Quantified Concept-Forming Operators*

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Abstract. Generalized quantifiers allow to decrease the outermost character of the universal and existential quantifiers. Concept-forming operators in Formal Concept Analysis have a universal character. As a consequence, some noise in the data can dramatically alter the final result. This paper introduces a first approximation to quantified formal-concept operators with the main goal of decreasing this universal character and be less sensitive to noisy data.

Keywords: Concept lattices, fuzzy sets, general quantifier, adjoint triples

1 Introduction

Formal Concept Analysis (FCA), introduced by Ganter and Wille in the eighties, has become an appealing research topic both from theoretical [18, 22] and applicative perspectives [1, 16, 17, 19, 24, 26]. Specifically, FCA is a mathematical tool in charge of extracting information from databases containing a set of attributes A and a set of objects B related to each other by a binary relation $R \subseteq A \times B$. These pieces of information are called concepts and a hierarchy can be established on them providing an algebraic structure called concept lattice. From the concept lattice, a mathematical development for the conceptual data analysis and processing of knowledge can be carried out. Soon after its introduction, a number of different approaches for its generalization were introduced [2, 5, 7]. One of the most general fuzzy approaches is the multi-adjoint concept lattice [20, 21, 23], which will be considered in this paper. The multi-adjoint concept lattice framework considers different adjoint triples in the definition of the concept-forming operations, providing interesting properties [11, 13].

Generalized quantifiers [6, 8, 9, 14, 15, 25] have been proposed in order to solve the theoretical drawback of the usual universal and existencial quantifiers. Specifically, these quantifiers try to introduce intermeditate quantifiers between the existential quantifier, where just one element is enough to result the truth, and

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the universal quantifier, where all elements have to fulfill a given formula in order to result the truth. As a consequence, more proper quantifiers can be used in applications where the outermost quantifiers are very restrictive. For example, notions as "Most" or "Many" can be implemented and so, considered in applications.

This paper is inspired by [9, 25] in order to develop different notions and results on multi-adjoint concept lattices, based on the monadic quantifiers of type $\langle 1 \rangle$ determined by fuzzy measures [15]. Specifically, we have introduced the definition of generalized quantifier in this general framework as a natural extension, which also satisfies the main properties. Moreover, we have used the extended definition to present a generalization of the usual fuzzy concept-forming operators [3, 4, 7, 23] in order to decrease the drastic (universal) character of these operators, which are very sensitive to noise data. As a consequence, the new introduced operators offer a complementary alternative to the original concept-forming operators to obtain more robust, stable and valuable information from datasets. Although the new operators generalize the original one, they do not form an antitone Galois connection, in general. Therefore, different sufficient conditions and alternative definitions will be studied in the future in order to ensure Galois connections.

The paper is organized as follows. In the second section, we recall the theory related to multi-adjoint concept lattices. In the third section, we generalize the notion of generalized quantifier by using adjoint triples. In addition, we use the extended definition in order to present a generalization of the usual fuzzy concept-forming operators. In the last section, we list some directions where our approach could be further developed.

2 Preliminaries

To begin with, we will recall the notion of adjoint triple which play an important role in this work. Adjoint triples arise as a generalization of triangular norms and their residuated implications. These operators are the basic calculus operators in different frameworks such as multi-adjoint logic programming, multi-adjoint fuzzy relation equations, multi-adjoint fuzzy rough sets and multi-adjoint concept lattices. It is worth noting that these operators increase the flexibility of the mentioned frameworks, since the conjunctors are not required to be either commutative or associative.

Definition 1. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\& \colon P_1 \times P_2 \to P_3$, $\swarrow \colon P_3 \times P_2 \to P_1$, $\nwarrow \colon P_3 \times P_1 \to P_2$ be mappings. We say that $(\&, \swarrow, \nwarrow)$ is an* adjoint triple *with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the following double equivalence is satisfied:*

$$x \leq_1 z \swarrow y \quad \textit{iff} \quad x \,\&\, y \leq_3 z \quad \textit{iff} \quad y \leq_2 z \nwarrow x$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$. The previous double equivalence is called adjoint property.

