# A Substructure based Lower Bound for Eternal Vertex Cover Number\*

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Abstract. The eternal vertex cover (EVC) problem is to compute the minimum number of guards to be placed on the vertices of a graph so that any sequence of attacks on its edges can be defended by dynamically reconfiguring the guards. The problem is NP hard in general and polynomial time algorithms are unknown even for simple graph classes like cactus graphs and bipartite graphs. A major difficulty is that only few lower bounds, other than the trivial lower bound of vertex cover, is known in general and the known bounds are too weak to yield useful results even for the graph classes mentioned above. We introduce the notion of substructure property in the context of the EVC problem and derive a new lower bounding technique for the problem based on the property. We apply the technique to cactus graphs and chordal graphs and obtain new algorithms for solving the eternal vertex cover problem in linear time for cactus graphs and quadratic time for a family of graphs that includes all chordal graphs and cactus graphs.

Keywords: Eternal vertex cover, Substructure property, Cactus, Chordal graphs

### 1 Introduction

Eternal vertex cover (EVC) problem is described by a two player attack-defense game played on a graph G [1] using k guards. Initially, the defender places guards on a vertex cover of G, defining an *initial configuration*. In each subsequent round, an attacker attacks an edge of her choice. In response, the defender reconfigures guards by moving them in parallel such that: a) each guard is either unmoved or moved to an adjacent vertex b) at least one guard moves across the attacked edge c) the resultant *configuration* is a vertex cover of  $G^{-1}$ . The game proceeds to the next round with this configuration. The set of all configurations encountered in the game defines an eternal vertex cover class (*evc class*) of size k for G. The *eternal vertex cover number*, evc(G), is the minimum number k of guards required

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<sup>&</sup>lt;sup>1</sup> Another version of the problem places an additional constraint that at most one guard can be on any vertex in a configuration. The results of this paper and those cited hold true in both versions.

to eternally defend any arbitrary sequence of attacks. Clearly,  $evc(G) \ge mvc(G)$ , the vertex cover number of G. An evc class of G whose configurations have evc(G) guards is called a *minimum evc class* of G.

Fomin et al. [2] showed that computation of evc(G) is NP-hard, but is in PSPACE. It is unknown whether the problem is in NP for bipartite graphs [2]. The problem remains open even for graphs of treewidth two. For any graph G, evc(G) is at most twice the size of a maximum matching in G [2] and is at most cvc(G) + 1, where cvc(G) is the minimum cardinality of a connected vertex cover of G [1]. Dynamic variants of other classical graph parameters like dominating set [3,4,5,6] and independent set [7,8] and their relationship with eternal vertex cover number [9,10] are well known in literature.

Efficient algorithms for computing evc(G) were known for elementary classes like trees, cycles, grids [1] and some simple generalized tree structures [11]. Recently, a quadratic time algorithm for biconnected chordal graphs was obtained [12]. All the graph classes for which polynomial time algorithms for eternal vertex cover has been obtained so far satisfies  $\text{evc}(G) \in \{\text{mvc}_X(G), \text{mvc}_X(G)+1\}$ , where  $\text{mvc}_X(G)$  is the minimum cardinality of a vertex cover of G that contains all cut vertices of G. In other words,  $\text{mvc}_X(G)$  is a close lower bound for evc(G)for these classes.

A cactus is a connected graph in which any two simple cycles have at most one vertex in common. The class of cactus graphs includes trees and cycles, but also contains graphs for which  $\operatorname{evc}(G) \notin \{\operatorname{mvc}_X(G), \operatorname{mvc}_X(G) + 1\}$ . In Section 6, we show an infinite family of cactus graphs for which  $\operatorname{evc}(G) > \frac{3}{2} \operatorname{mvc}_X(G)$ . This shows that known lower bound techniques fails even on simple graph classes like cactus graphs.

We formulate a new lower bounding technique based on the substructure property (Definition 4) using which efficient algorithms are obtained for the EVC problem in some graph classes violating the condition  $evc(G) \in \{mvc_X(G), mvc_X(G) + 1\}$ . This is achieved by generalizing a recursive procedure for trees by Klostermeyer and Mynhardt [1], to larger graph classes that satisfy the substructure property. The generalization yields a new quadratic time algorithm for computing evc(G) for a graph class that includes all chordal graphs and cactus graphs. It is also shown that evc(G) of cactus graphs can be computed in linear time.

### 2 Some basic observations

**Definition 1.** Let G be a graph and  $S \subseteq V(G)$ . The minimum cardinality of a vertex cover of G that contains all vertices of S is denoted by  $mvc_S(G)$ . The minimum integer k such that there is a defense strategy on G using k guards with all vertices of S being occupied in each configuration is denoted by  $evc_S(G)$ .

When  $S = \{v\}$ , we use  $\operatorname{mvc}_v(G)$  and  $\operatorname{evc}_v(G)$  respectively instead of  $\operatorname{mvc}_S(G)$ and  $\operatorname{evc}_S(G)$ . From the above definition, it is clear that  $\operatorname{evc}(G) \leq \operatorname{evc}_S(G)$ .

**Definition 2.** If G is a graph and  $x \in V(G)$ , we use  $G_x^+$  to denote the graph obtained by adding an additional vertex which is made adjacent only to x.

**Observation 1**  $\operatorname{evc}(G_x^+) \leq \operatorname{evc}(G) + 1.$ 

For proofs of some observations that are omitted from the main text, the reader may refer to the draft version [13].

**Definition 3 (x-components and x-extensions).** Let x be a cut vertex in a connected graph G and H be a component of  $G \setminus x$ . Let  $G_0$  be the induced subgraph of G on the vertex set  $V(H) \cup \{x\}$ . Then,  $G_0$  is called an x-component of G and G is called an x-extension of  $G_0$ .

