Existence of mild solution for stochastic differential equations with fractional derivative driven by multiplicative noise

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Abstract
This paper focuses on the study of the existence of a mild solution to time and space-fractional stochastic equation perturbed by multiplicative white noise. The required results are obtained by means of Krasnoselskii's fixed point theorem.

Keywords
stochastic equation, mild solution, Expectation, Krasnoselskii' s fixed point theorem.

1. Introduction

In this paper, we are interested in the existence of solutions for nonlinear fractional difference equations

$$^{c}D_{0^+}^{\alpha}\left[u-h\left(u\right)\right]=\Delta u\left(t\right)+u\cdot\nabla u+g\left(u\right)W\left(t\right),\quad x\in D,\ t>0, \tag{1}$$

subject to the initial condition

$$u\left(x,0\right)=u_0\left(x\right),\ x\in D,\ t=0, \tag{2}$$

and the Dirichlet boundary conditions

$$u\left(x,t\right)=0,\ x\in\partial D, \tag{3}$$

where $D\subset\mathbb{R}^d$, $u\left(x,t\right)$ represents the velocity field of the fluid, the state $u\left(\cdot\right)$ takes values in a separable real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$, the term $g\left(u\right)W\left(t\right)=\frac{d}{dt}W\left(t\right)$ describes a state dependent random noise, where $W\left(t\right)_{t\in[0,T]}$ is a F_t-adapted Wiener process defined in completed probability space (Ω,F,P) with expectation E and associate with the normal filtration $F_t=\sigma\left\{W\left(s\right)\ :\ 0\leq s\leq t\right\}$. The operator Δ is the Laplacian. Here, $^{c}D_t^{\alpha}$ denotes the Caputo type derivative of order α $(0<\alpha<1)$ for the function $u\left(x,t\right)$with respect time t which is defined by

$$\begin{cases} ^{c}D_t^{\alpha}u\left(x,t\right)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^{\alpha}}, & 0<\alpha<1,\\ \frac{\partial u(x,t)}{\partial t}, & \alpha=1,\end{cases} \tag{4}$$

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where $\Gamma(\cdot)$ stands for the Gamma function.

The existence and non-existence of solutions for the Navier-Stokes equations (NSEs) have been discussed in [13]. Chemin et al. [5] studied the global regularity for the large solutions to the NSEs. Miura [19] focused on the uniqueness of mild solutions to the NSEs. Germain [9] presented the uniqueness criteria for the solutions of the Cauchy problem associated to the NSEs. However, The existence and uniqueness of solutions for the stochastic Navier-Stokes equations (SNSEs) with multiplicative Gaussian noise were proved in [8], [20]. The large deviation principle for SNSEs with multiplicative noise had been established in [23], [27], [31], [32]. Just mention a few, the study of time-fractional Navier-Stokes equations has become a hot topic of research due to its significant role in simulating the anomalous diffusion in fractal media [12], [16], [33].

There has been a widespread interest during the last decade in constructing a stochastic integration theory with respect to fractional Brownian motion (FBM) and solving stochastic differential equations driven by FBM. On the other hand, time-fractional differential equations are found to be quite effective in modelling anomalous diffusion processes as its can characterize the long memory processes [22], [25], [28], [29], [30]. Hence, Burgers equation with time-fractional can be adapted to describe the memory effect of the wall friction through the boundary layer [35]. Furthermore, the analytical solutions of the time- and space-fractional Burgers equations have been investigated by variational iteration method [11] and Adomian decomposition method [21].

The existence of solution for partial neutral integro-differential equation with infinite delay in infinite dimensional spaces has been extensively studied by many authors. Ezzinbi and al. [7] investigated the existence and regularity of solutions for some partial functional integro-differential equations in Banach spaces. Cui and Yan [4] investigated the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of Sadovskii's fixed point theorem and in another paper [1], Balasubramaniam et al. discussed the existence of mild and strong solutions of semilinear neutral functional differential evolution equations with nonlocal conditions by using fractional power of operators and Krasnoselskii fixed point theorem, Djourdem and Bouteraa [6] studied the existence of a mild solution to time and space-fractional stochastic equation perturbed by multiplicative with noise via Sadovskii's fixed point theorem. In particular, the stability theory of stochastic differential equations has been popularly applied in variety fields of science and technology. Several authors have established the stability results of mild solutions for these equations by using various techniques, we refer the reader to [2], [10], [17], [26].

The main contribution of this paper is to establish the existence of mild solution for the problem (1)-(3). Using mainly the Krasnoselskii's fixed point theorem. The rest f paper organised as follows, In Section 2, we will introduce some notations and preliminaries, which play a crucial role in our theorem analysis. In Section 3, the existence results on a mild solutions are derived.