The following properties are obtained straightforwardly from the adjoint property [10, 12].

Proposition 1. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, then the following properties are satisfied:*

1. *$\&$ is order-preserving on both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.*
3. *$\bot_1 \& y = \bot_3$, $\top_3 \swarrow y = \top_1$, for all $y \in P_2$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.*
4. *$x \& \bot_2 = \bot_3$ and $\top_3 \nwarrow x = \top_2$, for all $x \in P_1$, when $(P_2, \leq_2, \bot_2, \top_2)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.*
5. *$z \nwarrow \bot_1 = \top_2$ and $z \swarrow \bot_2 = \top_1$, for all $z \in P_3$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_2, \leq_2, \bot_2, \top_2)$ are bounded posets.*

Once we have introduced the notion of adjoint triple, we are in a position to present multi-adjoint concept lattices. The philosophy of the multi-adjoint paradigm was applied to the formal concept analysis in order to obtain a new general approach framework that could conveniently accommodate different fuzzy approaches given in the literature. The mentioned general approach is called multi-adjoint concept lattices and their main notions will be recalled below [23].

Definition 2. *A* multi-adjoint frame *is a tuple $(L_1, L_2, P, \&_1, \dots, \&_n)$ where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P, for all $i \in \{1, \dots, n\}$.*

Definition 3. *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a* context *is a tuple (A, B, R, σ) such that A and B are non-empty sets, R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon A \times B \to \{1, \dots, n\}$ is a mapping which associates any element in $A \times B$ with some particular adjoint triple in the frame.*

After fixing a multi-adjoint frame and a context for that frame, the concept-forming operators are denoted as ${}^\uparrow\colon L_2^B \longrightarrow L_1^A$ and ${}^\downarrow\colon L_1^A \longrightarrow L_2^B$ and are defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as

$$g^\uparrow(a) = \bigwedge_{b \in B} \Big(R(a, b) \swarrow^{\sigma(a,b)} g(b)\Big) \tag{1}$$

$$f^\downarrow(b) = \bigwedge_{a \in A} \big(R(a, b) \nwarrow_{\sigma(a,b)} f(a)\big) \tag{2}$$

A *multi-adjoint concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^\uparrow = f$ and $f^\downarrow = g$. A hierarchy can be defined on the whole set of multi-adjoint concepts, which gives rise to the concept lattice.

Definition 4. *Given a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ), a* multi-adjoint concept lattice *is the set*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A, g^\uparrow = f, f^\downarrow = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (equivalently $f_2 \preceq_1 f_1$).

3 Formal-concept operators with generalized quantifiers

This section will generalize the notion of generalized quantifier given in [25] by using adjoint triples. Specifically, we will consider adjoint triples defined on the complete lattice $([0,1], \leq)$ assuming that their conjunctors have 1 as left and right identity element.

Before presenting the notion of quantifier, we need to present the concept of fuzzy measure invariant with respect to the cardinality.

Definition 5. *Let $\mathcal{U}$ be a finite universe and $\mathrm{P}(\mathcal{U})$ be the powerset of $\mathcal{U}$. We will say that the mapping $\mu\colon \mathrm{P}(\mathcal{U}) \to [0,1]$ is a* fuzzy measure*, if it is an increasing mapping satisfying that $\mu(\varnothing) = 0$ and $\mu(\mathcal{U}) = 1$. We will say that the fuzzy measure μ* is invariant with respect to the cardinality*, if the following condition holds:*

$$\textit{If } |A| = |B| \textit{ then } \mu(A) = \mu(B), \textit{ for all } A, B \in \mathrm{P}(\mathcal{U})$$

where $|\cdot|$ denotes the cardinality of a set.