**Definition 4 (Substructure property).** Let x be an arbitrary non-cut vertex of a graph G. If the following is true for any arbitrary x-extension G' of G, then G satisfies substructure property:

• if  $\operatorname{evc}(G_x^+) \leq \operatorname{evc}_x(G)$ , then in every eternal vertex cover C' of G', the number of guards on V(G) is at least  $\operatorname{evc}_x(G) - 1$  and

• if  $\operatorname{evc}(G_x^+) > \operatorname{evc}_x(G)$ , then in every eternal vertex cover C' of G', the number of guards on V(G) is at least  $\operatorname{evc}_x(G)$ .

**Observation 2** If a graph G satisfies substructure property and x is a non-cut vertex of G, then  $\operatorname{evc}_x(G) \leq \operatorname{evc}(G_x^+) \leq \operatorname{evc}_x(G) + 1$ .

**Definition 5.** Let G be a graph and x be a vertex of G. Then, G is Type 1 with respect to x if  $evc(G_x^+) = evc_x(G)$  and G is Type 2 with respect to x if  $evc(G_x^+) > evc_x(G)$ .

Remark 1. A tree is Type 1 with respect to all its vertices. An even (respectively, odd) cycle is Type 1 (respectively, Type 2) with respect to all its vertices. A complete graph  $K_n$ , with  $n \ge 3$  is Type 2 with respect to all its vertices.

*Remark 2.* By Observation 2, if G satisfies substructure property and x is a non-cut vertex of G, then G is either Type 1 or Type 2 with respect to x.

**Definition 6** ([14]). Let x be a cut vertex of a connected graph G. The set of x-components of G will be denoted as  $C_x(G)$ . For  $i \in \{1,2\}$ , we define  $T_i(G,x)$  to be the set of all x-components of G that are Type i with respect to x.

**Definition 7.** For a cut vertex x of connected graph G, we define

$$\chi(G, x) = 1 + \sum_{G_i \in T_1(G, x)} (\operatorname{evc}_x(G_i) - 2) + \sum_{G_i \in T_2(G, x)} (\operatorname{evc}_x(G_i) - 1)$$

**Lemma 1.** Let G be a connected graph and x be a cut-vertex of G such that each x-component of G satisfies the substructure property. If all x-components of G are Type 2, then  $evc(G) = evc_x(G) = \chi(G, x)$ . Otherwise,  $evc(G) = evc_x(G) = 1 + \chi(G, x)$ .

# 3 Computation of the Type of vertices

Lemma 1 indicates the possibility of a recursive method to compute eternal vertex cover number of graphs whose x-components satisfy the substructure property. However, this makes it necessary to also compute  $\operatorname{evc}_x(G)$  and the type of each x-component of G, with respect to x. Since a cut vertex of G is not a cut-vertex in its x-components, a general method to find the type of a graph with respect to any arbitrary vertex of the graph (including non-cut vertices) is necessary. This section addresses this issue systematically.

**Observation 3 (Type with respect to a cut vertex)** Let G be a connected graph and x be a cut-vertex of G such that each x-component of G satisfies the substructure property. Then, G is Type 1 with respect to x if at least one of the x-components of G is Type 1 with respect to x. Otherwise, G is Type 2 with respect to x.

*Proof.* Note that when a pendent edge xv is added to x, that edge is a Type 1 component with respect to x. Further,  $evc_x$  of this x-component is 2. Using these facts in the expressions given in Lemma 1 immediately yields the observation.  $\Box$ 

**Observation 4** Let G be a graph that satisfies substructure property and x be any vertex of G. If  $evc(G) < evc_x(G)$ , then G is Type 1 with respect to x.

*Proof.* If  $evc(G) < evc_x(G)$ , then by Observation 1, we have  $evc(G_x^+) \le evc(G) + 1 \le evc_x(G)$ . By Remark 2 and Observation 3,  $evc(G_x^+) \ge evc_x(G)$ . Therefore,  $evc_x(G) = evc(G_x^+)$ . From this, the observation follows.

The following observation is useful for deciding the type of a graph with respect to a pendent vertex.

**Lemma 2** (Type with respect to a pendent vertex). Let G be a graph and  $x \in V(G)$ . Let  $H = G_x^+$  with v being the vertex in  $V(H) \setminus V(G)$ . Suppose each x-component of H satisfies substructure property. Then, H is Type 1 with respect to v if and only if G is Type 1 with respect to x.

Proof. In any eternal vertex cover class of H with  $\operatorname{evc}_v(H)$  guards in which v is occupied in every configuration<sup>2</sup>, x must also be occupied in every configuration (otherwise, an attack on the edge vx cannot be defended maintaining a guard on v). Hence, the induced configurations on G define an eternal vertex cover class of G in which x is occupied in every configuration. It follows that  $\operatorname{evc}_v(H) \geq \operatorname{evc}_x(G) + 1$ . Moreover, from an eternal vertex cover class of G with x always occupied, we can get an eternal vertex cover class of H with v always occupied by placing an additional guard at v. Hence,  $\operatorname{evc}_v(H) \leq \operatorname{evc}_x(G) + 1$ . Thus,  $\operatorname{evc}_v(H) = \operatorname{evc}_x(G) + 1$ . Note that, by Observation 1,  $\operatorname{evc}(H_v^+) \leq \operatorname{evc}(H) + 1$ .

<sup>&</sup>lt;sup>2</sup> If more than one guard is allowed on a vertex, we still can assume without loss of generality that v has only one guard in any configuration. Instead of placing more than one guard on v, it is possible to place all but one of those guards on x.

First, suppose G is Type 1 with respect to x. Then,  $\operatorname{evc}(H) = \operatorname{evc}_x(G)$  and we get  $\operatorname{evc}_v(H) = \operatorname{evc}_x(G) + 1 = \operatorname{evc}(H) + 1 = \operatorname{evc}(H_v^+)$ , which means that H is Type 1 with respect to v.