2. Preliminaries

In this section, we give some notions and certain important preliminaries, which will be used in the subsequent discussions. Let $(\Omega, F, P, \{F\}_{t\geq 0})$ be a filtered probability space with a normal filtration, where P is a probability measure on (Ω, F) and F is the Borel $\sigma-$algebra . Let $\{F\}_{t\geq 0}$ satisfying that F_0 contains all P-null sets. Theoperator A is the infinitesimal generator of a strongly continuous semigroup on a separable real Hilbert space H.
Denote the basic functional space $L^p(D),\ 1 \leq p < \infty$ and $H^s(D)$ by the usual Lebesgue and Sobolev space, respectively. We assume that A is the negative Laplacian $-\Delta$ in a bounded domain with zero Dirichlet boundary conditions in Hilbert space $H = L^2(D)$, which are given by

$$A = -\triangle,\ D(A) = H_0^1(D) \cap H^2(D),$$

since the operator A is self-adjoint, i.e., there exist the eigenvectors e_k corresponding to eigenvalues λ_k such that

$$Ae_k = \lambda_k e_k,\ e_k = \sqrt{2} sin(k\pi),\ \lambda_k = \pi^2 k^2,\ k \in \mathbb{N}^+.$$

For any $\sigma > 0$, let H^σ be the domain of the fractional power $A^{\frac{\sigma}{2}} = (-\Delta)^{\frac{\sigma}{2}}$, which can be defined by

$$\sigma > 0, \quad A^{\frac{\sigma}{2}} e_k = \gamma_k^{\frac{\sigma}{2}} e_k,\ k = 1, 2, \ldots$$

and

$$H^\sigma = D\left(A^{\frac{\sigma}{2}}\right) = \left\{ v \in L^2(D),\ s.t.\ \|v\|_{H^\sigma}^2 = \sum_{k=1}^{\infty} \gamma_k^{\frac{\sigma}{2}} v_k^2 < \infty \right\},$$

where $v_k = \langle v, e_k \rangle$ with the inner product $\langle \cdot, \cdot \rangle$ in $L^2(D)$, the norm $\|H^\sigma v\| = \left\|A^{\frac{\sigma}{2}} v\right\|$, the bilinear operator $B(u, v) = u \cdot \nabla v$ and $\mathcal{D}(B) = H_0^1(D)$ with the slight abuse of notation $B(u) = B(u, u)$. Then we can rewrite the equation (1)-(3) as follows in the abstract form

$$\begin{cases} {}^cD_t^\alpha [u(t) - h(u(t))] = Au(t) + B(u(t)) + g(u(t)) \frac{W(t)}{dt},\ t > 0, \\ \qquad u(0) = u_0, \end{cases} \tag{5}$$

where $\{W(t),\ t \geq 0\}$ is a Q-Wiener process with linear bounded covarience operator Q such that a trace class operator Q denote $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots$, then the Wiener process is given by

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k,$$

where $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of real-valued standard Brownian motions.

Let $L_0^2 = L^2\left(Q^{\frac{1}{2}}(H), H\right)$ be a Hilbert-Schmidt space of operators from $Q^{\frac{1}{2}}(H)$ to H with the norm

$$\|\phi\|_{L_0^2} = \left\|\phi Q^{\frac{1}{2}}\right\|_{H^\sigma} = \left(\sum_{n=1}^{\infty} \phi Q^{\frac{1}{2}} e_n\right)^{\frac{1}{2}},$$

i.e.,

$$L_0^2 = \left\{ \phi \in L(H) \; : \; \sum_{n=1}^{\infty} \left\| \lambda_n^{\frac{1}{2}} \phi Q^{\frac{1}{2}} e_n \right\|^2 < \infty \right\},$$

where $L(H)$ is the space of bounded linear operators from H to H.
For an arbitrary Banach space B, we denote

$$\|v\|_{L^p(\Omega,B)} = \left(E \|v\|_B^p \right)^{\frac{1}{p}}, \; \forall v \in L^p(\Omega, F, P, B), \; for \; any \; p \geq 2.$$

We shall also need the following result with respect to the operator A (see [28]).

For any $\nu > 0$, an analytic semigroup $T(t) = e^{-tA}, \; t \geq 0$ is generated by the operator A on L^p, there exists a constant C_ν dependent on ν such that

$$\|AT(t)\|_{L(L^p)} \leq C_\nu t^{-\nu}, \; t > 0,$$

in which $L(B)$ denotes the Banach space of all bounded operators from B to itself.

Next we will introduce the following lemma to estimate the stochastic integrals, which contains the Burkhoder-Davis-Gundy's inequality.

[15] For any $0 \leq t_1 < t_2 \leq T$ and $p \geq 2$ and for any predictable stochastic process $v \; : [0,T] \times \Omega \longrightarrow L_0^2$ which satisfies

$$E\left[\left(\int_0^T \|v(s)\|_{L_0^2}^2 \, ds \right)^{\frac{p}{2}} \right] < \infty,$$

then, we have

$$E\left[\left\| \int_{t_1}^{t_2} v(s) \, dW(s) \right\|^p \right] < C(p) \, E\left[\left(\int_{t_1}^{t_2} \|v(s)\|_{L_0^2}^2 \, ds \right)^{\frac{p}{2}} \right].$$

Ispired by the definition of the mild solution to the time-fractional differential equations (see [24], [34]), we give the following definition of mild solution for our time-fractional stochastic equation.