An example of fuzzy measure invariant with respect to the cardinality is given by the mapping $\mu_{\mathrm{rc}}\colon \mathrm{P}(\mathcal{U}) \to [0,1]$ defined as $\mu_{\mathrm{rc}}(A) = \frac{|A|}{|\mathcal{U}|}$, for all $A \in \mathrm{P}(\mathcal{U})$. Notice that, if $\varphi\colon [0,1] \to [0,1]$ is an increasing mapping such that $\varphi(0) = 0$ and $\varphi(1) = 1$, then the mapping $\mu_\varphi\colon \mathrm{P}(\mathcal{U}) \to [0,1]$ defined as $\mu_\varphi(A) = \varphi(\mu_{rc}(A))$, for all $A \in \mathrm{P}(\mathcal{U})$, is also a fuzzy measure invariant with respect to the cardinality. The former fuzzy measure μ_{rc} is called *relative cardinality* whereas the latter fuzzy measure μ_φ is called *relative cardinality modified by φ* or simply *modified relative cardinality*. Both fuzzy measures were exemplified in [25].

Definition 6. *Let $\mathcal{U}$ be a non-empty finite universe, $\mathrm{F}(\mathcal{U}) = [0,1]^{\mathcal{U}}$ be the set of fuzzy sets of $\mathcal{U}$ on $[0,1]$, $\mathrm{P}(\mathcal{U})$ be the powerset of $\mathcal{U}$, $\mu\colon \mathrm{P}(\mathcal{U}) \to [0,1]$ be a fuzzy measure invariant with respect to the cardinality and $(\&, \swarrow, \nwarrow)$ be an adjoint triple w.r.t $([0,1], \leq)$ such that $x \mathbin{\&} 1 = 1 \mathbin{\&} x = x$, for all $x \in [0,1]$.*

- *A mapping $\mathrm{Q}_\mu\colon \mathrm{F}(\mathcal{U}) \to [0,1]$ defined, for all $C \in \mathrm{F}(\mathcal{U})$, as:*

$$\mathrm{Q}_\mu(C) = \bigvee_{D \in \mathrm{P}(\mathcal{U}) \setminus \{\varnothing\}} \left(\left(\bigwedge_{u \in D} C(u) \right) \mathbin{\&} \mu(D) \right) \tag{3}$$

  is called right quantifier determined by the fuzzy measure μ.
- *A mapping ${}_\mu\mathrm{Q}\colon \mathrm{F}(\mathcal{U}) \to [0,1]$ defined, for all $C \in \mathrm{F}(\mathcal{U})$, as:*

$${}_\mu\mathrm{Q}(C) = \bigvee_{D \in \mathrm{P}(\mathcal{U}) \setminus \{\varnothing\}} \left(\mu(D) \mathbin{\&} \left(\bigwedge_{u \in D} C(u) \right) \right) \tag{4}$$

  is called left quantifier determined by the fuzzy measure μ.

From now on, all results will be formulated with respect to a right quantifier determined by a fuzzy measure μ. For that reason, the right quantifier will be called simply quantifier. Notice that, all presented results can be translated analogously for a left quantifier.

The following proposition shows that the universal and existential quantifiers can be obtained from Definition 6 considering the minimum and maximum fuzzy measures $\mu_\forall$ and $\mu_\exists$, respectively, which are defined as follows:

$$\mu_\forall(D) = \begin{cases} 1 & \text{if } D = \mathcal{U} \\ 0 & \text{otherwise} \end{cases} \qquad \mu_\exists(D) = \begin{cases} 0 & \text{if } D = \varnothing \\ 1 & \text{otherwise} \end{cases}$$