Now, suppose G is Type 2 with respect to x. Then,  $\operatorname{evc}(H) = \operatorname{evc}_x(G) + 1$ . Further, by Observation 3, if x is a cut vertex in G, then all the x-components of G are Type 2 with respect to x. Irrespective of whether x is a cut-vertex of G or not, by Lemma 1,  $\operatorname{evc}(H_v^+) = 2 + \operatorname{evc}_x(G)$ . Hence,  $\operatorname{evc}(H_v^+) = \operatorname{evc}_v(H) + 1$ . Thus, if G is Type 2 with respect to x, then H is Type 2 with respect to v.  $\Box$ 

Now, we give some observations that are useful for deciding the type of a graph with respect to a degree-2 vertex.

**Lemma 3.** Let G be any graph and suppose v is a degree-2 vertex in G such that its neighbors  $v_1$ ,  $v_2$  are not adjacent. Let G' be the graph obtained by deleting v and adding an edge between  $v_1$  and  $v_2$ . Then,  $evc_v(G) = evc(G') + 1$ .

**Definition 8.** Let X be the set of cut vertices of a graph G. If B is a block of G, the set of B-components of G is defined as

 $\mathcal{C}_B(G) = \{G_i : G_i \in \mathcal{C}_x(G) \text{ for some } x \in X \cap V(B), G_i \text{ edge disjoint with } B\}.$ If P is a path in G, then the set of P-components of G is defined as  $\mathcal{C}_P(G) = \{G_i : G_i \in \mathcal{C}_x(G) \text{ for some } x \in X \cap V(P), G_i \text{ edge disjoint with } P\}.$ 

**Definition 9.** For a block B (respectively, a path P) of connected graph G, the type of a B-component (respectively, P-component) is its type with respect to the common vertex it has with B (respectively, P). For  $i \in \{1,2\}$ , we define  $T_i(G,B)$  (respectively,  $T_i(G,P)$ ) to be the set of all B-components (respectively, P-components) of G that are Type i.

If B is a block (respectively, P is a path) of a connected graph G, such that all B-components (respectively, P-components) of G satisfy substructure property, then we can easily obtain a lower bound on the total number of guards on  $\bigcup_{G_i \in \mathcal{C}_B(G)} V(G_i)$  (respectively, on  $\bigcup_{G_i \in \mathcal{C}_P(G)} V(G_i)$ ) in any eternal vertex cover of G or its extensions. The notation introduced below is to abstract this lower bound.

**Definition 10.** For a block B of connected graph G, we define  $\chi(G, B)$  to be

$$|V(B) \cap X| + \sum_{\substack{G_i \in T_1(G,B)\\x_i \in V(G_i) \cap V(B)}} (\operatorname{evc}_{x_i}(G_i) - 2) + \sum_{\substack{G_i \in T_2(G,B)\\x_i \in V(G_i) \cap V(B)}} (\operatorname{evc}_{x_i}(G_i) - 1).$$

Similarly, for a path P of connected graph G, we define  $\chi(G, P)$  to be

$$|V(P) \cap X| + \sum_{\substack{G_i \in T_1(G,P) \\ x_i \in V(G_i) \cap V(B)}} (\operatorname{evc}_{x_i}(G_i) - 2) + \sum_{\substack{G_i \in T_2(G,P) \\ x_i \in V(G_i) \cap V(B)}} (\operatorname{evc}_{x_i}(G_i) - 1).$$

Remark 3. If B is a block (respectively, P is a path) of a connected graph G, such that all B-components (respectively, P-components) of G satisfy substructure

property, then the total number of guards on

 $\bigcup_{G_i \in \mathcal{C}_B(G)} V(G_i) \text{ (respectively, on } \bigcup_{G_i \in \mathcal{C}_P(G)} V(G_i) \text{) in any eternal vertex cover of } G \text{ or its extensions is at least } \chi(G, B) \text{ (respectively, } \chi(G, P)).$ 

**Definition 11 (Vertex bunch of a path).** Let P be a path in a connected graph G. The vertex set  $V(P) \cup \bigcup_{G_i \in C_P(G)} V(G_i)$  is the vertex bunch of P in G, denoted by  $VB_G(P)$ .

**Definition 12 (Eventful path).** Let G be a connected graph and X be the set of cut vertices of G. A path P in a graph G is an eventful path if (i) P is either an induced path in G or a path obtained by removing an edge from an induced cycle in G (ii) the endpoints of P are in X and (iii) any subpath P' of P with both endpoints in X has  $|V(P') \setminus X|$  even.

**Lemma 4.** Let G be a connected graph and let P be an eventful path in G. Let X be the set of cut vertices of G. If each P-component in  $C_P(G)$  satisfies the substructure property, then in any eternal vertex cover configuration of G, the total number of guards on  $\operatorname{VB}_G(P)$  is at least  $\frac{|V(P)\setminus X|}{2} + \chi(G, P)$ . Moreover, if  $\operatorname{VB}_G(P) \neq V(G)$  and the number of guards on  $\operatorname{VB}_G(P)$  is exactly equal to the above expression, then at least one of the neighbors of the endpoints of P outside  $\operatorname{VB}_G(P)$  has a guard on it.

*Proof.* Consider any subpath P' of P such that both endpoints of P' are in X and none of its intermediate vertices are from X. Let C be an eternal vertex cover configuration of G. Since P is eventful,  $|V(P') \setminus X|$  is even and in any vertex cover of G, at least  $\frac{|V(P') \setminus X|}{2}$  internal vertices of P' must be present. Using this along with the substructure property of P-components proves the first part of the lemma.