An F_t-adapted stochastic process $(u(t), \; t \in [0,T])$ is called a mild solution to (5) if the following integral equation is satisfied

$$u(t) = E_\alpha(t) u_0 + h(u(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s)) \, ds$$
$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u(s)) \, dW(s), \tag{6}$$

where the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are defined, respectively, by

$$E_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta)\, T(t^\alpha \theta)\, d\theta,$$

and

$$E_{\alpha,\alpha}(t) = \int_0^\infty \alpha\theta\zeta_\alpha(\theta)\, T(t^\alpha \theta)\, d\theta,$$

where $T(t) = e^{-tA},\ \ t \geq 0$ is an analytic semi group generated by the operator $-A$ and the Mainardi's Wright-type function with $\alpha \in (0,1)$ is given by

$$\zeta_\alpha(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^k}{k!\Gamma(1-\alpha(1+k))}.$$

[3] For any $\alpha \in (0,1)$ and $-1 < \nu < \infty$, it is not difficult to verity that

$$\zeta_\alpha(\theta) \geq 0 \ and \ \int_0^\infty \theta^\nu \zeta_\alpha(\theta)\, d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \tag{7}$$

for all $\theta \geq 0$.

The operators and $\{E_\alpha(t)\}_{t\geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t\geq 0}$ in (7) have the following properties.

For any $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are linear and bounded operators. Moreover, for $0 < \alpha < 1$ and $0 \leq \nu < 2$, there exists a constant $C > 0$ such that $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are defined, respectively, by

$$\|E_\alpha(t)\chi\|_{H^\nu} \leq Ct^{-\frac{\alpha\nu}{2}}\|\chi\|, \quad \left\|E_{\alpha,\alpha}(t)\chi\right\|_{H^\nu} \leq Ct^{-\frac{\alpha\nu}{2}}\|\chi\|. \tag{8}$$

For $T > 0$ and $0 \leq \nu < 2$, by means of Lemma 2 and Lemma 2, we have

$$\begin{aligned}
\|E_\alpha(t)\chi\|_{H^\nu} &\leq \int_0^\infty \zeta_\alpha(\theta)\|A_\nu T(t^\alpha\theta)\chi\|\, d\theta \\
&\leq \int_0^\infty C_\nu t^{-\frac{\alpha\nu}{2}}\theta^{-\nu}\zeta_\alpha(\theta)\|\chi\|\, d\theta \\
&= \frac{C_\nu\Gamma(1-\nu)}{\Gamma(1-\alpha\nu)} t^{-\frac{\alpha\nu}{2}}\|\chi\|, \quad \chi \in L^2(D),
\end{aligned}$$

and

$$\begin{aligned}
\left\|E_{\alpha,\alpha}(t)\chi\right\|_{H^\nu} &\leq \int_0^\infty \alpha\theta\zeta_\alpha(\theta)\|A_\nu T(t^\alpha\theta)\chi\|\, d\theta \\
&\leq \int_0^\infty C_\nu \alpha t^{-\frac{\alpha\nu}{2}}\theta^{1-\nu}\zeta_\alpha(\theta)\|\chi\|\, d\theta
\end{aligned}$$

$$= \frac{C_\nu \alpha \Gamma(2-\nu)}{\Gamma(1-\alpha\nu)} t^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \chi \in L^2(D),$$

so, $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are linear and bounded operators. The proof is completed.

For any $t > 0$, the operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are strongly continuous. Moreover, for $0 < \alpha < 1$ and $0 \le \nu < 2$ and $0 \le t_1 < t_2 \le T$, there exists a constant $C > 0$ such that

$$\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} \le C(t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \tag{9}$$

and

$$\left\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\right\|_{H^\nu} \le C(t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|. \tag{10}$$

For any $0 < T_0 \le t_1 < t_2 \le T$, it iseasy to deduce that

$$\int_{t_1}^{t_2} \frac{dT(t^\alpha\theta)}{dt} dt = T(t_2^\alpha\theta) - T\left(t_1^\theta\right)$$

$$= \int_{t_1}^{t_2} \alpha t^{\alpha-1} \theta A T(t^\alpha\theta)\, dt,$$

and by (7) and Lemma 2, we have

$$\begin{aligned}
\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} &= \|A_\nu(E_\alpha(t_2) - E_\alpha(t_1))\chi\| \\
&= \left\| \int_0^\infty \zeta_\alpha(\theta) A_\nu\left(T(t_2^\alpha\theta) - T\left(t_1^\theta\right)\right)\chi d\theta \right\| \\
&\le \int_0^\infty \alpha\theta\zeta_\alpha(\theta) \int_{t_1}^{t_2} t^{\alpha-1} \|A_{2+\nu} T(t^\alpha\theta)\chi\|_{L^2}\, dt d\theta \\
&\le \int_0^\infty C_\nu \alpha \theta^{-\frac{\nu}{2}} \zeta_\alpha(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1} dt \right) \|\chi\|\, d\theta \\
&= \frac{2C_\nu \Gamma\left(1-\frac{\nu}{2}\right)}{\nu\Gamma\left(1-\frac{\alpha\nu}{2}\right)} \left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}} \right) \|\chi\| \\
&\le \frac{2C_\nu \Gamma\left(1-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu}\Gamma\left(1-\frac{\alpha\nu}{2}\right)} (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \ \chi \in L^2(D).
\end{aligned}$$

