Architecture of some models for optimization problems under conditions of hybrid uncertainty

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Abstract

In the article, a model of a minimal risk portfolio under conditions of hybrid uncertainty of possibilistic-probabilistic type is developed and studied. In this model, the interaction of fuzzy parameters is described by both the strongest and weakest triangular norms. Models of acceptable portfolios are based on the principle of expected possibility or on the basis of fulfilling the restriction on the possibility/necessity and probability of the level of portfolio return that is acceptable to an investor. Equivalent deterministic analogs of the models are constructed.

Keywords 1

Minimum risk portfolio, hybrid uncertainty, possibility, necessity, expected return, possibility/probability constraint, strongest t-norm, weakest t-norm

1. Introduction

The article presents the architecture of some models for optimization problems under conditions of hybrid uncertainty of possibilistic-probabilistic type and some indirect methods for their solving, complementing the results previously obtained in [1-6].

In our work attention is paid to the study of situations when the interaction of fuzzy model factors is described by both the strongest and weakest t-norms, which allows us to assess the range of risk changes and the behavior of a set of acceptable portfolios, that is, to manage uncertainty when making investment decisions. In order to remove probabilistic uncertainty from acceptable portfolios model, the principle based on the expected possibility is used. Uncertainty of possibilistic (fuzzy) type is removed by imposing requirements for the possibility/necessity of fulfilling restrictions on the acceptable level of expected profitability of the portfolio. The relationship between models of acceptable portfolios of different architectures is established and investigated. In a number of relevant papers devoted to the problem of portfolio selection, only the situation when the interaction of fuzzy factors is described by the strongest triangular norm (t-norm) is studied (see, for example, [7]). Theoretical results and conclusions are confirmed by numerical calculations.

2. Necessary concepts and notations

In the context of works [8-16], we introduce a number of definitions and concepts from the theory of possibilities. Let further $(\Gamma, \mathrm{P}(\Gamma), \tau)$ and $(\Omega, \mathrm{B}, \mathbf{P})$ be possibility and probability spaces, respectively, in which Ω is the space of elementary events $\omega \in \Omega$, Γ is the model space with elements $\gamma \in \Gamma$, B is the σ-algebra of events, $\mathrm{P}(\Gamma)$ is the set of all subsets of Γ, $\tau \in \{\pi, \nu\}$, π and ν are measures of possibility and necessity, respectively, and $\mathbf{P}$ is the probability measure; $\mathbb{E}^1$ is the number line.

We define a fuzzy random variable and its distribution as follows [9, 17].

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Definition 1. Fuzzy random variable $Y(\omega,\gamma)$ is a real function $Y: \Omega \times \Gamma \to \mathbb{E}^1$ σ-measurable for each fixed γ, and

$$\mu_Y(\omega,t) = \pi\{\gamma \in \Gamma: Y(\omega,\gamma) = t\}, \forall t \in \mathbb{E}^1$$

is called its distribution function.

From definition 1 it follows that the distribution function of a fuzzy random variable depends on a random parameter, i.e. it is a random function.

Definition 2. Let Y(ω, γ) – be a fuzzy random variable. Its expected value **E**[Y] is a fuzzy value that has a possibilities distribution function

$$\mu_{\mathbf{E}[Y]}(t) = \pi\{\gamma \in \Gamma: \mathbf{E}[Y(\omega,\gamma)] = t\}, \forall t \in E^1,$$

where **E** — is a mathematical expectation operator

$$\mathbf{E}[Y(\omega,\gamma)] = \int_\Omega Y(\omega,\gamma)\mathbf{P}(d\omega).$$

The distribution function of the expected value of a fuzzy random variable is no longer dependent on the random parameter and therefore is fuzzy.

We define, following [10], second-order moments. Let X and Y be fuzzy random variables.

Definition 3. The covariance of fuzzy random variables X and Y is defined as follows:

$$cov(X,Y) = \frac{1}{2}\int_0^1 \left(cov\left(X_\omega^-(\alpha), Y_\omega^-(\alpha)\right) + cov\left(X_\omega^+(\alpha), Y_\omega^+(\alpha)\right)\right) d\alpha,$$

where $X_\omega^-(\alpha), Y_\omega^-(\alpha), X_\omega^+(\alpha), Y_\omega^+(\alpha)$ – are boundaries of α-level sets of fuzzy variables X_ω, Y_ω, respectively.