Now, suppose  $\operatorname{VB}_{G}(P) \neq V(G)$  and the number of guards in the configuration C on  $\operatorname{VB}_{G}(P)$  is exactly equal to the expression given in the lemma. Now, for contradiction, let us assume that none of the neighbors of the endpoints of P outside  $\operatorname{VB}_{G}(P)$  has a guard in C. Consider an attack on an edge xv, where x is an endpoint of P and v is a neighbor of x outside  $\operatorname{VB}_{G}(P)$ . To defend this attack, a guard must move from x to v. Note that, no guards can move to  $\operatorname{VB}_{G}(P)$  from outside  $\operatorname{VB}_{G}(P)$ . Hence, while defending the attack, the number of guards on  $\operatorname{VB}_{G}(P)$  decreases at least by one. But, then the new configuration will violate the first part of the lemma. Hence, it must be the case that at least one of the neighbors of the endpoints of P outside  $\operatorname{VB}_{G}(P)$  has a guard in C.  $\Box$ 

**Definition 13 (Maximal uneventful path).** Let G be a connected graph and let X be the set of cut vertices of G. A path P in G is a maximal uneventful path in G if (i)  $V(P) \cap X = \emptyset$  (ii) |V(P)| is odd and (iii) P is a maximal induced path in G satisfying the above two conditions.

The next lemma is applicable to any connected graph G that contains a block B which is a cycle such that all B-components satisfy the substructure property. Since each block of a cactus is either a cycle or an edge, this lemma will be useful for computing the eternal vertex cover number of cactus graphs. The proof of the lemma makes use of the fact that B can be partitioned into a collection of edge disjoint paths which are either eventful paths or maximal uneventful paths.

**Lemma 5.** Let B be a cycle forming a block of a connected graph G and let X be the set of cut vertices of G. Suppose each B-component  $G_i$  of G that belongs to  $C_B(G)$  satisfies the substructure property. If  $T_1(G, B) = \emptyset$ , then  $evc(G) = \left\lceil \frac{|V(B)\setminus X|}{2} \right\rceil + \chi(G, B)$ . Otherwise,  $evc(G) = \left\lceil \frac{|V(B)\setminus X|+1}{2} \right\rceil + \chi(G, B)$ .

*Proof.* Let |V(B)| = n and  $|X \cap V(B)| = k$ . For each *B*-component  $G_i$  of *G* that belongs to  $\mathcal{C}_B(G)$ , let  $x_i$  be the vertex that  $G_i$  has in common with *B*. Let  $\mathcal{C}$  be a minimum eternal vertex cover of *G*. Note that the condition stated in Lemma 4 has to simultaneously hold for all subpaths of the cycle that are eventful in *G*.

Let  $l \geq 0$  be the number of maximal uneventful paths in B and let  $P_1, P_2, \ldots, P_l$  be these listed in the cyclic order along B. To protect the edges within each  $P_i, V(P_i)$  should contain at least  $\operatorname{mvc}(P_i) = \lfloor \frac{|V(P_i)|}{2} \rfloor$  guards. If there are exactly  $\lfloor \frac{|V(P_i)|}{2} \rfloor$  guards on  $V(P_i)$ , the end vertices of  $P_i$  are not occupied and alternate vertices in  $P_i$  are occupied by guards. We divide the proof into four cases, based on the parity of n - k and whether  $T_1(G, B)$  is empty or not. We prove one of the cases here.

Let us consider the case when  $T_1(G, B) = \emptyset$  and n - k is odd. In this case, from Definition 12, it follows that l is odd. First, suppose l = 1. Let P be the subpath obtained from B by deleting the edges of  $P_1$ . It is easy to see that P is an eventful path. If  $V(P_1)$  contains only  $\lfloor \frac{|V(P_1)|}{2} \rfloor$  guards, then the end vertices of  $P_1$  are not occupied by guards and the condition stated in Lemma 4 cannot hold for P. Therefore, the lemma holds when l = 1. Now, suppose l > 1. Then, if  $V(P_i)$ and  $V(P_{i+1})$  (+ is mod l) respectively contains only  $\lfloor \frac{|V(P_i)|}{2} \rfloor$  and  $\lfloor \frac{|V(P_{i+1})|}{2} \rfloor$ guards and the vertex bunch of the path P between the last vertex of  $P_i$  and the first vertex of  $P_{i+1}$  contains exactly as many guards as mentioned in the first part of Lemma 4, the condition stated in the second part of Lemma 4 cannot hold for P. Since this is true for all  $1 \leq i \leq l$ , and the condition stated in Lemma 4 has to simultaneously hold for all subpaths of the cycle that are eventful in G, a simple counting argument shows that number of guards in C should be at least  $\left\lfloor \frac{n-k}{2} \right\rfloor + \chi(G, B)$ .

Thus, we know that  $\operatorname{evc}(G)$  is at least the expression given above. If  $T_1(G, B) = \emptyset$ , it is easy to show that these many guards are also sufficient. The guards on V(B) can defend any attack on edges of B while keeping  $X \cap V(B)$  always occupied. Attacks on edges of B-components can be handled, maintaining  $x_i$  always occupied and having exactly  $\operatorname{evc}_{x_i}(G_i)$  guards on each B-component  $G_i$ .

The proof of the other cases are similar and is omitted.

The following lemma gives a method to compute the type of a graph with respect to degree-two vertices in blocks which are cycles.

**Lemma 6.** Let B be a cycle of n vertices, forming a block of a connected graph G. Let X be the set of cut vertices of G and let  $k = |X \cap V(B)|$ . Suppose each

B-component in  $C_B(G)$  satisfies the substructure property. Let  $v \in V(B) \setminus X$ . The type of G with respect to v can be computed as follows.

