_{X \in \mathcal{S}_{\mathcal{X}}} (\bigwedge_{p_\gamma \in X} p_\gamma \wedge \bigwedge_{p_\gamma \in \mathcal{P}_\phi \setminus X} \neg p_\gamma)\Big)$$

Furthermore, we denote the right part of the outer conjunction as $\phi_{\mathcal{S}_{\mathcal{X}}}$. Then, we denote the satisfiability of an temporal abstraction in the notion of t-satisfiability.

Definition 8 (t-satisfiability). *Let ϕ be an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL formula and $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$. We say that $\phi^{\mathbf{p}}$ is t-satisfiable w.r.t. $\mathcal{S}_{\mathcal{X}}$ if $\phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}}$ has a model.*

Then, we check the existence of nested interpretations sequence that induces $\mathcal{S}_{\mathcal{X}}$. Furthermore, we ensure that they respect rigid names for both meta and object levels.

Definition 9 (c-admissibility). *We say that $\mathcal{S}_{\mathcal{X}} = \{X_1, \ldots, X_n\}$ such that $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$ is c-admissible w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$ if there exist nested interpretations $\mathfrak{I}^{(1)} = (\mathbb{C}, \cdot^{\mathfrak{I}^{(1)}}, \Delta, \{\mathcal{I}^{(1)}_c\}_{c\in\mathbb{C}}), \ldots, \mathfrak{I}^{(n)} = (\mathbb{C}, \cdot^{\mathfrak{I}^{(n)}}, \Delta, \{\mathcal{I}^{(n)}_c\}_{c\in\mathbb{C}})$ such that for any i, j such that $0 \leq i \leq j \leq n$: (1) $c^{\mathfrak{I}^{(i)}} = c^{\mathfrak{I}^{(j)}}$ holds for every $c \in \mathsf{M}_{\mathsf{I}}$; (2) $a^{\mathcal{I}^{(i)}_c} = a^{\mathcal{I}^{(j)}_d}$ holds for every $a \in \mathsf{O}_{\mathsf{I}}$ and all $c, d \in \mathbb{C}$; (3) $A^{\mathcal{I}^{(i)}_c} = A^{\mathcal{I}^{(j)}_d}$ holds for every $A \in \mathsf{O}_{\mathsf{RC}}$ and all $c, d \in \mathbb{C}$; (4) $r^{\mathcal{I}^{(i)}_c} = r^{\mathcal{I}^{(j)}_d}$ holds for every $r \in \mathsf{O}_{\mathsf{RR}}$ and all $c, d \in \mathbb{C}$; (5) every $\mathfrak{I}^{(i)}, 1 \leq i \leq n$ is a model of the Boolean m-KB*

$$\mathfrak{C}_{X_i} := (\bigwedge_{p_\gamma \in X_i} \gamma \wedge \bigwedge_{p_\gamma \in \mathcal{P}_\phi \setminus X_i} \neg\gamma, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$$

We have two defined properties to be tested in order to check the satisfiability of an $\mathcal{L}_M[\![\mathcal{L}_O]\!]-$LTL formula. This is a similar idea with DL-LTL satisfiability checking, but we check the existence of appropriate nested interpretations instead of DL interpretations.

Lemma 2. *Let $\mathfrak{D} = (\phi, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ be a t-KB, then $\mathfrak{D}$ is satisfiable w.r.t. rigid names iff there exists a set $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$ such that $\phi^{\mathbf{p}}$ is t-satisfiable w.r.t. $\mathcal{S}_{\mathcal{X}}$, and $\mathcal{S}_{\mathcal{X}}$ is c-admissible w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$.*

In fact, this result is enough to give us a procedure to decide satisfiability in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL. One can use a similar approach in DL-LTL [3] for $\mathcal{ALC}$-LTL by guessing an appropriate set $\mathcal{S}_{\mathcal{X}}$, then do the checking for two separate satisfiability problems. However, consider the c-admissibility checking of $\mathcal{S}_{\mathcal{X}}$. A naive approach is to build an m-KB

$$\mathfrak{C}_{\mathcal{S}_{\mathcal{X}}} := (\bigwedge_{X_i \in \mathcal{S}_{\mathcal{X}}} (\bigwedge_{p_\gamma \in X_i} \gamma^{(i)} \wedge \bigwedge_{p_\gamma \in \mathcal{P}_\phi \setminus X_i} \neg\gamma^{(i)}), \bigcup_{X_i \in \mathcal{S}_{\mathcal{X}}} \mathcal{R}^{(i)}_{\mathsf{M}}, \bigcup_{X_i \in \mathcal{S}_{\mathcal{X}}} \mathcal{R}^{(i)}_{\mathsf{O}})$$

with an appropriate renaming technique and then do the consistency checking. Since checking the m-KB $\mathfrak{C}$ satisfiability can be done in time doubly exponential in the size of the $\mathfrak{C}$ and the size of $\mathfrak{C}$ is exponential in the size of $\mathfrak{D}$, this yields a 3-ExpTime decision procedure.

To this end, we split again the c-admissibility into two subproblems. Instead of checking the c-admissibility as a whole, we separate the satisfiability of meta level abstraction and object level admissibility. We guess a mapping $\mathbf{y} : \mathcal{S}_{\mathcal{X}} \mapsto 2^{\mathcal{E}_\phi}$ that denotes possible O-axioms in referring meta concepts that are consistent together. For convenience, we say that we guess $\mathcal{S} = \{(X_1, \mathcal{Y}_1), ..., (X_n, \mathcal{Y}_n)\}$ that built in such way, where $X_i \in \mathcal{S}_{\mathcal{X}}$ and $\mathcal{Y}_i = \mathbf{y}(X_i)$ for each $1 \leq i \leq n$.

We recall the approach in [6], by introducing abstraction on the context level. This is done by introducing a fresh concept name E_α to collect all contexts that are in $[\![\alpha]\!]$. We define $Ax_{\mathsf{O}}(\phi)$ be the set of all object axioms α such that $[\![\alpha]\!]$ occurring in ϕ. Let $\mathbf{f}$ be a bijection mapping every occurring referring meta concept $[\![\alpha]\!]$ in ϕ to a meta concept name E_α. We assume that E_α does not occur in ϕ and we define $\mathcal{E}_\phi := \{E_\alpha \mid [\![\alpha]\!] \in Ax_{\mathsf{O}}(\phi)\}$.