Definition 4. The variance of a fuzzy random variable Y is

$$\mathbf{D}[Y] = cov(Y,Y). \tag{1}$$

The mathematical expectation, variance, and covariance of fuzzy random variables determined in accordance with the considered approach inherit the main properties of similar characteristics of real random variables.

An LR-type distribution is often used for modeling fuzzy variables [11], which for a fuzzy variable $Y(\gamma)$ is usually written as $\mu_Y(t) = \left[\underline{m}, \overline{m}, \underline{d}, \overline{d}\right]_{LR}$. Further we will simply write $Y(\gamma) = \left[\underline{m}, \overline{m}, \underline{d}, \overline{d}\right]_{LR}$. Here $\underline{m}, \overline{m}$ are left and right boundaries of the tolerance interval, $\underline{d}, \overline{d}$ are the coefficients of fuzziness, while $\underline{m} \le \overline{m}$ and $\underline{d} > 0, \overline{d} > 0$, $L(t)$ and $R(t)$ are left and right shape functions for the possibility distribution.

We will use triangular norms (t-norms) to aggregate fuzzy information. These norms generalize "min" operation inherent in operations on fuzzy sets and fuzzy variables [12]. The following t-norms are of particular interest:

$$T_M(x,y) = \min\{x,y\} \ and \ T_W(x,y) = \begin{cases} \min\{x,y\}, if \max\{x,y\} = 1, \\ 0, else, \end{cases}$$

T_M is called the strongest, and T_W is called the weakest t-norm.

3. Mathematical models of a minimal risk portfolio under hybrid uncertainty

3.1. Portfolio return under hybrid uncertainty of possibilistic-probabilistic type

Under conditions of hybrid uncertainty of possibilistic-probabilistic type, the return on an investment portfolio can be represented by a fuzzy random function

$$R_p(w,\omega,\gamma) = \sum_{i=1}^n R_i(\omega,\gamma) w_i, \tag{2}$$

which is a linear function of equity shares $w = (w_1, \dots, w_n)$ in the portfolio. Here $R_i(\omega,\gamma)$ are fuzzy random variables that model the returns of individual financial assets with the help of shift-scale representation [9]:

$$R_i(\omega,\gamma) = a_i(\omega) + \sigma_i(\omega) Z_i(\gamma). \tag{3}$$

Further we assume that fuzzy variables $Z_i(\gamma) = \left[\underline{m}_i, \overline{m}_i, \underline{d}_i, \overline{d}_i\right]_{LR}$ in representation (3) are mutually T-related, where $T \in \{T_M, T_W\}$, and $a_i(\omega), \sigma_i(\omega)$ are shift and scale coefficients – random variables defined on a probability space (Ω, B, **P**), with $\sigma_i(\omega) \geq 0$. Then possibilities distribution of the portfolio return (2) takes the following form

$$R_p^T(w,\omega,\gamma) = \left[\underline{m}_{R_p}(w,\omega), \overline{m}_{R_p}(w,\omega), \underline{d}_{R_p^T}(w,\omega), \overline{d}_{R_p^T}(w,\omega)\right]_{LR}, \tag{4}$$

where

$$\underline{m}_{R_p}(w,\omega) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right) w_i, \qquad \overline{m}_{R_p}(w,\omega) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right) w_i,$$

and the coefficients of fuzziness take the form depending on the type of T:

$$\underline{d}_{R_p^M}(w,\omega) = \sum_{i=1}^{n} \sigma_i(\omega)\underline{d}_i w_i, \quad \overline{d}_{R_p^M}(w,\omega) = \sum_{i=1}^{n} \sigma_i(\omega)\overline{d}_i w_i,$$

when $T = T_M$, and

$$\underline{d}_{R_p^W}(w,\omega) = \max_{i=1\dots n}\left\{\sigma_i(\omega)\underline{d}_i w_i\right\}, \quad \overline{d}_{R_p^W}(w,\omega) = \max_{i=1\dots n}\left\{\sigma_i(\omega)\overline{d}_i w_i\right\},$$

in case of $T = T_W$. Further we will denote $R_p^T(w,\omega,\gamma)$ as $R_p^M(w,\omega,\gamma)$ when $T = T_M$ and $R_p^W(w,\omega,\gamma)$ when $T = T_W$.