- If  $T_1(G, B) \neq \emptyset$  and n k is even, then  $\operatorname{evc}_v(G) = \operatorname{evc}(G) = \operatorname{evc}(G_v^+)$ . If  $T_1(G, B) \neq \emptyset$  and n k is odd, then  $\operatorname{evc}_v(G) = \operatorname{evc}(G) + 1 = \operatorname{evc}(G_v^+)$ . In both cases, G is Type 1 with respect to v.
- If  $T_1(G, B) = \emptyset$  and n k is even, then  $\operatorname{evc}_v(G) = \operatorname{evc}(G) + 1 = \operatorname{evc}(G_v^+)$ and G is Type 1 with respect to v. If  $T_1(G, B) = \emptyset$  and n - k is odd, then  $\operatorname{evc}_v(G) = \operatorname{evc}(G), \operatorname{evc}(G_v^+) = \operatorname{evc}_v(G) + 1$  and G is Type 2 with respect to v.

### 4 Computing eternal vertex cover number of cactus

### **Theorem 1.** Every cactus graph satisfies substructure property.

*Proof.* Let G be a cactus graph. The proof is using an induction on the number of cut-vertices in G.

In the base case, G is a cactus without a cut vertex. Then, G is either a single vertex, a single edge or a simple cycle. In all these cases, the lower bound for the number of guards on V(G), specified by substructure property is equal to the vertex cover number of the respective graphs. Hence, the theorem holds in the base case.

Now, let us assume that the theorem holds for any cactus with at most k cut-vertices. Let G be a cactus with k+1 cut vertices, for  $k \ge 0$  and let X be the set of cut vertices of G. Let v be a non-cut vertex of G and G' be an arbitrary v-extension of G. We need to show that in any eternal vertex cover configuration of G', the number of guards on V(G) is as specified by the substructure property. Since v is a non-cut vertex of G that is in some block B of G, where B is a cycle. We want to compute a lower bound on the number of guards on V(G) in an arbitrary eternal vertex cover configuration  $\mathcal{C}'$  of G'.

First, consider the case where v is a degree-one vertex of G. Let w be the neighbor of v in G and let  $H = G \setminus v$ . Since G has at least one cut-vertex, w must be a cut vertex in G. There are two possibilities depending on whether w is a cut-vertex of H or not.

When w is a cut-vertex of H, consider any w-component  $H_i$  of H.  $H_i$  is a cactus and the number of its cut vertices is less than that of G. Hence,  $H_i$ satisfies the substructure property. Further, G' is a w-extension of  $H_i$ . In  $\mathcal{C}'$ , the number of guards on  $V(H_i)$  is at least  $\operatorname{evc}_w(H_i) - 1$  if  $H_i \in T_1(H, w)$  and it is  $\operatorname{evc}_w(H_i)$  if  $H_i \in T_2(H, w)$ . The total number of guards on V(H) is at least  $1 + \sum_{H_i \in T_1(H,w)} (\operatorname{evc}_w(H_i) - 2) + \sum_{H_i \in T_2(H,w)} (\operatorname{evc}_w(H_i) - 1)$ . If v is occupied in  $\mathcal{C}'$ , then the total number of guards on V(G) is at least the same as that of  $\operatorname{evc}(G)$ as given by Lemma 1. If v is not occupied in  $\mathcal{C}'$ , then to defend an attack on the edge wv, a guard from V(H) must move to v. Hence, in  $\mathcal{C}'$ , the number of guards in V(H) should have been one more than the minimum mentioned earlier. Hence, in this case also, the number of guards on V(G) in  $\mathcal{C}'$  would have been at least  $\operatorname{evc}(G)$ . By Lemma 1,  $\operatorname{evc}(G_v^+) = \operatorname{evc}(G) + 1$ . By Lemma 2, the type of G with respect to v is the same as the type of H with respect to w. Hence, if H is Type 1 with respect to w, we have  $\operatorname{evc}(G_v^+) = \operatorname{evc}_v(G) = \operatorname{evc}(G) + 1$ . We have seen that the number of guards on V(G) is at least  $\operatorname{evc}(G) = \operatorname{evc}_v(G) - 1$ . If H is Type 2 with respect to w, then we have  $\operatorname{evc}(G_v^+) > \operatorname{evc}_v(G)$ . Since  $\operatorname{evc}(G_v^+) = \operatorname{evc}(G) + 1$ , in this case we must have  $\operatorname{evc}_v(G) = \operatorname{evc}_v(G)$ . We have seen that the number of guards on V(G) is at least  $\operatorname{evc}(G) = \operatorname{evc}_v(G)$ . Hence, the substructure property holds in both cases.

When w is not a cut vertex of H, G' is a w-extension of H. By substructure property of H, it follows that in configuration  $\mathcal{C}'$ , the number of guards on V(G) must be at least  $\operatorname{evc}_w(H)$  if H is Type 1 with respect to w and at least  $\operatorname{evc}_w(H) + 1$  if H is Type 2 with respect to w. By Lemma 1, when H is Type 1 with respect to  $w, \operatorname{evc}(G) = \operatorname{evc}_w(H)$  and when H is Type 2 with respect to  $w, \operatorname{evc}(G) = \operatorname{evc}_w(H) + 1$ . By Lemma 2, the type of G with respect to vis the same as the type of H with respect to w. Hence, if H is Type 1 with respect to w, we have  $\operatorname{evc}(G_v^+) = \operatorname{evc}_v(G) = \operatorname{evc}(G) + 1$ . The number of guards on V(G) is at least  $\operatorname{evc}_w(H) = \operatorname{evc}_v(G) - 1$ . If H is Type 2 with respect to w, we have  $\operatorname{evc}(G_v^+) > \operatorname{evc}_v(G)$ . Since  $\operatorname{evc}(G_v^+) = \operatorname{evc}(G) + 1$ , in this case we must have  $\operatorname{evc}_v(G) = \operatorname{evc}(G)$ . The number of guards on V(G) is at least  $\operatorname{evc}_w(H) + 1 = \operatorname{evc}_v(G)$ . Hence, the substructure property holds in both cases.