Definition 10 (Context Level Abstraction). *Let $\mathfrak{C} = (\mathcal{C}, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ be an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-KB and function $\mathbf{f} : Ax_O(\phi) \rightarrow \mathcal{E}_\phi$ be a bijection where $\mathbf{f}(\alpha) = E_\alpha$.*

- *The outer abstraction of $\mathfrak{C}$ is the KB $\mathfrak{C}^{\mathbf{f}} = (\mathcal{C}^{\mathbf{f}}, \mathcal{R}_{\mathsf{M}})$ over M, where $\mathcal{C}^{\mathbf{f}}$ is obtained from $\mathcal{C}$ by replacing every occurrence of a referring meta concept $[\![\alpha]\!]$ by E_α.*
- *Given an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-structure $\mathfrak{I} = (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c\in\mathbb{C}})$, the contextual abstraction of $\mathfrak{I}$, denoted by $\mathfrak{I}^{\mathbf{f}}$ is the M-interpretation $\mathfrak{I}^{\mathbf{f}} = (\mathbb{C}, \cdot^{\mathfrak{I}^{\mathbf{f}}})$ where*
  - *for every $x \in (\mathsf{M}_{\mathsf{C}} \setminus \mathcal{E}_\phi) \cup \mathsf{M}_{\mathsf{R}} \cup \mathsf{M}_{\mathsf{I}}$, we have $x^{\mathfrak{I}^{\mathbf{f}}} = x^{\mathfrak{I}}$, and*
  - *for every $E_\alpha \in \mathcal{E}_\phi$, we have $E_\alpha^{\mathfrak{I}^{\mathbf{f}}} = [\![\alpha]\!]^{\mathfrak{I}}$.*

Similar with the temporal abstraction, we state the connection between a nested interpretation and its abstraction in the following lemma. The complete proof can be found in [7].

Lemma 3. *Let $\mathfrak{I} = (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c\in\mathbb{C}})$ be a nested interpretation such that $\mathfrak{I} \models \mathcal{R}_{\mathsf{O}}$. Then, $\mathfrak{I}$ is a model of $\mathfrak{C}$ iff $\mathfrak{I}^{\mathbf{f}}$ is a model of $\mathfrak{C}^{\mathbf{f}}$.* □

As in the temporal case, some combinations of referring meta concepts are not compatible due to the O-axioms that they represent. We recall the notion of *weakly respect* to restrict the nested-interpretations with allowed combinations of O-axioms.

Definition 11 (Weakly Respect). *Let $\mathcal{U} \subseteq \mathsf{N}_{\mathsf{C}}$ and let $\mathcal{V} \subseteq 2^{\mathcal{U}}$. The N-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ weakly respects $(\mathcal{U}, \mathcal{V})$ if $\mathcal{V} \subseteq \mathcal{Z}$ where $\mathcal{Z} := \{Y \subseteq \mathcal{U} \mid$ there exists $d \in \Delta^{\mathcal{I}}$ with $d \in (C_{\mathcal{U},\mathcal{V}})\}$ and $C_{\mathcal{U},\mathcal{V}} := \bigsqcap_{A\in Y} A \sqcap \bigsqcap_{A\in\mathcal{U}\setminus Y} \neg A$. It respects $(\mathcal{U}, \mathcal{V})$ if $\mathcal{V} = \mathcal{Z}$.*

In [6], the notion of outer consistency was introduced to express the existence of the abstracted $\mathcal{L}_M$-interpretation over M. However, we have to extend this notion since we have many abstracted $\mathcal{L}_M$-KB over M to be checked for satisfiability. Furthermore, we recall a lemma from [6] to be used in the proof.

Definition 12 (Outer Consistency). *We say that* $\mathfrak{C}^{\mathbf{f}}$ *is* outer consistent *w.r.t.* $\mathcal{Y} \subseteq 2^{\mathcal{E}_\phi}$ *if there exists a model of* $\mathfrak{C}^{\mathbf{f}}$ *that weakly respects* $(\mathcal{E}_\phi, \mathcal{Y})$*. Furthermore, we say that* $\mathcal{S}$ *is* conjointly outer consistent *iff there exist* $\mathcal{J}^{(1)}, ..., \mathcal{J}^{(n)}$ *such that for each* i*,* $1 \le i \le n$ *we have that* $\mathcal{J}^{(i)}$ *is a model of* $\mathfrak{C}^{\mathbf{f}}_{X_i}$ *that weakly respects* $(\mathcal{E}_\phi, \mathcal{Y}_i)$*.*

Lemma 4. *For every* M*-interpretation* $\mathcal{H} = (\Gamma, \cdot^{\mathcal{H}})$ *the following two statements are equivalent: (1) there exists a model* $\mathcal{J}$ *of* $\mathfrak{C}$ *with* $\mathcal{J}^{\mathbf{f}} = \mathcal{H}$*; (2)* $\mathcal{H}$ *is a model of* $\mathfrak{C}^{\mathbf{f}}$ *and the set* $\{X_d \mid d \in \Gamma\}$ *is admissible, where* $X_d := \{A \in \mathcal{E}_\phi \mid d \in A^{\mathcal{H}}\}$*.*

The existence of rigid object names does not matter for checking outer consistency since the next procedure will make verify it. Thus, it is possible to check if the consistency of each $\mathfrak{C}^{\mathbf{f}}_{X_i}$ (w.r.t. $(\mathcal{E}_\phi, \mathcal{Y})$) separately. The next step is to check whether it is possible all of $\mathcal{L}_O$-axiom combinations over O in guessed referring meta concept are satisfiable w.r.t. object-level rigid names. We check them, as a KB over O that represent such combinations. We recall the notion of object-level admissibility from [6,7], and extend it to our setting.