Also

$$\begin{aligned}
\left\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\right\|_{H^\nu} &= \left\|A_\nu(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\right\| \\
&= \left\| \int_0^\infty \alpha\theta\zeta_\alpha(\theta) A_\nu\left(T(t_2^\alpha\theta) - T\left(t_1^\theta\right)\right)\chi d\theta \right\|
\end{aligned}$$

$$\leq \int_0^\infty \alpha^2\theta^2\zeta_\alpha(\theta)\int_{t_1}^{t_2} t^{\alpha-1}\left\|A_{2+\nu}T\left(t^\alpha\theta\right)\chi\right\|_{L^2} dtd\theta$$

$$\leq \int_0^\infty C_\nu\alpha^2\theta^{1-\frac{\nu}{2}}\zeta_\alpha(\theta)\left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1}dt\right)\|\chi\|\, d\theta$$

$$= \frac{2\alpha C_\nu\Gamma\left(2-\frac{\nu}{2}\right)}{\nu\Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)}\left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}}\right)\|\chi\|$$

$$\leq \frac{2C_\nu\Gamma\left(2-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu}\Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)}\left(t_2-t_1\right)^{\frac{\alpha\nu}{2}}\|\chi\|,\ \chi\in L^2(D).$$

It is obviously to see that the term

$$\left\|\left(E_\alpha(t_2)-E_\alpha(t_1)\right)\chi\right\|_{H^\nu} \longrightarrow 0,$$

and

$$\left\|\left(E_{\alpha,\alpha}(t_2)-E_{\alpha,\alpha}(t_1)\right)\chi\right\|_{H^\nu} \longrightarrow 0,$$

as $t_1 \longrightarrow t_2$ which mean that the operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are strongly continuous.

3. Existence results

In this section, we present our main results on the existence of mild solutions of problem (5) and we define the following space

$$K = \left\{ u\ :\ u\in C\left([0,T],H^\nu\right),\ \sup_{t\in[0,T]}\|u\| < \infty \right\}.$$

To do this, we make the following hypotheses:
(H_1) A is the infinitesimal generator of a strongly continuous semigroup $\{T(t),\ t\geq 0\}$ on H. We will also suppose that the operator $E_\alpha(t),\ t>0$ is compact.
(H_2) The function $g\ :\ \Omega\times H \longrightarrow L_0^2$ satisfies the following global Lipshitz and growth conditions:

$$\|g(v)\|_{L_0^2} \leq C\|u\|,\ \|g(u)-g(v)\|_{L_0^2} \leq C\|u-v\|,$$

for any $u\in H,\ v\in H$.

(H_3) The initial value $u_0\ :\ \Omega \longrightarrow H^\nu$ is a F_0- measurable random variable, it hold that

$$\|u_0\|_{L^p(\Omega,H^\nu)} < \infty,\ for\ any\ 0\leq\nu<\alpha<2.$$

(H_4) The function $h\ :\ L_0^2 \longrightarrow L_0^2$ is continuous and there exists $L_h>0$ such that

$$E\left\|h(u_1(t))-h(u_2(t))\right\|_{L_0^2}^p \leq L_h\left\|u_1(t)-u_2(t)\right\|_{L_0^2}^p,\ t\in[0,T],\ u_1,\ u_2\in L_0^2,$$

and

$$E\left\|h\left(u\left(t\right)\right)\right\|_{L_0^2}^p \leq L_h E\left\|u\left(t\right)\right\|_{L_0^2}^p,\ t\in[0,T],\ u\in L_0^2.$$

(H_5) Let $C>0$ be a real number, then the bounded bilinear operator $B\ :\ L^2(D)\longrightarrow H^{-1}(D)$ satisfies the following properties

$$\left\|B\left(u\right)\right\|_{H^{-1}}\leq C\left\|u\right\|^2,$$

and

$$\left\|B\left(u\right)-B\left(v\right)\right\|_{H^{-1}}\leq C\left(\left\|u\right\|+\left\|v\right\|\right)\left\|u-v\right\|,$$

for any $u,\ v\in L^2(D)$.
Our main results is based on the following Krasnoselskii fixed points theorem [14]. [14] Let X be a Banach space, C a closed, bounded, convex and nonemty subset of X. Consider the operators F_1 and F_2 such that
(i) $F_1u+F_2v\in C$ whenever $u,\ v\in C$,
(iii) F_1 is a contraction mapping.
(ii) F_2 is compact and continuous.
Then there exists $z\in C$ such that $z=F_1z+F_2z$.