Now, consider the case when v is a degree-two vertex of G that is in some block B of G, where B is a cycle. Suppose  $|V(B)| = n_b$ . Let X be the set of cut vertices of G and  $k_b = |X \cap V(B)|$ . As noted earlier, we have to compute a lower bound on the number of guards on V(G) in an arbitrary eternal vertex cover configuration  $\mathcal{C}'$  of G'. Let p and q respectively be the clockwise and anticlockwise nearest vertices to v in  $X \cap V(B)$ . Let P be the path in B between p and q that does not contain v. Note that, every B-component of G satisfies substructure property by our induction hypothesis and G' is an extension for each of them. Hence, we have a lower bound on the number of guards on  $V(G_i)$ , for each  $G_i \in \mathcal{C}_B(G)$ . Similarly, the condition stated in Lemma 4 needs to be satisfied for each eventful subpath P' of P.

We have three possibilities to consider. When  $T_1(G, B) = \emptyset$ , by similar arguments as in the proof of Lemma 5, we can see that the number of guards on V(G) in  $\mathcal{C}'$  must be at least  $\operatorname{evc}(G)$ . By Lemma 6, when  $n_b - k_b$  is even, G is Type 1 with respect to v and  $\operatorname{evc}_v(G) = \operatorname{evc}(G) + 1$ . Since the number of guards on V(G) is at least  $\operatorname{evc}(G) = \operatorname{evc}_v(G) - 1$ , we are done. Similarly, when  $n_b - k_b$ is odd, G is Type 2 with respect to v and  $\operatorname{evc}_v(G) = \operatorname{evc}(G)$  and we are done. When  $T_1(G, B) \neq \emptyset$  and  $n_b - k_b$  is even, by similar arguments we can see that the number of guards on V(G) in  $\mathcal{C}'$  must be at least  $\operatorname{evc}(G) - 1$ . By Lemma 6, G is Type 1 with respect to v and  $\operatorname{evc}(G) = \operatorname{evc}_v(G)$ . Since the number of guards on V(G) is at least  $\operatorname{evc}(G) - 1 = \operatorname{evc}_v(G) - 1$ , we are done. When  $T_1(G, B) \neq \emptyset$ and  $n_b - k_b$  is odd, by similar arguments we can see that the number of guards on V(G) in  $\mathcal{C}'$  must be at least  $\operatorname{evc}(G) - 1$ , we are done. When  $T_1(G, B) \neq \emptyset$ and  $n_b - k_b$  is odd, by similar arguments we can see that the number of guards on V(G) in  $\mathcal{C}'$  must be at least  $\operatorname{evc}(G)$ . By Lemma 6, G is Type 1 with respect to v and  $\operatorname{evc}_v(G) = \operatorname{evc}(G) + 1$ . Since the number of guards on V(G) is at least  $\operatorname{evc}(G) = \operatorname{evc}_v(G) - 1$ , we are done. Thus, in all cases, the lower bound on the number of guards on V(G) in an arbitrary eternal vertex cover configuration  $\mathcal{C}'$  of G' satisfies the condition stated in substructure property. Hence, G satisfies substructure property.

Thus, by induction, it follows that every cactus satisfies substructure property.

Now, we have all ingredients for designing a recursive algorithm for the computation of eternal vertex cover number of a cactus, using Lemma 1. Our algorithm will take a cactus G and a vertex v of G and output evc(G),  $evc_v(G)$  and the type of G with respect to v. If G is a cycle or an edge or a vertex, the answer is trivial and can be computed in linear time.

In other cases, G has at least one cut vertex. If v is a cut vertex, then we call the algorithm recursively on each v-component of G along with vertex v. Then, we can use Lemma 1 to compute evc(G) and  $evc_v(G)$  in constant time from the result of the recursive call. Using the same information from recursive calls, the type of G with respect to v can also be computed using Observation 3. If v is a pendent vertex and w is its neighbor in G, then we recursively call the algorithm on  $(G \setminus v, w)$ . By Lemma 2, the type of G with respect to v is the same as the type of  $G \setminus v$  with respect to w. Moreover, from the proof of Lemma 2, we have  $\operatorname{evc}_v(G) = \operatorname{evc}_w(G \setminus v) + 1$ . Further,  $\operatorname{evc}(G) = \operatorname{evc}_w(G \setminus v)$ , when G is Type 1 with respect to v and  $evc(G) = evc_w(G \setminus v) + 1$ , when G is Type 2 with respect to v. Thus, from the results of the recursive call on  $(G \setminus v, w)$ , the output can be computed in constant time. In the remaining case, v is a vertex that belongs to a cycle B in G. In this case, we recursively call the algorithm for each B-component of G, along with the respective cut vertices it shares with B. Using this information, we can compute evc(G) using Lemma 5 in time proportional to the number of B-components. We can also compute  $evc_n(G)$  and the type of G with respect to v, using Lemma 6 in time proportional to the number of B-components.

Thus, the algorithm works in all cases and runs in time linear in the size of G. Hence, we have the following result.

# **Theorem 2.** Eternal vertex cover number of a cactus G can be computed in time linear in the size of G.

From the upper bound arguments in the proofs discussed in Section 2 and Section 3, we can see that with evc(G) guards determining configurations of guards to keep defending attacks on G is straightforward.

