Numerical Simulation of a Game-Theoretic Model of Environmental Pollution Problem ⋆ ⋆⋆

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Abstract. In this paper, we present a numerical simulation of the game-theoretic model of an environmental pollution problem. This model is formalized by a noncooperative two-person differential game in Banach space with separated dynamics of the agents and continuous payoff functions depending on a game trajectory. The numerical simulation is based on the dynamic programming approach and the finite difference method. Some numerical results are provided for two-dimensional dynamic conflict model of an environmental pollution problem.

Keywords: Noncooperative differential games · Numerical simulation · Environmental pollution.

1 Introduction

Application of the game-theoretic approach plays a special role in mathematical modeling of environmental problems. The dynamic games that deal with similar or related problems were investigated in many works, see e.g. [3–6, 9–12, 15] and references therein.

In [14], the existence of ε-Nash equilibrium in a dynamic conflict model of an environmental pollution problem which is formalized by the noncooperative n-person differential game in Banach space was proved.

In this paper, we present a numerical simulation for the two-dimensional dynamic conflict model of an environmental pollution problem. Enterprises (agents) contaminate a water reservoir by dumping a pollutant (harmful substance) of the same type during the production process.

This model is formalized by a noncooperative two-person differential game in Banach space with separated dynamics of the agents and continuous payoff functions depending on a game trajectory. For simplicity, we consider the case of separated dynamics of agents, in which the dynamics of each agent is described by the initial boundary value problem for the parabolic equation involving Dirac

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measure. The existence of ε-Nash equilibrium in our model follows from the results of [14].

The paper gives a formal mathematical description of the model and related numerical method and presents simulation results.

2 The Model

A closed water reservoir (e.g. lake) is considered. Two enterprises dump a pollutant (harmful substance) of the same type into this water reservoir during the process of production. Both enterprises dump the pollutant at some prescribed points of the reservoir.

Furthermore, it is assumed that the reservoir has a water intake. The level of pollution at the water intake point (the total concentration of the pollutant released by all enterprises) must not exceed the maximum permissible value. It is assumed that if this value is exceeded, all the enterprises will pay a fine (penalty) as a percentage of their income. It is also assumed that the company has certain expenses associated with the cleaning of the pollutant.

The spread of the harmful substances in the reservoir occurs by diffusion. Besides, a pollutant is decomposed with the rate $r > 0$.

The total income of an enterprise depends on its volume of production, which is tightly linked with its total volume of dumped pollutant. Besides, the total income depends on the overall cleaning expenses and possible pollution fines. The aim of each enterprise is to maximize the total income for finite period of time.

3 Differential Game

We will study the differential two-person game $\Gamma(c_0, T)$ with a prescribed duration $T < \infty$ and an initial position of the game c_0. Let $I = \{i\} = \{1, 2\}$ be a set of the agents (enterprises).

We will consider the closed water reservoir as a rectangle $\overline{\Omega} = [0, d_1] \times [0, d_2]$.

Let $z^i(x_1, x_2, t)$ be the pollutant concentration of the agent i at the point $(x_1, x_2) \in \overline{\Omega}$ at the moment $t \in [0, T]$. Denote by $(\bar{x}_1^i, \bar{x}_2^i) \in \Omega$ the prescribed point of dumping the pollutant of the agent $i \in I$ and by (x_w, y_w) the coordinates of the water intake location inside the domain Ω.

Let us denote by $u_i(t)$ the intensity of dumping the pollutant of the agent $i \in I$ at the moment t. We assume that the intensity of dumping the pollutant satisfies the following conditions:

$$0 \leq u_i(t) \leq G_i(t), \quad i \in I, \tag{1}$$

at any moment $t \in [0, T]$. Here $G_i(t) > 0$ is a given square integrable function which describes the maximal intensity of dumping the pollutant of the agent i at the moment t. Let us assume that the production expenditures per unit product of the enterprise i are constant and equal to $M_i > 0$, $i \in I$.

The dynamics of the agent $i = 1, 2$ in the game $\Gamma(c_0, T)$ is described by the initial boundary value problem for the following differential equation on the domain $\Omega = (0, d_1) \times (0, d_2)$:

$$\frac{\partial z^i}{\partial t} = D\frac{\partial^2 z^i}{\partial x_1^2} + D\frac{\partial^2 z^i}{\partial x_2^2} - rz^i + u_i\psi_i(x_1, x_2), \quad x \in \Omega,\ t > 0. \tag{2}$$

Here $D > 0$ is the diffusion coefficient; $r > 0$ is the coefficient characterizing the pollution decomposition; $u_i = u_i(t)$ is a control function of the agent i, satisfying the condition (1). The function $\psi_i(x_1, x_2) = \delta(x_1 - \bar{x}_1^i, x_2 - \bar{x}_2^i)$ gives the location of the agent i inside the domain Ω.