Definition 13 (Object Level Admissibility). *Let* $\mathcal{Y}_i \subseteq 2^{\mathcal{E}_\phi}$*. We say that* $\mathcal{Y}_i = \{Y_{(i,1)}, \ldots, Y_{(i,k)}\}$ *o-admissible if there exists* O*-interpretations* $\mathcal{I}^{(i,1)} = (\Delta, \cdot^{\mathcal{I}^{(i,1)}}), \ldots, \mathcal{I}^{(i,k)} = (\Delta, \cdot^{\mathcal{I}^{(i,k)}})$ *such that: (1)* $x^{\mathcal{I}^{(i,j)}} = x^{\mathcal{I}^{(i,j')}}$ *for all* $x \in \mathsf{O}_{\mathsf{RC}} \cup \mathsf{O}_{\mathsf{RR}} \cup \mathsf{O}_{\mathsf{I}}$ *and for all* j *and* j' *and (2) every* $\mathcal{I}^{(i,j)}$*,* $1 \le j \le k$ *is a model of the* $\mathcal{L}_O$*-KB* $\mathfrak{B}_{Y_{(i,j)}} = (\mathcal{B}_{Y_{(i,j)}}, \mathcal{R}_{\mathsf{O}})$ *over* O *where*

$$\mathcal{B}_{Y_{(i,j)}} := \bigwedge_{A_{[\![\alpha]\!]} \in Y_{(i,j)}} \alpha \wedge \bigwedge_{A_{[\![\alpha]\!]} \in \mathcal{E}_\phi \setminus Y_{(i,j)}} \neg \alpha$$

We say that $\mathcal{S}_{\mathcal{Y}} = \{\mathcal{Y}_1, ..., \mathcal{Y}_n\}$ *is conjointly o-admissible if every* $\mathcal{Y}_i$*,* $1 \le i \le n$ *is admissible and* $x^{\mathcal{I}^{(i,j)}} = x^{\mathcal{I}^{(i',j')}}$ *for all* $x \in \mathsf{O}_{\mathsf{RC}} \cup \mathsf{O}_{\mathsf{RR}} \cup \mathsf{O}_{\mathsf{I}}$ *for any* $(i,j), (i',j') \in \mathrm{Ind}_{\mathcal{S}}$ *where* $\mathrm{Ind}_{\mathcal{S}} = \{(i,j) \mid X_i \in \mathcal{S}_{\mathcal{X}}$ *and* $Y_j \in \mathcal{Y}_i\}$*.*

Then, we have two properties that determine c-admissibility of $\mathcal{S}_{\mathcal{X}}$, shown by the following lemma.

Lemma 5. $\mathcal{S}_{\mathcal{X}}$ *is c-admissible iff there exists a mapping* $\mathbf{y} : \mathcal{S}_{\mathcal{X}} \mapsto 2^{\mathcal{E}_\phi}$ *and* $\mathcal{S}_{\mathcal{Y}} = \{\mathbf{y}(X_i) \mid X_i \in \mathcal{S}_{\mathcal{X}})\}$ *such that (1)* $\mathcal{S}$ *is conjointly outer consistent; and (2)* $\mathcal{S}_{\mathcal{Y}}$ *is conjointly o-admissible.*

3.2 Satisfiability in expressive $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL

In this subsection, we consider cases where both $\mathcal{L}_M$ and $\mathcal{L}_O$ are between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. We show that these cases are well-behaved in the sense that have same complexity as consistency problem in underlying ConDLs.

Theorem 1. *The consistency problem in* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *w.r.t. rigid names is* 2-ExpTime*-complete for if both* $\mathcal{L}_M$ *and* $\mathcal{L}_O$ *are between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$*.*

Proof. We begin with showing the hardness for $\mathcal{ALC}[\![\mathcal{ALC}]\!]$ case. It is easy to see that an $\mathcal{ALC}[\![\mathcal{ALC}]\!]$-LTL t-KB is an $\mathcal{ALC}[\![\mathcal{ALC}]\!]$ m-KB. Since $\mathcal{ALC}[\![\mathcal{ALC}]\!]$ consistency problem w.r.t. rigid names is already 2-ExpTime-hard [6], we have that $\mathcal{ALC}[\![\mathcal{ALC}]\!]$-LTL (and more expressive $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL) consistency checking w.r.t. rigid names is 2-ExpTime-hard.

We prove the upper bound for $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ case. Let $\mathfrak{D} = (\phi, \mathcal{R}_\mathsf{M}, \mathcal{R}_\mathsf{O})$ be an $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL t-KB. We enumerate all possible sets $\mathcal{S}_\mathcal{X} \subseteq 2^{\mathcal{P}_\phi}$ which can be done in double exponential time in the size of ϕ, hence $\mathfrak{D}$. For each $X_i \in \mathcal{S}_\mathcal{X}$, we enumerate $\mathcal{Y}_i \subseteq 2^{\mathcal{P}_\phi}$ to build $\mathcal{S}$, which can be done, again, in double exponential time. In overall, we need double exponential time to guess a proper set $\mathcal{S}$. Then, we check t-satisfiability of $\phi^\mathbf{p}$ w.r.t. $\mathcal{S}_\mathcal{X}$. As argued in [3], this can be done in time exponential in the size of ϕ (and hence $\mathfrak{D}$) by constructing an appropriate Büchi automaton.

The next procedure is checking c-admissibility of $\mathcal{S}$. First part is checking whether $\mathcal{S}$ is conjointly outer consistent. It is easy to see that there are exponentially many pair $(X_i, \mathcal{Y}_i)$ to be checked. Furthermore, the size of $\mathfrak{C}^\mathbf{f}_{X_i}$ is polynomial in the size of $\mathfrak{D}$, while $\mathcal{Y}_i$ are at most exponential in the size of $\mathfrak{D}$. Exploiting Lemma 15 in [6], checking each of them can be done in exponential time. In overall, checking meta-level outer consistency can be done in time exponential in the size of ϕ.

The last property to be checked is o-admissibility. We have exponentially many $Y_{(i,j)}$ and each of them induces an $\mathcal{SHOQ}$-formula of size polynomial. Hence, the overall size of $\mathfrak{B}^\mathsf{O}_\mathcal{S}$ is exponential in the size of $\mathfrak{D}$. Since checking the satisfiability of an $\mathcal{ALC}$-KB can be done in exponential time of the size of the input, this yields a 2-ExpTime procedure. In overall we have two procedures that can be done in time doubly exponential to check c-admissibility of $\mathcal{S}$. □

We adjust the approach to be in NExpTime in the presence of only rigid concepts. The idea is to guess rigid concept membership for named individuals. On the other hand, the case without rigid names does not suffer from exponential blow up from $\mathfrak{C}_{\mathcal{S}_\mathcal{X}}$ since we can check each possible X_i separately.