Proposition 2. *Given a non-empty finite universe $\mathcal{U}$, the quantifiers $\mathrm{Q}_\forall$ and $\mathrm{Q}_\exists$ determined by the minimum and maximum fuzzy measures $\mu_\forall$ and $\mu_\exists$, respectively, we have that $\mathrm{Q}_\forall$ and $\mathrm{Q}_\exists$ represent the universal and existencial quantifiers. That is, for all $C \in \mathrm{F}(\mathcal{U})$, the following equalities are satisfied:*

$$\mathrm{Q}_\forall(C) = \bigwedge_{u \in \mathcal{U}} C(u)$$
$$\mathrm{Q}_\exists(C) = \bigvee_{u \in \mathcal{U}} C(u)$$

From a computational point of view, the notion of quantifier given in Equation (3) of Definition 6 is not suitable. This fact is due to the computation over all sets from $\mathrm{P}(\mathcal{U}) \setminus \{\varnothing\}$ is required. In order to compute the quantifier in a more efficient way, we propose an alternative computation procedure which makes use of the property of being invariant with respect to the cardinality of fuzzy measures.

Theorem 1. *Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be a universe, with $|\mathcal{U}| = n$, and Q_μ be a quantifier determined by a fuzzy measure μ invariant with respect to the cardinality. Then,*

$$\mathrm{Q}_\mu(C) = \bigvee_{i=1}^{n} C(u_{\pi(i)}) \mathbin{\&} \mu(\{u_1, ..., u_i\}), \quad C \in \mathrm{F}(\mathcal{U})$$

where π is a permutation on $\{1, 2, \ldots, n\}$ such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \cdots \geq C(u_{\pi(n)})$.

As a direct consequence of Theorem 1, the following result proposes to compute the quantifier considering a fuzzy measure built from the relative cardinality, which was illustrated at the beginning of Section 3.

Corollary 1. *Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be a universe, $\varphi\colon [0,1] \to [0,1]$ be an increasing mapping such that $\varphi(0) = 0$, $\varphi(1) = 1$ and μ be a fuzzy measure built from the relative cardinality, by using φ. Then,*

$$\mathrm{Q}_\mu(C) = \bigvee_{i=1}^{n} C(u_{\pi(i)}) \mathbin{\&} \varphi(i/n), \quad C \in \mathrm{F}(\mathcal{U})$$

where π is a permutation on $\{1,2,\ldots,n\}$ such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \cdots \geq C(u_{\pi(n)})$.

After introducing a generalization of the definition of generalized quantifier in the multi-adjoint concept lattices framework and showing its main properties, we will focus on using the extended definition to present a generalization of the usual fuzzy concept-forming operators.

Definition 7. *Given a multi-adjoint frame and a context for that frame, μ_A, μ_B two fuzzy measures on A and B, respectively, which are invariant with respect to the cardinality and Q_A, Q_B two quantifiers determined by the fuzzy measures μ_A and μ_B, respectively, the* quantified concept-forming operators *are denoted as ${}^{\uparrow_{\mathrm{Q}_A}}\colon [0,1]^B \longrightarrow [0,1]^A$ and ${}^{\downarrow^{\mathrm{Q}_B}}\colon [0,1]^A \longrightarrow [0,1]^B$, where $[0,1]^B$ and $[0,1]^A$ denote the set of fuzzy subsets $g\colon B \to [0,1]$ and $f\colon A \to [0,1]$, respectively, and are defined, for all $g \in [0,1]^B$, $f \in [0,1]^A$ and $a \in A$, $b \in B$, as:*

$$g^{\uparrow_{\mathrm{Q}_B}}(a) = \bigvee_{X \in \mathrm{P}(B)\setminus\{\varnothing\}} \left(\bigwedge_{b \in X} R(a,b) \swarrow^{\sigma(a,b)} g(b) \right) \& \ \mu_B(X) \tag{5}$$

$$f^{\downarrow^{\mathrm{Q}_A}}(b) = \bigvee_{Y \in \mathrm{P}(A)\setminus\{\varnothing\}} \left(\bigwedge_{a \in Y} R(a,b) \nwarrow_{\sigma(a,b)} f(a) \right) \& \ \mu_A(Y) \tag{6}$$

Applying Corollary 1, we obtain the following characterization of the quantified concept-forming operators, where the fuzzy measures μ_A and μ_B are defined from two fuzzy sets φ_A and φ_B, as follows:

$$\mu_A(\{a_1,...,a_j\}) = \varphi_A(j/m)$$

$$\mu_B(\{b_1,...,b_i\}) = \varphi_B(i/n)$$

for all subset $\{a_1,...,a_j\} \subseteq A$, $\{b_1,...,b_i\} \subseteq B$, with $|A| = m$ and $|B| = n$.