### 5 Extension to a superclass of chordal graphs

It may be noticed that most of the intermediate results stated in this paper are generic, though we have stated Lemma 5 and Lemma 6 in a way suitable for handling cactus graphs. In this section, we show how to extend the method used for cactus graphs to a graph class consisting of connected graphs in which each block is a cycle, an edge or a biconnected chordal graph. It may be noted that this class contains all chordal graphs and cactus graphs. To generalize the proof of Theorem 1 for this class, the base case of the proof needs to be modified to handle biconnected chordal graphs as well. The following observation addresses this requirement.

### **Observation 5** Every biconnected chordal graph satisfies substructure property.

*Proof.* Let G be a biconnected chordal graph and  $v \in V(G)$ . Let G' be a vextension of G and C be an eternal vertex cover configuration of G'. If the number of guards on V(G) in C is less than  $mvc_v(G)$ , then v is not occupied in C. In this configuration, an attack on an edge of G adjacent to v cannot be defended, because when a guard on a vertex of G moves to v, some edge of G will be without guards. Therefore, in any arbitrary eternal vertex cover configuration C of G', the number of guards on V(G) must be at least  $mvc_v(G)$ .

By a result in [12],  $\operatorname{evc}_v(G) \in \{\operatorname{mvc}_v(G), \operatorname{mvc}_v(G) + 1\}$  and  $\operatorname{evc}(G_v^+) = \operatorname{mvc}(G_v^+) + 1 = \operatorname{mvc}_v(G) + 1$ . If  $\operatorname{evc}_v(G) = \operatorname{mvc}_v(G)$  and  $\operatorname{evc}(G_v^+) = \operatorname{mvc}_v(G) + 1$ , then G is Type 2 with respect to v and we need at least  $\operatorname{evc}_v(G) = \operatorname{mvc}_v(G)$ guards on V(G). If  $\operatorname{evc}_v(G) = \operatorname{mvc}_v(G) + 1 = \operatorname{evc}(G_v^+)$ , then G is Type 1 with respect to v and we need at least  $\operatorname{evc}_v(G) - 1 = \operatorname{mvc}_v(G)$  guards on V(G). Thus, in both cases, the requirements of substructure property are satisfied by G.

The following lemma is a suitable modification of Lemma 5 and Lemma 6 to handle the new class.  $\hfill \Box$ 

**Lemma 7.** Let B be a biconnected chordal graph forming a block of a connected graph G and let X be the set of cut vertices of G. Suppose each B-component  $G_i$  of G that belongs to  $C_B(G)$  satisfies the substructure property. If  $T_1(G, B) = \emptyset$ , then  $\operatorname{evc}(G) = \operatorname{evc}_{X \cap V(B)}(B) + \chi(G, B) - |X \cap V(B)|$  and  $\operatorname{evc}(G) = \operatorname{mvc}_{X \cap V(B)}(B) +$  $1 + \chi(G, B) - |X \cap V(B)|$  otherwise. Further, if  $\chi(G, B)$  is known, then for any  $v \in V(B)$ , the type of G with respect to v can be computed in time quadratic in the size of B.

*Proof.* Since *B* is a biconnected chordal graph, if for every  $v \in V(B) \setminus X$ ,  $mvc_{(X \cap V(B))\cup\{v\}}(B) = mvc_{X \cap V(B)}(B)$ , then  $evc_{X \cap V(B)}(B) = mvc_{X \cap V(B)}(B)$ and  $evc_{X \cap V(B)}(B) = mvc_{X \cap V(B)}(B) + 1$  otherwise [12]. Consider any eternal vertex cover class *C* of *G*. By substructure property of *B*-components of *G*, it follows that in any configuration of *C*, the number of guards on  $\bigcup_{G_i \in C_B} V(G_i)$ is at least  $\chi(G, B)$ . To cover edges of the induced subgraph of *G* on  $V(B) \setminus X$ , the number of guards required is at least  $mvc(B \setminus X) = mvc_{X \cap V(B)}(B) - |X \cap V(B)|$ . Hence, the total number of guards on V(G) must be at least  $mvc_{X \cap V(B)}(B) - |X \cap V(B)| + \chi(G, B)$ . Further, if there is a vertex  $v \in V(B) \setminus X$ for which  $mvc_{(X \cap V(B))\cup\{v\}}(B) \neq mvc_{X \cap V(B)}(B)$ , then in any configuration of *C* in which *v* is occupied, the total number of guards on V(G) must be at least  $mvc_{X \cap V(B)}(B) + 1 - |X \cap V(B)| + \chi(G, B)$ . Hence, in all cases,  $evc(G) \ge$  $evc_{X \cap V(B)}(B) + \chi(G, B) - |X \cap V(B)|$ .

When  $T_1(G, B) = \emptyset$ , it is easy to show that  $\operatorname{evc}(G) \leq \operatorname{evc}_{X \cap V(B)}(B) + \chi(G, B) - |X \cap V(B)|$ . When  $T_1(G, B) \neq \emptyset$ , by repeated attacks on edges of  $V(G_i)$  for some  $G_i \in T_1(G, B)$ , eventually a configuration which requires  $\chi(G, B) + 1$ 

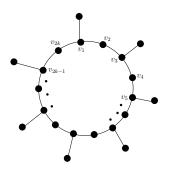
guards on  $\bigcup_{G_i \in \mathcal{C}_B} V(G_i)$  can be forced. Hence,  $\operatorname{evc}(G) \geq \operatorname{mvc}_{X \cap V(B)}(B) + 1 + \chi(G, B) - |X \cap V(B)|$ . Since  $\operatorname{evc}_{X \cap V(B)}(B) \leq \operatorname{mvc}_{X \cap V(B)}(B) + 1$ , it is not difficult to also show that  $\operatorname{evc}(G) \leq \operatorname{mvc}_{X \cap V(B)}(B) + 1 + \chi(G, B) - |X \cap V(B)|$ . In both cases, the value of  $\operatorname{evc}(G)$  is as stated in the lemma.