To remove the uncertainty of probabilistic type in accordance with approach of [3] it is necessary to identify the possibilities distribution of the mathematical expectation of the function $R_p^T(w,\omega,\gamma)$, that is, to calculate its parameters. Expected return in portfolio models is a fuzzy value for a fixed w. This follows from the results of the theorems.

Theorem 1. *Let $T = T_M$. Then expected portfolio return $\hat{R}_p^M(w,\gamma)$ is characterized by the possibilities distribution function*

$$\hat{R}_p^M(w,\gamma) = \mathbf{E}\left[R_p^M(w,\omega,\gamma)\right] = \left[\underline{m}_{\hat{R}_p}(w), \overline{m}_{\hat{R}_p}(w), \underline{d}_{\hat{R}_p^M}(w), \overline{d}_{\hat{R}_p^M}(w)\right]_{LR},$$

where

$$\underline{m}_{\hat{R}_p}(w) = \sum_{i=1}^{n} \left(\hat{a}_i + \hat{\sigma}_i \underline{m}_i\right) w_i, \quad \overline{m}_{\hat{R}_p}(w) = \sum_{i=1}^{n} \left(\hat{a}_i + \hat{\sigma}_i \overline{m}_i\right) w_i,$$

$$\underline{d}_{\hat{R}_p^M}(w) = \sum_{i=1}^{n} \hat{\sigma}_i \underline{d}_i w_i, \ \overline{d}_{\hat{R}_p^M}(w) = \sum_{i=1}^{n} \hat{\sigma}_i \overline{d}_i w_i, \ \hat{a}_i = \mathbf{E}[a_i(\omega)], \hat{\sigma}_i = \mathbf{E}[\sigma_i(\omega)].$$

Theorem 2. *Let $T = T_W$. Then expected portfolio return $\hat{R}_p^W(w,\gamma)$ is characterized by the possibilities distribution function*

$$\hat{R}_p^W(w,\gamma) = \mathbf{E}\left[R_p^W(w,\omega,\gamma)\right] = \left[\underline{m}_{\hat{R}_p}(w), \overline{m}_{\hat{R}_p}(w), \underline{d}_{\hat{R}_p^W}(w), \overline{d}_{\hat{R}_p^W}(w)\right]_{LR},$$

where

$$\underline{d}_{\hat{R}_p^W}(w) = \mathbf{E}\left[\max_{i=1\dots n}\{\sigma_i(\omega)\underline{d}_i w_i\}\right], \ \overline{d}_{\hat{R}_p^W}(w) = \mathbf{E}\left[\max_{i=1\dots n}\{\sigma_i(\omega)\overline{d}_i w_i\}\right].$$

3.2. Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

In accordance with the classical Markowitz [18] approach, we need to construct a portfolio risk function in the minimal risk portfolio model. The expected return or portfolio return can be entered into a system of restrictions. Since the expected return of a portfolio in the case of fuzzy random data is a fuzzy value, in order to remove the uncertainty of possibilistic type in a system of restrictions that defines the set of acceptable portfolios, one can introduce a restriction on possibility/necessity of the level of expected return acceptable to an investor. Then the generalized Markowitz model of acceptable portfolios can be represented as

$$F_p^{\tau\mathbf{E}}(w) = \begin{cases} \tau\{\hat{R}_p^T(w,\gamma)\ \mathcal{R}\ m_d\} \ge \alpha, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n, \end{cases}$$

where $\mathbb{E}_+^n = \{x \in \mathbb{E}^n : x \ge 0\}$, $\hat{R}_p^T(w,\gamma)$ – expected return, $\mathcal{R}$ – crisp relation $\{\ge, =\}$; $\alpha \in (0, 1]$, m_d – level of profitableness, acceptable to an investor, $T \in \{T_M, T_W\}$.

The following theorems allow us to construct equivalent deterministic analogs of acceptable portfolio models.