In the proof of main result, we need the following Lemmas. Assume that conditions (H_1) and (H_2) hold. Let Φ_1 and Φ_2 be two operators defined respectively for each $u\in K$ by

$$\begin{aligned}\Phi_1(u)&=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(t-s)B(u(s))\,ds,\\ \Phi_2(u)&=\int_0^t S_\alpha(t-s)f(u(s))\,dW(s).\end{aligned}\tag{11}$$

Then Φ_1 and Φ_2 are continuous and map K into itself.
It is obvious that Φ_1 is continuous. Next we show that $\Phi_1(K)\subset K$. By (H_1) and (H_2), from the equation (11) and by applying Holder inequality, we have

$$\begin{aligned}E\left\|(\Phi_1u)(t)\right\|_{H^\nu}^p&=E\left\|\int_0^t (t-s)^{\alpha-1}A_1E_{\alpha,\alpha}(t-s)A_{\nu-1}B(u(s))\,ds\right\|_{H^\nu}^p\\ &\leq C_\alpha^p\left(\int_0^t (t-s)^{\frac{p\left(\frac{\alpha-1}{2}\right)}{p-1}}ds\right)^{p-1}\int_0^t E\left[\left\|A_{\nu-1}B(u(s))\right\|^p\right]ds\\ &\leq C^pC_\alpha\left[\frac{2(p-1)}{p-2}\right]^{p-1}(T)^{\frac{p-2}{2}}\int_0^t E\left[\left\|u(t)\right\|_{H^\nu}^p\right]\\ &=\gamma_1\int_0^t E\left[\left\|u(s)\right\|_{H^\nu}^p\right]ds\qquad,\end{aligned}\tag{12}$$

where $\gamma_1 = C^p C_\alpha \left[\frac{2(p-1)}{p-2}\right]^{p-1} (T)^{\frac{p-2}{2}}$. This complete the proof.

By using also the Holder inequality and Lemma 2, we obtain

$$E\left\|(\Phi_2 u)(t)\right\|_{H^\nu}^p = E\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\, g(u(s))\, dW(s)\right\|_{H^\nu}^p$$

$$\leq C(p)\, E\left[\left(\int_0^t \left\|(t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\right\|^2 \|A_\nu g(u)\|_{L_0^2}^2\, ds\right)^{\frac{p}{2}}\right]$$

$$\leq C(p)\, C_\alpha^p \left(\int_0^t (t-s)^{\frac{2p(\alpha-1)}{p-2}}\right)^{\frac{p-2}{2}} \int_0^t E\|A_\nu g(u)\|_{L_0^2}^p\, ds$$

$$\leq C(p)\, C_\alpha^p \left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p-2}{2}} \int_0^t E\|A_\nu g(u)\|_{L_0^2}^p\, ds$$

$$= \gamma_2 \int_0^t E\left[\|u(s)\|_{H^\nu}^p\right] ds, \tag{13}$$

where $\gamma_2 = C(p)\, C_\alpha^p C^p \left[\frac{p-2}{p(2\alpha-1)-2}\right]^{\frac{p-2}{2}}$.
That is $\Phi_2(K) \subset K$.

Assume that conditions (H_1) and (H_4) hold. Let Φ_3 be the operator defined by for each $u \in K$

$$(\Phi_3 u)(t) = E_\alpha(t)\, u_0 + h(u(t)).$$

Then Φ_3 is continuous and maps K into K. The continuity of Φ_3 follows from (H_4). Next, we show that $\Phi_3(Y) \subset Y$. By (H_1), (H_5) and from (13), we have

$$E\|(\Phi_3 u)(t)\|_{L_0^2}^p \leq E\|h(u(t))\|_{L_0^2}^p \leq L_h E\|u(t)\|_{L_0^2}^p.$$

So, we conclude $\Phi_3(K) \subset K$.

Assume that conditions (H_1) and (H_2) hold. Then

$$E\left[\|E_\alpha(t)\, u_0\|_{H^\nu}\right] \leq E\left[\|u_0\|_{H^\nu}\right].$$

By Lemma 2, we have

$$E\left[\|E_\alpha(t)\, u_0\|_{H^\nu}\right] \leq E\left[\int_0^\infty \zeta_\alpha(\theta) \left(\|A_\nu T(t^\alpha \theta)\, u_0\|^2\right)^{\frac{1}{2}} d\theta\right]$$

$$\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\sum_{n=1}^\infty \left\langle A_\nu e^{-t^\alpha \theta A} u_0, e_n\right\rangle^2\right)^{\frac{1}{2}} d\theta\right]$$

$$\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\sum_{n=1}^\infty \left\langle A_\nu u_0, e^{-t^\alpha \theta \lambda_n^{\frac{\nu}{2}}}, e_n\right\rangle^2\right)^{\frac{1}{2}} d\theta\right]$$

$$\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\|u_0\|_{H^\nu}\, d\theta\right] = E\left[\|u_0\|_{H^\nu}\right].$$

First, we define a map $F : K \longrightarrow C([0,T], H^\nu)$ in the following manner: for any $u \in K$,

$$(Fu)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(u(s))\, dW(s)$$

$$+E_\alpha(t) u_0(s) + h(u(t)).$$

Now, we set $F = F_1 + F_2$, where

$$(F_1 u)(t) = E_\alpha(t) u_0(s) + h(u(t)),$$

and

$$(F_2 u)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(u(s))\, dW(s),$$

for $t \in [0,T]$.