Let the function $z^i(x_1, x_2, t)$ satisfies the following boundary conditions of impenetrability:

$$\frac{\partial z^i}{\partial x_1} = 0, \quad x_1 = 0,\ x_2 \in [0, d_2],\ t > 0, \tag{3}$$

$$\frac{\partial z^i}{\partial x_1} = 0, \quad x_1 = d_1,\ x_2 \in [0, d_2],\ t > 0, \tag{4}$$

$$\frac{\partial z^i}{\partial x_2} = 0, \quad x_1 \in [0, d_1],\ x_2 = 0,\ t > 0, \tag{5}$$

$$\frac{\partial z^i}{\partial x_2} = 0, \quad x_1 \in [0, d_1],\ x_2 = d_2,\ t > 0, \tag{6}$$

and the following initial condition:

$$z^i(x_1, x_2, 0) = c_0^i(x_1, x_2), \quad x_1 \in [0, d_1],\ x_2 \in [0, d_2],\ t = 0, \tag{7}$$

where $c_0^i(x_1, x_2)$ is some given function describing the initial distribution of the pollutant concentration of the agent i in the water reservoir at the initial moment $t = 0$.

Definition 1. *A measurable function $u_i = u_i(t)$, satisfying the condition (1) for all $t \in [0, T]$ is called the admissible control of the agent $i \in I$. Let us denote by $\overline{\mathcal{U}}_i \subset L_p(0, T),\ i \in I$, the set of admissible controls (measurable functions) $u_i(t),\ t \in [0, T]$.*

Let the function $f_i(t) \geq 0$ for all $t \in [0, T]$ determine the amount of the penalty of the agent i for exceeding the maximum permissible value of pollution at the water intake point (x_w, y_w) as follows:

$$f_i(t) = \begin{cases} 0, & \text{if } \sum\limits_{j=1}^{2} z^j(x_w, y_w, t) \leq C_w, \\ \dfrac{z^i(x_w, y_w, t)}{\sum\limits_{j=1}^{2} z^j(x_w, y_w, t)} \cdot \dfrac{\sum\limits_{j=1}^{2} z^j(x_w, y_w, t) - C_w}{C_w}, & \text{if } \sum\limits_{j=1}^{2} z^j(x_w, y_w, t) > C_w, \end{cases}$$

where C_w is the maximum permissible value of pollution at the water intake point.

Denote by $q_i = q_i(t)$ the volume of production of the agent i at the moment t. Let $\bar{P}_i > 0$ be the price of the product of the agent i.

Assume that the intensity of dumping the pollutant $u_i(t)$ linearly depends on the production volume of the agent i:

$$u_i(t) = \alpha q_i(t), \quad \alpha > 0.$$

Let $\bar{p} > 0$ be the payment for the discharge of a unit of pollutant. Then the payoff of the agent i at time T is defined by the following functional:

$$H_i(z, u_i) = \int_0^T P_i u_i(\tau)(1 - p f_i(\tau)) d\tau - \int_0^T M_i u_i(\tau) d\tau, \tag{8}$$

where $z = (z^1, z^2)$, $P_i = \bar{P}_i/\alpha$, $p = \bar{p}/P_i$, $M_i > 0$ is the production expenditures per unit product of the enterprise i. The goal of the agent i is to maximize $H_i(\cdot)$.

Game $\Gamma(c_0, T)$ is a particular case of a noncooperative n-person differential game in Banach space that was studied in [14]. Below, for the sake of completeness, let us recall the main results of [14].

The dynamics of the agent $i \in I = \{i\} = \{1, \ldots, n\}$ in the n-person differential game $\Gamma(c_0, T)$ is described by the initial boundary value problem for the following differential equation:

$$\begin{aligned}\frac{\partial z^i}{\partial t} = \frac{\partial}{\partial x}\left(D(x, y, t)\frac{\partial z^i}{\partial x}\right) + \frac{\partial}{\partial y}\left(D(x, y, t)\frac{\partial z^i}{\partial y}\right) - \\ -r z^i + u_i \psi_i(x, y), \quad (x, y) \in \Omega \in \mathrm{R}^2,\ t > 0.\end{aligned} \tag{9}$$

Here $D(x, y, t) > 0$ is the diffusion coefficient; $r > 0$ is the coefficient characterizing the pollution decomposition; $u_i \in U_i$ is a control parameter of the agent i, $U_i \subset R^{m_i}$ is a compact set in Euclidean space. The function $\psi_i(x, y) = \delta(x - x_i, y - y_i)$ gives the location of the agent i inside the domain Ω.