Theorem 2. *The consistency problem in* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *w.r.t rigid concepts is* NExpTime-*complete for if both* $\mathcal{L}_M$ *and* $\mathcal{L}_O$ *are between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$.

Theorem 3. *The consistency problem in* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL *without rigid names is* ExpTime-*complete for if both* $\mathcal{L}_M$ *and* $\mathcal{L}_O$ *are between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$.

4 Using Lightweight $\mathcal{EL}$

In this section, we consider the case where at least one of $\mathcal{L}_M$ and $\mathcal{L}_O$ are $\mathcal{EL}$.

4.1 The case of $\mathcal{L}_M[\![\mathcal{EL}]\!]$-LTL and $\mathcal{EL}[\![\mathcal{L}_O]\!]$-LTL

We consider the case where at least one of underlying DLs is $\mathcal{EL}$. First, we consider the case where rigid concepts and role names are present.

Theorem 4. *Satisfiability in* $\mathcal{L}_M[\![\mathcal{EL}]\!]$-LTL *w.r.t. rigid names is* NExpTime-*complete if* $\mathcal{L}_M$ *is between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$.

Proof. (Sketch) The hardness follows immediately from NExpTime-completeness of the consistency problem in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ [6]. To show the upper bound, we consider the $\mathcal{SHOQ}[\![\mathcal{EL}]\!]$ case. Most of the approach is similar to the case of $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL, except in checking o-admisibility. The fact that $\mathfrak{B}^{\mathsf{O}}_{\mathcal{S}}$ is a conjunction of $\mathcal{EL}$-literals that can be decided in polynomial time yields an exponential time algorithm for checking o-admissibility instead of 2-ExpTime as in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL case. □

We have several other cases that are easy consequences of existing result in this work and previous studies in ConDLs.

Theorem 5. *Satisfiability in* $\mathcal{EL}[\![\mathcal{L}_O]\!]$-LTL *w.r.t. rigid names is* 2-ExpTime-*complete if* $\mathcal{L}_M$ *is between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$.

Theorem 6. *Satisfiability in* $\mathcal{L}_M[\![\mathcal{EL}]\!]$-LTL *and* $\mathcal{EL}[\![\mathcal{L}_O]\!]$-LTL *w.r.t. rigid concepts is* NExpTime-*complete if* $\mathcal{L}_M$ *and* $\mathcal{L}_O$ *are between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$.

Theorem 7. *Satisfiability in* $\mathcal{L}_M[\![\mathcal{EL}]\!]$-LTL *and* $\mathcal{EL}[\![\mathcal{L}_O]\!]$-LTL *without rigid names is* ExpTime-*complete if* $\mathcal{L}_M$ *and* $\mathcal{L}_O$ *are between* $\mathcal{ALC}$ *and* $\mathcal{SHOQ}$.

4.2 The case of $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL

The consistency of an $\mathcal{EL}[\![\mathcal{EL}]\!]$-KB is trivial if only conjunctions of m-axioms are allowed. Obviously, such assumption is not relevant anymore since LTL provides Boolean propositional operators. We consider beforehand the case of conjunctions of $\mathcal{EL}[\![\mathcal{EL}]\!]$-literals, i.e., $\mathcal{EL}[\![\mathcal{EL}]\!]$ m-axioms and negated $\mathcal{EL}[\![\mathcal{EL}]\!]$ m-axioms. In this setting, it is enough to consider satisfiability of $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL formula since both $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$ are always empty.

Claim. Satisfiability of conjunctions of $\mathcal{EL}[\![\mathcal{EL}]\!]$-literals without rigid names can be checked in polynomial time.

Theorem 8. *Satisfiability in* $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL *w.r.t. rigid names is* NExpTime-*complete.*

Proof. We show a reduction from $\mathcal{EL}$-LTL to show the hardness. Note that although an $\mathcal{EL}$-LTL formula is an $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL formula without referring meta concepts, it is not possible to directly consider rigid names since we do not have rigid meta names. However, we can still reduce the problem by moving them to the object level. Given an $\mathcal{EL}$-LTL formula ϕ, we define an $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL formula ϕ' by replacing any $\mathcal{EL}$-axiom α in ϕ with $\{c\} \sqsubseteq [\![\alpha]\!]$ where c is a fresh M-individual. Then, we can define the rigid (object) concepts and role names in $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL problem as rigid concept and role names in $\mathcal{EL}$-LTL problem, respectively. Then, it is easy to see that ϕ is satisfiabile w.r.t. rigid names iff ϕ' is satisfiable w.r.t. rigid names.

We show the problem is in NExpTime for the upper bound. To check for the satisfiability, we guess a set $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$ and we check whether $\phi^{\mathbf{P}}$ is t-satisfiable w.r.t. $\mathcal{S}_{\mathcal{X}}$ and $\mathcal{S}_{\mathcal{X}}$ is c-admissible. For t-satisfiability, we again may use the same argumentation as in Theorem 3 can be done in time exponential. For checking c-admissibility, we can build m-KB $\mathfrak{C}_{\mathcal{S}_{\mathcal{X}}}$ and then check for the satisfiability. We have that $\mathfrak{C}_{\mathcal{S}_{\mathcal{X}}}$ is of exponential size in the size of ϕ and $\mathfrak{C}_{\mathcal{S}_{\mathcal{X}}}$ is a conjunction of $\mathcal{EL}[\![\mathcal{EL}]\!]$-literals. Using Claim 4.2, this yields an exponential time procedure for checking c-admissibility. Thus, we have NExpTime upper bound in overall. □

The case with rigid concept names is an easy consequence of existing result. On the other hand, we exploit the same idea of checking satisfiability $\mathcal{EL}$-LTL for the case without rigid names since checking satisfiability of $\mathcal{EL}[\![\mathcal{EL}]\!]$-literals conjunction can be done also in polynomial time.