Theorem 3. *Let in the constraint model* $F_p^{\tau\boldsymbol{E}}$ $\tau = '\pi'$, $\mathcal{R} = '\ge'$. *Then the equivalent deterministic model of acceptable portfolios has the form:*

$$F_p^{\pi\mathbf{E}}(w) = \begin{cases} \sum_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \overline{m}_i) w_i + \overline{d}_{\hat{R}_p^T}(w) * R^{-1}(\alpha) \ge m_d, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n. \end{cases}$$

In case when portfolio return (2) is included in the system of restrictions, the hybrid uncertainty can be removed by imposing a limit on the possibility/necessity and probability of an acceptable level of return. Formally, the mathematical model of such a constraint can be written as:

$$F_p^{\tau\mathbf{P}}(w) = \begin{cases} \tau\{\mathbf{P}\{R_p(w,\omega,\gamma)\ \mathcal{R}\ m_d\} \ge p_0\} \ge \alpha_0, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n, \end{cases}$$

where **P** – probability measure, $p_0 \in (0,\ 1]$ – probability level.

3.3. Assessment of portfolio risk with hybrid uncertainty

In accordance with the indicated approach to determining second-order moments, we can determine the variance of the portfolio to assess its risk. We now define the variance for the t-norm T_W. To do this, we will use formula (1) to find the covariance of two fuzzy random variables. If all random parameters of distributions are independent, then after series of simplifications we get:

$$D_p^W(w) = \frac{1}{2}\sum_{i=1}^{n} w_i^2 \left(2\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)]\left(\underline{m}_i^2 + \overline{m}_i^2\right)\right)$$
$$+ \frac{1}{2}\mathbf{D}\left[\max_{j=1\dots n}\{\sigma_j(\omega)\overline{d}_j w_j\}\right]\int_0^1 \left(R^{-1}(\alpha)\right)^2 d\alpha$$
$$+ \frac{1}{2}\mathbf{D}\left[\max_{j=1\dots n}\{\sigma_j(\omega)\underline{d}_j w_j\}\right]\int_0^1 \left(L^{-1}(\alpha)\right)^2 d\alpha$$
$$+ \sum_{i=1}^{n} w_i \left(\int_0^1 R^{-1}(\alpha)d\alpha\, cov\left(\left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right), \max_{j=1\dots n}\{\sigma_j(\omega)\overline{d}_j w_j\}\right)\right.$$
$$\left. - \int_0^1 L^{-1}(\alpha)d\alpha\, cov\left(\left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right), \max_{j=1\dots n}\{\sigma_j(\omega)\underline{d}_j w_j\}\right)\right).$$

For T=T_M, the formula is obtained in the same way.

3.4. Minimum risk portfolio models

Based on the results presented in sections 3.1, 3.2, and 3.3, the minimum risk portfolio models can be written as:

$$D_p^T(w) \to min, \tag{6}$$
$$w \in F_p(w), \tag{7}$$

where $F_p(w) \in \{F_p^{\mu\mathbf{E}}, F_p^{\nu\mathbf{E}}, F_p^{\mu\mathbf{P}}, F_p^{\nu\mathbf{P}}\}$. Further we assume that the minimum risk portfolio models use the same t-norm in the criteria and constraints. Let's move on to their research.

4. Minimal risk portfolio under hybrid uncertainty and numerical calculations

We consider an example of two-dimensional portfolio ($n = 2$). Let $Z_1 = [2.2, 2.2, 0.3, 0.3]_{LR}$, $Z_2 = [1.2, 1.2, 0.4, 0.4]_{LR}$, $L(t) = R(t) = \max\{0, 1 - t\}, t \geq 0, \alpha = 0.75$. Recall that all $a_i(\omega), \sigma_i(\omega)$ are independent random variables with a uniform distribution on the segment [0,1]. We first specify the minimum risk portfolio models for the weakest t-norm. Under the assumptions made, the equivalent deterministic analog of the minimum risk portfolio (6)-(7) in the context of the possibility measure takes the form:

$$\frac{73}{150}w_1^2 + \frac{61}{300}w_2^2 + \frac{1}{3}\left(EMax2(dw) - \left(EMax(dw)\right)^2\right) \to min,$$
$$F_p^{\pi\mathbf{E}}(w) = \begin{cases} 1.6w_1 + 1.1w_2 + 0.25 * EMax(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0, \end{cases}$$

and in the context of a necessity measure:

$$\frac{73}{150}w_1^2 + \frac{61}{300}w_2^2 + \frac{1}{3}\left(EMax2(dw) - \left(EMax(dw)\right)^2\right) \to min,$$
$$F_p^{\nu\mathbf{E}}(w) = \begin{cases} 1.6w_1 + 1.1w_2 - 0.75 * EMax(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0. \end{cases}$$