Assume $(H_2), (H_4), (H_5)$ hold and $0 < \nu < \alpha \leq 2,\ p \geq 2$, Then

$$E\|E_\alpha(t_2) - E_\alpha(t_1)\|_{H^\nu}^p \leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E\|u_0\|^p.$$

We set

$$I_1 = F_1(t_2) - F_1(t_1) = E_\alpha(t_2) u_0 - E_\alpha(t_2) u_0$$

For any $p \geq 2$, by vertue of Lemma 2, it follows that

$$E\left[\|I_1\|_{H^\nu}^p\right] = E\left[A\|E_\alpha(t_2) u_0 - E_\alpha(t_2) u_0\|^p\right]$$

$$\leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E\|u_0\|^p.$$

It is obviously to see that the term $\|(F_1(t_2) - F_1(t_1))\|_Y \longrightarrow 0$ as $t_1 \longrightarrow t_2$ which mean that the operators F_1 is strongly continuous.

Assume $(H_2), (H_4), (H_5)$ hold and $0 < \nu < \alpha \leq 2,\ p \geq 2$, then the operator F_2 is uniformly bounded.

From Lemma 11, using the estimate (12) and by means of extension of Gronwall's lemma, we have

$$\sup_{t\in[0,T]} E\left[\|F_2(u(t))\|_{H^\nu}^p\right] \le \infty,$$

that is the operator F_2 is uniformly bounded.

Assume $(H_2),(H_4),(H_5)$ hold and $0 < \nu < \alpha \le 2,\ p \ge 2$. Then the operator F_2 is equicontinuous.

For any $0 \le t_1 < t_2 \le T$, from

$$(F_2u)(t_2) - (F_2u)(t_1) = \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) B(u(s))\, ds$$

$$- \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) B(u(s))\, ds + \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) g(u)\, dW(s).$$

$$- \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) g(u)\, dW(s) = I_2 + I_3, \tag{14}$$

where

$$\begin{aligned}
I_2 &= \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) B(u(s))\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) B(u)\, d(s) \\
&= \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2 - s) - E_{\alpha,\alpha}(t_1 - s)\right] B(u(s))\, ds \\
&\quad + \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right] E_{\alpha,\alpha}(t_2 - s) B(u(s))\, ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) B(u(s))\, ds \\
&= I_{21} + I_{22} + I_{23},
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
I_3 &= \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) f(u(s))\, dWs - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) f(u)\, dW(s) \\
&= \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2 - s) - E_{\alpha,\alpha}(t_1 - s)\right] f(u(s))\, dW(s)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]E_{\alpha,\alpha}(t_2-s)f(u(s))\,dW(s)\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(t_2-s)f(u(s))\,dW(s)\\
=\;& I_{31}+I_{32}+I_{33}.
\end{aligned}
\tag{16}
$$

For the first term I_{21} in (15), applying the assumptions (H_5) and Lemma 2 and Holder inequality, we have

$$
\begin{aligned}
E\left[\|I_{21}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\left[E_{\alpha,\alpha}(t_2-s)-E_{\alpha,\alpha}(t_1-s)\right]B(u(s))\,ds\right\|^p\right]\\
&\le C_{\alpha\nu}^p(t_2-t_1)^{\frac{p\alpha(\nu+1)}{2}}\left(\int_0^{t_1}(t_1-s)^{\frac{p(\alpha-1)}{p-1}}\,ds\right)^{p-1}\int_0^t E\left[\|A_{-1}B(u(s))\|_{H^1}^p\right]ds\\
&\le C^pC_{\alpha\nu}^pT^{p\alpha}\left(\frac{p-1}{p\alpha-1}\right)^{p-1}\left(\sup_{t\in[0,T]}E\left[\|u(s)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(\nu+1)}{2}}.
\end{aligned}
\tag{17}
$$