Theorem 9. *Satisfiability in $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL w.r.t. rigid concept is* NExpTime-*complete and without rigid names is* PSpace-*complete.*

5 Conclusion

We have introduced and investigated a family of languages to describe temporal properties over contextual knowledge. The formula of the language can be constructed using LTL operators over ConDL axioms. The underlying DLs that are used in particular are $\mathcal{EL}$, $\mathcal{ALC}$ and $\mathcal{SHOQ}$. We have shown that most of considered members of the family are well-behaved in the sense that the satisfiability problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL has the same complexity as consistency problem in underlying ConDL $\mathcal{L}_M[\![\mathcal{L}_O]\!]$, except for $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL cases.

For future work, we would like to investigate the use of resulting language in the setting of system verification. One of possible extensions is combining this work with ConDL-based actions as formalized in [17]. We would like to introduce a context-sensitive formal program based on ConDL-based actions. Then, a ConDL-LTLformula can be used to verify whether a property is satisfied in a (possibly non-deterministic) transition system induced by the program.

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A Proof of Lemma 1

Proof. Let $\mathfrak{T} = (\mathfrak{I}^{(i)})_{i\geq 0}$ be an $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure such that $\mathfrak{T} \models \mathcal{R}$ and $\mathfrak{T}^{\mathbf{p}} = (w^{(i)})_{i\geq 0}$ its propositional abstraction w.r.t. $\mathbf{p}$. We show that $\mathfrak{T}, i \models \phi$ iff $\mathfrak{T}^{\mathbf{p}}, i \models \phi^{\mathbf{p}}$ for every $i \geq 0$ by induction on the structure of ϕ. The base case is where ϕ is an axiom. Then for every $i \geq 0$, $\mathfrak{T}, i \models \phi$ iff $\mathfrak{I}^{(i)} \models \phi$ iff $p_\phi \in w_i$ iff $w^{(i)} \models \phi^{\mathbf{p}}$ iff $\mathfrak{T}^{\mathbf{p}}, i \models \phi^{\mathbf{p}}$.

Then if ϕ is of the form:

- $\neg\phi_1$ — for every $i \geq 0$, $\mathfrak{T}, i \models \neg\phi_1$ iff $\mathfrak{T}, i \not\models \phi_1$ iff $\mathfrak{T}^{\mathbf{p}}, i \not\models \phi_1^{\mathbf{p}}$ iff $\mathfrak{T}^{\mathbf{p}} \models (\neg\phi_1)^{\mathbf{p}}$.
- $\phi_1 \wedge \phi_2$ — for every $i \geq 0$, $\mathfrak{T}, i \models \phi_1 \wedge \phi_2$ iff $\mathfrak{T}, i \models \phi_1$ and $\mathfrak{T}, i \models \phi_2$ iff $\mathfrak{T}^{\mathbf{p}}, i \models \phi_1^{\mathbf{p}}$ and $\mathfrak{T}^{\mathbf{p}}, i \models \phi_2^{\mathbf{p}}$ iff $\mathfrak{T}^{\mathbf{p}}, i \models (\phi_1 \wedge \phi_2)^{\mathbf{p}}$.
- $\mathsf{X}\,\phi_1$ — for every $i \geq 0$, $\mathfrak{T}, i \models \mathsf{X}\,\phi_1$ iff $\mathfrak{T}, i+1 \models \phi_1$ iff $\mathfrak{T}^{\mathbf{p}}, i+1 \models \phi_1^{\mathbf{p}}$ iff $\mathfrak{T}^{\mathbf{p}}, i \models (\mathsf{X}\,\phi_1)^{\mathbf{p}}$.
- $\phi_1\,\mathsf{U}\,\phi_2$ — for every $i \geq 0$, $\mathfrak{T}, i \models \phi_1\,\mathsf{U}\,\phi_2$ iff there exists $k \geq i$ such that $\mathfrak{T}, k \models \phi_2$ and $\mathfrak{T}, i \models \phi_1$ for every j, where $i \leq j < k$ iff there exists $k \geq i$ such that $\mathfrak{T}^{\mathbf{p}}, k \models \phi_2^{\mathbf{p}}$ and $\mathfrak{T}^{\mathbf{p}}, j \models \phi_1^{\mathbf{p}}$ for every j, where $i \leq j < k$ iff $\mathfrak{T}^{\mathbf{p}}, i \models (\phi_1\,\mathsf{U}\,\phi_2)^{\mathbf{p}}$. □

B Proof of Lemma 2

Proof. $\Longrightarrow$ Assume that there exists a model $\mathfrak{T} = (\mathfrak{I}^{(i)})_{i\geq 0}$ of $\mathfrak{D}$ w.r.t. rigid names, and $\mathfrak{T}^{\mathbf{p}} = (w^{(i)})_{i\geq 0}$ its temporal abstraction. We define the induced set $\mathcal{S}_{\mathcal{X}} := \{w^{(i)} \mid i \geq 0\}$ which is finite since $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$. First we show t-satisfiability, i.e., there exists $\mathcal{M} = (w_i)_{i\geq 0}$ such that $\mathcal{M}, 0 \models \phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}}$. We show that in fact $\mathfrak{T}^{\mathbf{p}}, 0 \models \phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}}$. Due to Lemma 1, we have that $\mathfrak{T}^{\mathbf{p}}, 0 \models \phi^{\mathbf{p}}$. Furthermore, due to the construction we have that every $w^{(i)}$ satisfies one of disjunctions in $\phi_{\mathcal{S}_{\mathcal{X}}}$. Next, we show that $\mathcal{S}_{\mathcal{X}}$ is c-admissible w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$. Since $\mathfrak{T}$ obeys rigid individual assumptions and respects rigid names, condition 9 - 9 are satisfied. Due to the construction of $\mathcal{S}_{\mathcal{X}}$, we know that for every X_i, there exists $w^{(i)}$, $i \geq 0$ such that $w^{(i)} = \{p_\gamma \mid p_\gamma \in \mathcal{P}_\phi$ and $\mathfrak{I}^{(i)} \models \gamma\}$. It is easy to see that $\mathfrak{I}^{(i)} \models \mathfrak{C}_{X_i}$.