where

$$EMax(dw) \coloneqq \mathbf{E}\left[\max_{i=1\dots n}\{\sigma_i(\omega)d_i w_i\}\right] = \sum_{i=1}^{n} \frac{(dw)_{(i)}^{n-i+1}}{(n-i+1)(n-i+2)(dw)_{(i+1)}\cdots(dw)_{(n)}},$$

$$EMax2(dw) \coloneqq \mathbf{E}\left[\left(\max_{i=1\dots n}\{\sigma_i(\omega)d_i w_i\}\right)^2\right] = \sum_{i=1}^{n} \frac{2(dw)_{(i)}^{n-i+2}}{(n-i+2)(n-i+3)(dw)_{(i+1)}\cdots(dw)_{(n)}},$$

and $(\mathrm{d}w)_{(1)}, \dots, (\mathrm{d}w)_{(\mathrm{n})}$ is an ascending permutation of elements $\{\mathrm{d}_1 w_1, \dots, \mathrm{d}_\mathrm{n} w_\mathrm{n}\}$.

For comparative analysis, the same problem was considered for the strongest t-norm. The results are shown in Figure 1.

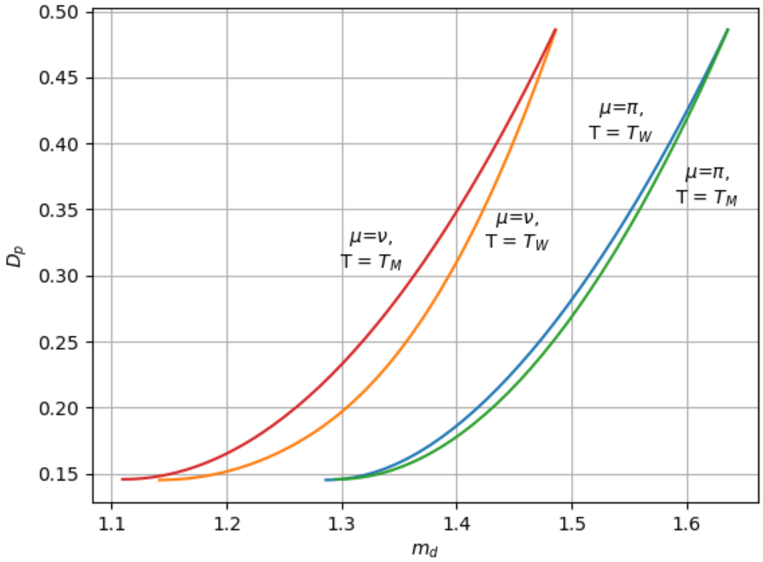

Figure 1: Sets of quasi-efficient portfolios depending on the measure of possibility/necessity and the t-norm

The first thing to note in Figure 1 is the behavior of quasi-efficient portfolios estimates in different contexts. In the context of possibility we have an optimistic decision making model, while in the context of necessity, we have a pessimistic one, which for a given level of expected return gives a significantly higher risk.

Secondly, as one can see in the context of possibility measure the weakest t-norm which has the property of reducing the uncertainty [9], narrows the scope of feasible solutions and makes the model more "strict" or "cautious", i.e. the risk at a fixed rate of return is increased slightly. In the context of necessity, the model behaves in the "opposite" way. Thus, we can say that the weakest t-norm reduces the "level of optimism" in the optimistic model and reduces the level of "pessimism" in the pessimistic model.

5. Conclusion

In this paper, a comprehensive study of the architecture of mathematical models of the minimal risk portfolio is carried out. For extremal t-norms (weakest and strongest) in the context of possibility/necessity the properties of models of acceptable portfolios are studied depending on the decision-making principles used in conditions of hybrid uncertainty of possibilistic-probabilistic type.

Based on the approach of Feng [10] formulas for assessing portfolio risk are specified in the contexts of the strongest and weakest t-norms. The obtained theoretical results and conclusions are consistent with numerical calculations.

In terms of further research, we intend to generalize the results of the article to the case when the acceptable level of portfolio return for an investor is a fuzzy value associated with the portfolio return by a fuzzy relation [19]. This will allow more "soft" and adequate modeling of an investor preferences.

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7. References

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