Using the assumptions (H_5) and Lemma 2 and Holder inequality, we have

$$
\begin{aligned}
E\left[\|I_{22}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-s)\right]B(u(s))\,ds\right\|^p\right]\\
&\le C_\alpha^p\left(\int_0^{t_1}\left\{\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times(t_2-s)^{\frac{-\alpha(\nu+1)}{2}}\right\}^{\frac{p}{p-1}}ds\right)^{p-1}\\
&\quad\times\int_0^t E\left[\|A_{-1}B(u(s))\|_{H^1}^p\right]ds\\
&\le C^pC_\alpha^pT\left(\frac{p-1}{p\left(\alpha-\frac{\alpha(\nu+1)}{2}\right)}\right)^{p-1}\left(\sup_{t\in[0,T]}E\left[\|u(s)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(1-\nu)-2}{2}},
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
E\left[\|I_{23}\|_{H^\nu}^p\right] &= E\left[\left\|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2-s) B(u(s))\, ds\right\|^p\right] \\
&\le C_\alpha^p \left(\int_{t_1}^{t_2} (t_2-s)^{\alpha-1-\frac{\alpha(\nu+1)}{2}}\, ds\right)^{p-1} \int_{t_1}^{t_2} E\left[\|A_{-1}B(u(s))\|_{H^1}^p\right] ds \qquad (19)\\
&\le C^p C_\alpha^p \left(\frac{p-1}{p\left(\alpha-\frac{\alpha(\nu+1)}{2}\right)-1}\right)^{p-1} \left(\sup_{t\in[0,T]} E\left[\|u(s)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(1-\nu)}{2}}.
\end{aligned}
$$

Next, by following similar arguments as in the proof of (17)-(19) and using Lemma 2 there holds,

$$
\begin{aligned}
E\left[\|I_{31}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1} (t_1-s)^{\alpha-1}\left[E_{\alpha,\alpha}(t_2-s)-E_{\alpha,\alpha}(t_1-s)\right] f(u(s))\, dWs\right\|^p\right] \\
&\le C(p) E\left[\left(\int_0^{t_1} \left\|(t_1-s)^{\alpha-1} A_\nu\left[E_{\alpha,\alpha}(t_2-s)-E_{\alpha,\alpha}(t_1-s)\right]\right\|^2 \|f(u(s))\|_{L_0^2}^2\, ds\right)^{\frac{p}{2}}\right] \\
&\le C(p) C_{\alpha\nu}^p (t_2-t_1)^{\frac{p\alpha\nu}{2}} \left(\int_0^{t_1} (t_1-s)^{\frac{2p(\alpha-1)}{p-2}}\, ds\right)^{\frac{p-2}{2}} \int_0^{t_1} E\|f(u(s))\|_{L_0^2}^p\, ds \\
&\le C^p C_{\alpha\nu}^p T^{\frac{2p\alpha-p-1}{2}} \left(\frac{p-1}{2p\alpha-p-2}\right)^{p-1} \left(\sup_{t\in[0,T]} E\left[\|u(s)\|^p\right]\right)(t_2-t_1)^{\frac{p\alpha\nu}{2}}, \qquad (20)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\|I_{32}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1} \left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-s)\right] f(u(s))\, dWs\right\|^p\right] \\
&\le C(p) E\left[\left(\int_0^{t_1} \left\|\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-s)\right]\right\|^2 \|f(u(s))\|_{L_0^2}^2\, ds\right)^{\frac{p}{2}}\right] \\
&\le C(p) C_\alpha^p \left(\int_0^{t_1} \left\{\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times(t_2-s)^{\frac{-\alpha\nu}{2}}\right\}^{\frac{2p}{p-2}}\, ds\right)^{\frac{p-2}{2}}
\end{aligned}
$$

$$\times \int_0^t E\left[\|f(u(s))\|_{L_0^2}^p\right] ds$$

$$\leq C(p)\, C^p C_\alpha^p T \left(\frac{2(p-2)}{2p\alpha(2-\nu)-2(p+2)}\right)^{\frac{p-2}{2}}$$

$$\times \left(\sup_{t\in[0,T]} E\left[\|u(t)\|^p\right]\right)(t_2-t_1)^{\frac{2p\alpha(2-\nu)-2(p+2)}{4}}, \tag{21}$$

and

$$E\left[\|I_{33}\|_{H^\nu}^p\right] = E\left[\left\|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2-s)\, B(u(s))\, ds\right\|^p\right]$$

$$\leq C(p)\, E\left[\left(\int_0^{t_1} \left\|(t_2-s)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2-s)\right\|^2 \|f(u(s))\|_{L_0^2}^2\, ds\right)^{\frac{p}{2}}\right]$$

$$\leq C(p)\, C_\alpha^p \left(\int_{t_1}^{t_2} (t_2-s)^{\alpha-1-\frac{\alpha\nu}{2}}\right)^{\frac{p-2}{2}} \times \int_{t_1}^{t_2} E\left[\|f(u(s))\|_{L_0^2}^p\right] ds$$

$$\leq C(p)\, C^p C_\alpha^p \left(\frac{2(p-2)}{2p\alpha(2-\nu)-2(p+2)}\right)^{\frac{p-2}{2}} \left(\sup_{t\in[0,T]} E\left[\|u(t)\|^p\right]\right)(t_2-t_1)^{\frac{2p\alpha(2-\nu)-2p}{4}}. \tag{22}$$

Taking expectation on the both side of (14) and in view of estimates (15) and (17) – (22), we conclude that

$$\|(F_2 u)(t_2) - (F_2 u)(t_1)\|_{L^p(\Omega, H^\nu)} \leq C(t_2-t_1)^\gamma,$$

where $\gamma = min\left\{\frac{\alpha\nu}{2}, \frac{\alpha p(1-\nu)-2}{2p}, \frac{2p\alpha(2-\nu)-2(p+2)}{4p}\right\}$ when $0 < t_2 - t_1 < 1$.
Otherwise, if $t_2 - t_1 \geq 1$, then we set $\gamma = max\left\{\frac{\alpha(\nu+1)}{2}, \frac{\alpha(2-\nu-1)}{2}, \frac{2p\alpha(2-\nu)-2p}{4p}\right\}$.