$\Longleftarrow$ Assume there exists a set $\mathcal{S}_{\mathcal{X}} \subseteq 2^{\mathcal{P}_\phi}$, such that $\phi^{\mathbf{p}}$ is t-satisfiable w.r.t. $\mathcal{S}_{\mathcal{X}}$ and $\mathcal{S}_{\mathcal{X}}$ is c-admissible w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$. Then, there exists an LTL-structure $\mathfrak{M} = (w^{(j)})_{j\geq 0}$ such that $\mathfrak{M} \models \phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}}$. We define $\mathcal{W} = \{w^{(i)} \mid i \geq 0\}$. Since $\mathfrak{M}$ is a model of $\phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}}$, then $\mathcal{W} \subseteq \mathcal{S}_{\mathcal{X}}$, hence $\mathcal{W}$ is c-admissible w.r.t. $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$. Then, there exists nested interpretations $\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(n)}$ that satisfies c-admissibility properties of $\mathcal{S}_{\mathcal{X}}$. We define a function $\mathbf{v} : \mathcal{W} \mapsto \{\mathfrak{I}^1, \ldots, \mathfrak{I}^n\}$ such that $\mathbf{v}(w^{(i)}) = \mathfrak{I}^{(j)}$ where $w^{(i)} = X^{(j)} \in \mathcal{S}_{\mathcal{X}}$. We construct $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL-structure $\mathfrak{T}_{\mathfrak{M}} := (\mathbf{v}(w^{(i)}))_{i\geq 0}$. Since $\mathfrak{M} = \mathfrak{T}^{\mathbf{p}}$ and $\mathfrak{M} \models \phi^{\mathbf{p}}$, we have that $\mathfrak{T}_{\mathfrak{M}} \models \phi$ due to Lemma 1. Furthermore, $\mathfrak{T}_{\mathfrak{M}} \models \mathcal{R}_{\mathsf{M}}$ and $\mathfrak{T}_{\mathfrak{M}} \models \mathcal{R}_{\mathsf{O}}$ due to condition 9 in Definition 9. Then, we have $\mathfrak{T}_{\mathfrak{M}} \models \mathfrak{D}$. Finally, $\mathfrak{T}_{\mathfrak{M}}$ obeys rigid individual assumptions and respect rigid names due to condition 9 - 9 in Definition 9. □

C Proof of Lemma 5

Proof. $\Longrightarrow$ Assume that $\mathcal{S}_{\mathcal{X}}$ is c-admissible. Let $\mathfrak{I}^{(1)}, ..., \mathfrak{I}^{(n)}$ be the nested interpretations satisfy the properties. For each X_i, we define $\mathcal{Y}_i := \{Y_c \mid c \in \mathbb{C}\}$, where $Y_c := \{E_\alpha \in \mathcal{E}_\phi \mid \mathcal{I}_c \models \alpha\}$ to build $\mathcal{S} = \{(X_1, \mathcal{Y}_i), ..., (X_n, \mathcal{Y}_n)\}$.

First, we show that $\mathcal{S}_{\mathcal{X}}$ is conjointly outer consistent since $(\mathfrak{I}^{(1)})^{\mathbf{f}}, ..., (\mathfrak{I}^{(n)})^{\mathbf{f}}$ satisfy the properties. We show that for any i, $(\mathfrak{I}^{(i)})^{\mathbf{f}})$ is a model of $\mathfrak{C}^{\mathbf{f}}_{X_i}$ and $(\mathfrak{I}^{(i)^{\mathbf{f}}})$ weakly respects $(\mathcal{E}_\phi, \mathcal{Y}_i)$. The former one is trivial due to the Lemma 3 and rigidity properties of c-admissibility. The later is also an easy consequence of the construction of $\mathcal{Y}_i$. Other properties are easy consequences of (1) - (3) in Definition 9.

Then, we show that $\mathcal{S}_{\mathcal{Y}}$ is conjointly o-admissible. We show that any $\mathcal{Y}_i$, $1 \leq i \leq n$ is o-admissible. Due to the construction, for any $Y_{(i,j)}$, $1 \leq j \leq k$ there exists $c \in \mathbb{C}$ such that $\mathcal{I}_c \models \alpha$ for any $\alpha \in Y_{(i,j)}$ and $\mathcal{I}_c \not\models \alpha'$ for any $\alpha' \in \mathcal{E}_\phi \setminus Y_{(i,j)}$. Then, it is easy to see that $\mathcal{I}^{(i,j)} \models \mathfrak{C}_{Y_{(i,j)}}$. Finally, it is easy to see that rigidity properties are transferred over from c-admissibility to o-admissibility.

$\Longleftarrow$ Assume that there exists $\mathcal{S} = \{(X_1, \mathcal{Y}_1), \ldots, (X_n, \mathcal{Y}_n)\}$ such that $\mathcal{S}$ is conjointly outer consistent and $\mathcal{S}_{\mathcal{Y}}$ is conjointly o-admissible. Due to Löwenheim-Skolem theorem, we can safely assume that all $\mathcal{I}^{(1)}, \cdots, \mathcal{I}^{(n)}$ have same domain $\mathbb{C}$. Furthermore, we can assume that individual names are interpreted the same. Then, there exists a model $\mathcal{I}^{(i)}$ of $\mathfrak{C}^{\mathbf{f}}_{X_i}$ that weakly respects $(\mathcal{E}_\phi, \mathcal{Y}_i)$. $\square$

D Proof of Theorem 2

Proof. We begin with showing the hardness for $\mathcal{ALC}[\![\mathcal{ALC}]\!]$ case. Checking consistency of an $\mathcal{ALC}[\![\mathcal{ALC}]\!]$ m-KB with rigid concepts is already NExpTime-hard. Thus, we have that checking the consistency of an $\mathcal{ALC}[\![\mathcal{ALC}]\!]$-LTL (and more expressive $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-LTL) t-KB w.r.t. rigid concepts is NExpTime-hard.

For the upper bound, we again have to adjust the previous result to work with our setting. Let $\mathfrak{D} = (\phi, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL t-KB. We can do the checking t-satisfiability and c-admissibility as in the case w.r.t. rigid names. This problem has been solved in reasoning over $\mathcal{ALC}$-LTL formula, where we build an appropriate Büchi automaton. The automaton can be checked in exponential time in the size of $\phi^{\mathbf{p}}$, hence in the size of ϕ. The full construction can be found in [3]. Thus, we need two exponential time procedures, and an exponential time algorithm for deciding consistency.