Proposition 3. *In the framework of Definition 7, the quantified concept-forming operators ${}^{\uparrow_{\mathrm{Q}_A}}\colon [0,1]^B \longrightarrow [0,1]^A$ and ${}^{\downarrow^{\mathrm{Q}_B}}\colon [0,1]^A \longrightarrow [0,1]^B$, satisfy:*

$$g^{\uparrow_{\mathrm{Q}_B}}(a) = \bigvee_{i=1}^{n} \left(R(a,b_{\pi(i)}) \swarrow^{\sigma(a,b_{\pi(i)})} g(b_{\pi(i)}) \right) \& \ \varphi_B(i/n) \tag{7}$$

$$f^{\downarrow^{\mathrm{Q}_A}}(b) = \bigvee_{j=1}^{m} \left(R(a_{\lambda(j)},b) \nwarrow_{\sigma(a_{\lambda(j)},b)} f(a_{\lambda(j)}) \right) \& \ \varphi_A(j/m) \tag{8}$$

for all $g \in [0,1]^B$, $f \in [0,1]^A$ and $a \in A$, $b \in B$, where π and λ are permutations such as:

$$R(a,b_{\pi(i+1)}) \swarrow^{\sigma(a,b_{\pi(i+1)})} g(b_{\pi(i+1)}) \leq R(a,b_{\pi(i)}) \swarrow^{\sigma(a,b_{\pi(i)})} g(b_{\pi(i)})$$
$$R(a_{\lambda(j+1)},b) \nwarrow_{\sigma(a_{\lambda(j+1)},b)} f(a_{\lambda(j+1)}) \leq R(a_{\lambda(j)},b) \nwarrow_{\sigma(a_{\lambda(j)},b)} f(a_{\lambda(j)})$$

The next result shows that the quantified concept-forming operators ${}^{\uparrow_{Q_A}}$ and ${}^{\downarrow^{Q_B}}$ form an antitone Galois connection when the quantifiers Q_A and Q_B are determined by the minimum fuzzy measure $\mu_\forall$.

Proposition 4. *Given the quantifiers $Q_A^\forall$ and $Q_B^\forall$ determined by the minimum fuzzy measure $\mu_\forall$ on the universes A and B, respectively, then the pair $({}^{\uparrow_{Q_B^\forall}}, {}^{\uparrow_{Q_A^\forall}})$ is an antitone Galois connection.*

However, this result does not need to be satisfied for arbitrary quantifiers. Hence, it would be interesting to study sufficient conditions to ensure that the pair $({}^{\uparrow_{Q_B}}, {}^{\uparrow_{Q_A}})$ be an antitone Galois connection.

4 Conclusions and future work

In this paper, we have introduced the monadic quantifiers of type $\langle 1 \rangle$ determined by fuzzy measures [15] in the general framework of multi-adjoint lattices. Moreover, we have proven that the new definitions also have the universal and existential quantifiers as particular cases. The second important result is the simplification of the generalized quantifier definition taking into account that fuzzy measures are invariant with respect to the cardinality. This result allows to develop more simple and efficient proofs in this framework. Finally, the natural and simple expressions of the quantified concept-forming operators have been presented, and we have also shown that they form a Galois connection when the minimum fuzzy measure is considered. However, this is not true in general. Hence, in the future, sufficient conditions or new definitions will be studied in order to provide Galois connections.

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