For deciding the type of G with respect to a vertex  $v \in V(B)$ , we need to compare  $\operatorname{evc}_v(G)$  and  $\operatorname{evc}(G_v^+)$ . For computing  $\operatorname{evc}(G_v^+)$ , we can use the formula given by the first part of the lemma for the graph  $\operatorname{evc}(G_v^+)$ . Using similar arguments as in the proof of the first part of the lemma, we get the following: if  $T_1(G,B) = \emptyset$ , then  $\operatorname{evc}_v(G) = \operatorname{evc}_{(X \cap V(B)) \cup \{v\}}(B) + \chi(G,B) - |X \cap V(B)|$ and  $\operatorname{evc}_v(G) = \operatorname{mvc}_{(X \cap V(B)) \cup \{v\}}(B) + 1 + \chi(G,B) - |X \cap V(B)|$  otherwise. By computing  $\operatorname{mvc}_{(X \cap V(B))}(B)$  and  $\operatorname{mvc}_{(X \cap V(B)) \cup \{v\}}(B)$  for each  $v \in V(B)$ ,  $\operatorname{evc}_v(G)$ and  $\operatorname{evc}(G_v^+)$  can be computed. Since minimum vertex cover of a chordal graph can be computed in linear time, the total time required for computing  $\operatorname{evc}_v(G)$ and  $\operatorname{evc}(G_v^+)$  this way is possible in time quadratic in the size of B. From the values of  $\operatorname{evc}_v(G)$  and  $\operatorname{evc}(G_v^+)$ , the type of G with respect to v can be inferred.

Now, similar arguments as in the proof of Theorem 1 and Theorem 2 yields:

**Theorem 3.** Suppose G is a connected graph in which each block is a cycle, an edge or a biconnected chordal graph. Then, G satisfies substructure property and the eternal vertex cover number of G can be computed in quadratic time.

# 6 A cactus family with $evc(G) > 1.5 mvc_X(G)$



**Fig. 1.** Pendent vertices are attached to alternate vertices of a cycle on 2k vertices. The cactus G has  $mvc_X(G) = k$  and  $evc(G) = k + \lceil \frac{k+1}{2} \rceil$ .

Until now, all graph classes with known polynomial time algorithms for eternal vertex cover had  $\text{evc}(G) \in \{\text{mvc}_X(G), \text{mvc}_X(G) + 1\}$ , where X is the set of all cut vertices of G. It is interesting to note that  $\text{mvc}_X(G)$  could be much smaller than evc(G) for some cactus graphs. An example of this is shown in Fig. 1. Using

the formula given by Lemma 5, we can show that the eternal vertex cover number of this graph G is  $k + \lceil \frac{k+1}{2} \rceil$ . However, for this graph,  $\text{mvc}_X(G)$  is only k making  $\text{evc}(G) > 1.5 \text{ mvc}_X(G)$ .

# 7 Discussion and open problems

Though the substructure property has been shown here for a family of graphs that include cactus and chordal graphs, we believe that this property or a generalization of it holds for fairly large classes of graphs. It is interesting to characterize graphs that satisfy this property. Deriving a generalization of the substructure property to outerplanar graphs and bounded treewidth graphs is also an interesting direction.

### References

- Klostermeyer, W., Mynhardt, C.: Edge protection in graphs. Australasian Journal of Combinatorics 45 (2009) 235 – 250
- Fomin, F.V., Gaspers, S., Golovach, P.A., Kratsch, D., Saurabh, S.: Parameterized algorithm for eternal vertex cover. Information Processing Letters **110**(16) (2010) 702 - 706
- Rinemberg, M., Soulignac, F.J.: The eternal dominating set problem for interval graphs. Information Processing Letters 146 (2019) 27–29
- Goldwasser, J.L., Klostermeyer, W.F.: Tight bounds for eternal dominating sets in graphs. Discrete Mathematics 308(12) (2008) 2589–2593
- Klostermeyer, W.F., Mynhardt, C.: Vertex covers and eternal dominating sets. Discrete Applied Mathematics 160(7) (2012) 1183–1190
- Goddard, W., Hedetniemi, S.M., Hedetniemi, S.T.: Eternal security in graphs. J. Combin. Math. Combin. Comput 52 (2005) 169–180
- Hartnell, B., Mynhardt, C.: Independent protection in graphs. Discrete Mathematics 335 (2014) 100 - 109
- 8. Caro, Y., Klostermeyer, W.: Eternal independent sets in graphs. Volume 3. (2016)
- 9. Klostermeyer, W.F., Mynhardt, C.M.: Graphs with equal eternal vertex cover and eternal domination numbers. Discrete Mathematics **311** (2011) 1371 1379
- Anderson, M., Carrington, J.R., Brigham, R.C., D.Dutton, R., Vitray, R.P.: Graphs simultaneously achieving three vertex cover numbers. Journal of Combinatorial Mathematics and Combinatorial Computing **91** (2014) 275 – 290
- Araki, H., Fujito, T., Inoue, S.: On the eternal vertex cover numbers of generalized trees. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E98.A (06 2015) 1153–1160
- Babu, J., Chandran, L.S., Francis, M., Prabhakaran, V., Rajendraprasad, D., Warrier, J.N.: On graphs with minimal eternal vertex cover number. In: Conference on Algorithms and Discrete Applied Mathematics (CALDAM), Springer (2019) 263–273
- 13. Babu, J., Prabhakaran, V., Sharma, A.: A linear time algorithm for computing the eternal vertex cover number of cactus graphs. CoRR **abs/2005.08058** (2020)
- Babu, J., Prabhakaran, V.: A new lower bound for the eternal vertex cover number of graphs. In: Computing and Combinatorics - 26th International Conference (COCOON), Springer (2020) 27–39