We non-deterministically guess the set $\mathcal{S}$ which the size is at most exponential in the size of $\mathfrak{D}$. We define $\mathsf{O}^{\phi}_{\mathsf{RC}} := \{A_1, \ldots, A_r\}$ as the set of rigid concept names occurring in ϕ and $\mathsf{O}^{\phi}_{\mathsf{I}}$ be the set of all individuals occurring in ϕ. We guess a set $\mathcal{Q} \subseteq 2^{\mathsf{O}^{\phi}_{\mathsf{RC}}}$, and a mapping $\mathbf{r} : \mathsf{O}^{\phi}_{\mathsf{I}} \mapsto \mathcal{Q}$, which both can be done within NExpTime. Then. we define

$$\widehat{\mathfrak{B}}_{Y_{(i,j)}} := (\mathcal{B}_{Y_{(i,j)}} \wedge \bigwedge_{a \in \mathsf{O}^{\phi}_{\mathsf{I}}} \big(\bigsqcap_{A \in \mathbf{r}(a)} A \sqcap \bigsqcap_{A \in \mathsf{O}^{\phi}_{\mathsf{RC}} \setminus \mathbf{r}(a)} \neg A \big)(a), \mathcal{R}_{\mathsf{O}})$$

As argued in previous approaches [6,3], we have that $\mathcal{S}_{\mathcal{Y}}$ is conjointly o-admissible iff for all $(i,j) \in \text{Ind}_{\mathcal{S}_{\mathcal{Y}}}$, $\widehat{\mathfrak{B}}_{Y_{(i,j)}}$ has a model that respects $(\mathsf{O}^{\phi}_{\mathsf{RC}}, \mathcal{Q})$. Since we have that $\widehat{\mathfrak{B}}_{Y_{(i,j)}}$ is of size polynomial in the size of ϕ, and checking the consistency of a $\mathcal{SHOQ}$ KB is in ExpTime, then this procedure can be done in exponential time. In overall, we have NExpTime procedure for the case w.r.t. rigid concept only. □

E Proof of Theorem 3

Proof. Notice that checking consistency of $\mathcal{ALC}[\![\mathcal{ALC}]\!]$-KB without rigid names is ExpTime-complete. It is easy to see that since any $\mathcal{ALC}[\![\mathcal{ALC}]\!]$ m-KB is also an $\mathcal{ALC}[\![\mathcal{ALC}]\!]$-LTL t-KB, ExpTime-hardness of the satisfiability problem is carried over.

Next, we prove the upper bound by considering $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ case. Let $\mathfrak{D} = (\phi, \mathcal{R}_{\mathsf{M}}, \mathcal{R}_{\mathsf{O}})$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL t-KB. Instead of refining each procedure in the case of rigid names, we use a different approach for checking c-satisfiability. Instead of breaking down *c-admissibility* into two procedures, we make use of the fact that $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ satisfiability without rigid names is in ExpTime. It is enough for us to have an appropriate $\mathcal{S}_{\mathcal{X}} = \{X_1, ..., X_n\}$. Instead of guessing $\mathcal{S}_{\mathcal{X}}$, we compute a maximal set $\widehat{\mathcal{S}}_{\mathcal{X}} := \{\mathfrak{C}_X \mid X \in 2^{\mathcal{P}_\phi}\}$. We can enumerate all possible subsets of $\mathcal{P}_\phi$ and for each of them we have an exponential time procedure for checking consistency of $\mathfrak{C}_X$. This yields an exponential time procedure to get $\widehat{\mathcal{S}}_{\mathcal{X}}$. First, we recall that in other cases we have shown that checking t-satisfiability is exponential of the size of $\mathfrak{D}$. Next, we have to show that $\widehat{\mathcal{S}}_{\mathcal{X}}$ is indeed c-admissible. Since each $\mathfrak{C}_{X_i}$ is consistent, there exists a model for each them with countably infinite domain. We can assume that they have the same meta-level domain $\mathbb{C}$ and object-level domain Δ. Moreover, we assume that M-individual names and O-individual names (for the same $c \in \mathbb{C}$) are interpreted the same. The conditions (9) and (9) are vacuously satisfied since there are no rigid names. Thus, we have that $\mathcal{S}_{\mathcal{X}}$ is c-admissible. In overall, we get an ExpTime procedure for checking consistency of a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL t-KB without rigid names. □

F Proof of Theorem 4

Proof. First, we show the hardness for the case $\mathcal{ALC}[\![\mathcal{EL}]\!]$-LTL. Obviously, an $\mathcal{ALC}[\![\mathcal{EL}]\!]$ m-KB is an $\mathcal{ALC}[\![\mathcal{EL}]\!]$-LTL t-KB without temporal operators. Then, NExpTime-hardness for consistency checking of $\mathcal{ALC}[\![\mathcal{EL}]\!]$-LTL immediately follows from NExpTime-completeness of the consistency problem in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ [6].

To show the upper bound, we consider the $\mathcal{SHOQ}[\![\mathcal{EL}]\!]$ case. Let $\mathfrak{D}$ be an $\mathcal{SHOQ}[\![\mathcal{EL}]\!]$-LTL t-KB. We guess a set $\mathcal{S} = \{(X_1, \mathcal{Y}_1), \ldots, (X_n, \mathcal{Y}_n)\}$ that can be done in exponential time. Then, we check t-satisfiability of $\phi^{\mathbf{p}}_{\mathcal{S}_{\mathcal{X}}}$ that can be done in ExpTime as shown in [3]. Checking whether $\mathcal{S}$ is conjointly outer consistent is again in ExpTime since there are exponentially many $\mathfrak{C}^{\mathbf{f}}_{X_i}$ to be checked and each of them takes exponential time. In overall, conjointly outer consistency can be tested in time exponentially of the size of $\mathfrak{D}$. Finally, notice that $\mathfrak{B}_{\mathcal{S}}^{\mathsf{O}}$ is a conjunction of $\mathcal{EL}$-literals. We recall that the satisfiability of conjunctions of $\mathcal{EL}$-literals can be decided in polynomial time. Since the size of $\mathfrak{B}_{\mathcal{S}}^{\mathsf{O}}$ is of exponential size in the size of ϕ, checking o-admissibility can be done in exponential time in the size of ϕ, and hence $\mathfrak{B}$. This yields the NExpTime upperbound for checking satisfiability in $\mathcal{ALC}[\![\mathcal{EL}]\!]$-LTL w.r.t. rigid names. □