Assume the conditions (H_1) and (H_2) hold. Then F maps K into itself.
Let the nonlinear operator F defined by, for $t \geq 0$,

$$(Fu)(t) = E_\alpha(t)\, u_0 + h(u(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\, B(u(s))\, ds$$

$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\, g(u)\, dW(s).$$

We prove that the operator F has a fixed point, which is a mild solution of the problem (1)-(2). We shall employ Theorem 3. For better readability, we divide the proof into two steps.

Step 1. $F : Y \longrightarrow C([0,T], H^{\sigma})$ is continuous. Let $\{u_n(t)\}_{n\geq 0}$ with $u_n \longrightarrow u$ $(n \longrightarrow \infty)$ in Y. Then there is a number $r > 0$ such that $E\|u_n(t)\|^2_{H^{\nu}} \leq r$ for all n and $a.e.$ $t \in [0,T]$, so $u_n \in B_r(0,Y) = \left\{ u \in Y : \sup_{t\in[0,T]} \|u\|_{H^{\sigma}} \right\}$ and $u \in B_r(0,Y)$. By the assumptions (H_2) and similar argument to obtain (12) and (13), we have

$$
\begin{aligned}
&E\|(Fu_n)(t) - (Fu)(t)\|^p_{H^{\nu}} \\
&\leq 3^{p-1}\|h(u_n(t)) - h(u(t))\|^p_{H^{\nu}} + 3^{p-1}E\|\Phi_1(u_n(t) - u(t))\|^p_{H^{\nu}} \\
&\quad + 3^{p-1}E\|\Phi_2(u_n(t) - u(t))\|^p_{H^{\nu}} \\
&\leq 3^{p-1}\|h(u_n(t)) - h(u(t))\|^p_{H^{\nu}} + 3^{p-1}(G\gamma_1 + K\gamma_2)\left(\int_0^t E\|u_n - u\|^p_{H^{\nu}}\, ds \right).
\end{aligned}
$$

Then, we have for all $t \in [0,T]$,

$$\|Fu_n - Fu\|^p_Y \longrightarrow 0, \ as\ n \longrightarrow \infty.$$

Therefore F is continuous.

Step 2. We decompose F as $F = F_1 + F_2$ where F_1 and F_2 defined above.

(1) F_1 is a contraction on Y. Let $u, v \in Y$. It follows from Lemma 3 that

$$
\begin{aligned}
E\|F_1u - F_1v\|^p_{H^{\nu}} &\leq L_h E\|u(s) - v(s)\|^p_{H^{\nu}} \\
&\leq L_h \sup_{s\in[0,T]} E\|u(s) - v(s)\,ds\|^p_{H^{\nu}} \\
&\leq L_h \|u(s) - v(s)\,ds\|^p_Y
\end{aligned}
$$

Taking supremum over t

$$\|F_1u - F_1v\|^p_Y \leq L_0\|u(s) - v(s)\|^p_Y,$$

where $L_0 = L_h < 1$.
Hence F_1 is a contraction on Y.

(2) F_2 is compact operator. Let $u, v \in Y$. It follows from (H_2), (H_5) and Lemma 3 that

$$
\begin{aligned}
E\|F_2u - F_2v\|^2_{H^{\nu}} \leq 2^{p-1}E&\left\| \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(t-s)A_{\nu}[g(u(s)) - g(v(s))]\,dW(s) \right\|^2_{H^{\nu}} \\
&+2^{p-1}E\left\| \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(t-s)A_{\nu}[B(u(s)) - B(v(s))]\,ds \right\|^p_{H^{\nu}}
\end{aligned}
$$

$$\leq (\gamma_1 + \gamma_2) E \left(\int_0^t \|u - v\|_{H^\nu}^2 \, ds \right),$$

which implies

$$\sup_{t\in[0,T]} E \|F_2 u - F_2 v\|_{H^\nu}^2 = (\gamma_1 + \gamma_2) \sup_{t\in[0,T]} E \|u - v\|_{H^\nu}^2 .$$

Since $0 < L = \gamma_1 + \gamma_2 < 1$, then F is contraction maping on Y.

From Lemma 3 and Lemma 3, the operator F_2 is relatively compact. together with Ascoli's theorem, we conclude that the operator F_2 is compact.
In view of Theorem 3, we conclude that F has at least one fixed point, which is a mild solution of the problem (1)-(2).

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