G Proof of Theorem 5

Proof. Obviously, the 2-ExpTime-hardness directly follows from 2-ExpTime-completeness of the consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$. Due to Theorem 1 and the fact that $\mathcal{EL}[\![\mathcal{SHOQ}]\!]$-LTL is a fragment of $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL, we can use the same procedure to yield the upper bound. □

H Proof of Theorem 6

Proof. Obviously, the NExpTime-hardness for $\mathcal{ALC}[\![\mathcal{EL}]\!]$-LTL and $\mathcal{EL}[\![\mathcal{ALC}]\!]$-LTL are immediate consequences of NExpTime-hardness of the consistency problem with only rigid concepts in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ and $\mathcal{EL}[\![\mathcal{ALC}]\!]$, respectively. The upper bound are consequences of Theorem 3 since both $\mathcal{SHOQ}[\![\mathcal{EL}]\!]$-LTL and $\mathcal{EL}[\![\mathcal{SHOQ}]\!]$-LTL are fragments of $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL. □

I Proof of Theorem 7

Proof. The ExpTime-hardness for $\mathcal{ALC}[\![\mathcal{EL}]\!]$-LTL and $\mathcal{EL}[\![\mathcal{ALC}]\!]$-LTL follows immediately from the fact that the consistency problem without rigid names in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ and $\mathcal{EL}[\![\mathcal{ALC}]\!]$ are ExpTime-hard already. The ExpTime upper bound are consequences of Theorem 2 since both $\mathcal{SHOQ}[\![\mathcal{EL}]\!]$-LTL and $\mathcal{EL}[\![\mathcal{SHOQ}]\!]$-LTL are fragments of $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-LTL. □

J Proof of Claim in Section 4.2

Proof. Due to Lemma 2, checking the satisfiability of an $\mathcal{EL}[\![\mathcal{EL}]\!]$ m-formula $\mathcal{C}$ can be done by checking the existence of a set $\mathcal{S} := \{X_1, \ldots, X_k\} \subseteq 2^{\mathcal{E}_{\mathfrak{C}}}$ such that $\mathcal{C}^{\mathbf{f}}$ is consistent w.r.t. $\mathcal{S}$ and $\mathcal{B}_{X_i}$ is consistent for any $1 \leq i \leq k$. It is not necessary to construct $\mathcal{S}$ explicitly. We recall that X_i represents a combination of referring meta concepts which a domain element can be member of.

We consider again checking the consistency of $\mathfrak{C}^{\mathbf{f}}$. Note that the abstraction $\mathfrak{C}^{\mathbf{f}}$ is a conjunction of $\mathcal{EL}$-literals. As shown in [7,8], checking the satisfiability of $\mathcal{EL}$-literals can be done in polynomial time by reducing it to the consistency problem of $\mathcal{ELO}_{\bot}$. We can modify this approach to solve our problem. We recall that the algorithm in [1] compute a mapping S from C_ϕ to $C_\phi \cup \{\bot\}$ where

C_ϕ is the set that contains top concept, all concepts names used in ϕ and all subconcepts of the form $\{a\}$ (nominal) appearing in ϕ and a similar mapping R to represent $\exists r.C$. Intuitively, the mapping S represents the subsumption relation in the following sense: $D \in S(C)$ implies $C \sqsubseteq D$. We can compute this mapping using the algorithm that can be done in polynomial time. Afterwards, we check whether for every C such that there exists $\{a\} \in S(C)$ for some individual a, $\mathfrak{B}_C^\phi$ is consistent where

$$\mathfrak{B}_C^\phi := \bigwedge_{E_\alpha \in S(C)} \alpha \wedge \bigwedge_{E_\alpha \in \mathcal{E}_\phi \setminus S(C)} \neg\alpha$$

we the combination of referring meta concepts in $S(C)$ are consistent. There is no need to check the one without nominal since such concept can be empty. Note that any $\mathfrak{B}_C^\phi$ is a conjunction of $\mathcal{EL}$-literals. Since there are at most polynomial number of C to be checked and each combination can be checked in polynomial time, this yields a polynomial time procedure to check the consistency of conjunction of $\mathcal{EL}[\![\mathcal{EL}]\!]$-literals. □

K Proof Theorem 9

Proof. Case w.r.t. rigid concepts : Since $\mathcal{EL}$-LTL with rigid concept names is NExpTime-hard, we can use the same reduction as in Theorem 8. Furthermore, there are only rigid concept names to be copied to corresponding $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL problem. The upper bound is immediately follows from the case of rigid (concept and role) names.

Case without rigid names : By Lemma 2, checking the satisfiability of an $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL formula ϕ can be done by checking the existence of a set $\mathcal{S} := \{X_1, \ldots, X_k\} \subseteq 2^{\mathcal{E}_\mathfrak{c}}$ such that $\phi^\mathbf{f}$ is consistent w.r.t. $\mathcal{S}_\mathcal{X}$ and $\mathcal{S}_\mathcal{X}$ is c-admissible. We employ the same idea of deciding $\mathcal{EL}$-LTL without rigid names [7,8]. Instead of guessing or building the set $\mathcal{S}_\mathcal{X}$, we can check the induced X_i of the world w_i on the fly. As argued in [8], we can exploit the periodic model property of $\phi^\mathbf{p}$ [16]. With a similar construction, we can build a modified $\mathcal{M}_{\phi^\mathbf{p}}$ for our objective. Instead of checking induced conjunction of $\mathcal{EL}$-literals for each world, we check conjunction of $\mathcal{EL}[\![\mathcal{EL}]\!]$-literals. However, by Claim 4.2, this can be done in polynomial time, i.e., the same complexity of checking $\mathcal{EL}$-literals. Thus, we have shown a PSpace algorithm that check satisfiability of an $\mathcal{EL}[\![\mathcal{EL}]\!]$-LTL formula. □