Let the function $z^i(x, y, t)$ satisfies the following boundary and initial conditions:

$$D(t, x, y)\frac{\partial z^i}{\partial m} = 0, \qquad (x, y) \in S, \quad t \in [0, T], \tag{10}$$

$$z^i(x, y, 0) = c_0^i(x, y), \quad (x, y) \in \Omega, \quad t = 0, \tag{11}$$

where m is an outward normal to the boundary surface $S \times [0, T]$, $c_0^i(x, y)$ is some given function describing the initial distribution of the pollutant concentration of the agent i in the water reservoir at the initial moment $t = 0$.

Let us represent the problem (9)–(11) as the initial-value problem for the following operator-differential equation

$$\frac{dc^i(t)}{dt} - A(t)c^i(t) = \nu^i(t),\ t \in [0, T], \tag{12}$$

$$c^i(0) = z_0^i = c_0^i, \tag{13}$$

where $c^i(t) = z^i(x, y, t)$, $\nu^i(t) = u_i(t) \cdot \delta(x - x_i, y - y_i)$. The operator $Ac = \partial_x(D(t, x, y)c_x) + \partial_y(D(t, x, y)c_y) - rc$ allows for the boundary condition (10).

The equation (12) involves the Dirac measure. The existence of a unique solution of abstract parabolic evolution equations involving Banach space-valued Radon measures is proved in [1].

We assume that the coeffitients D and r in (9)–(11) satisfy the following conditions $D(t, x, y) \in C([0, T]; C^1(\overline{\Omega}))$, $r \in L_\infty(0, T; L_q(\Omega))$. According to results of [1], the unique solution $c^i \in L_p(0, T; W_p^1(\Omega))$, $c_t^i \in L_p(0, T; (W_q^1(\Omega))')$, $i \in I$ of the problem (12)–(13) exists for all $\nu^i \in L_p(0, T; (W_q^1(\Omega))')$, for all admissible control $u_i \in \overline{\mathcal{U}}_i \subset L_p(0, T)$, and for all initial condition $c_0^i \in (W_q^{2/p-1}(\Omega))'$ and $q > 2$ $(1/p + 1/q = 1)$.

Let us denote by $F_i(c_0^i,\, t_0,\, t)$ the set of the points $c^i(\cdot) \in W_p^1(\Omega)$ for which there exists an admissible control $u_i(t)$ such that the game goes from the state $c^i(t_0) = c_0^i$ to the state $c^i(t + t_0)$ for the time interval $[t_0, t]$. The set $F_i(c_0^i,\, t_0,\, t)$ is a bounded set of the space $W_p^1(\Omega)$. It is known [7] that if the boundary of the domain Ω is smooth then a bounded set of the space $W_p^1(\Omega)$ is a compact set in $L_p(\Omega)$. This implies that $F_i(c_0^i,\, t_0,\, t)$ is a compact set for all $c_0^i \in (W_q^{2/p-1}(\Omega))'$, $t_0,\, t \in [0, T]$ as well. $F_i(c_0^i,\, t_0,\, t_0) = c_0^i$ for all $c_0^i \in (W_q^{2/p-1}(\Omega))'$, $t_0 \in [0, T]$. The set $F_i(c_0^i,\, t_0,\, t)$ has semigroup property.

The set $F_i(c_0^i,\, t_0,\, t)$ is called the attainability set of the player i, $i = \overline{1, n}$ from the initial state c_0^i on the time interval $[t_0, t]$.

Let us denote by $\widehat{F}_i(c_0^i,\, t_0,\, t)$, $i \in I$ the set of trajectories $\hat{c}^i(c_0^i, t - t_0)$ of (12)–(13) which start at c_0^i at the moment t_0 and which are defined on the time interval $[t_0, t]$. The set of trajectories $\widehat{F}_i(c_0^i,\, t_0,\, t)$ is compact e.g. in the space $L_p(0, T; W_p^{1-s}(\Omega))$ for any $s > 0$, and the function $\widehat{F}_i(c_0^i,\, t_0,\, t)$ is continuous in the corresponding Hausdorff metric.

At every moment $t \in [0, T]$ of the game $\Gamma(c_0, T)$ the agents know the realized trajectory of the game, the dynamics and the duration T of the game.

Let $\hat{c}^i(\cdot) \in \widehat{F}_i(c_0^i, 0,\, T)$ be the trajectory of (12)–(13) arising from a control u_i and $\Pi_\delta^i(\hat{c}^i)$ be the trajectory arising from the same control u_i delayed by δT. The following lemma describes the relation between these trajectories.

Lemma 1. *For each $\delta \in (0, 1]$ there exists a map $\Pi_\delta^i : \widehat{F}_i(c_0^i, 0,\, T) \to \widehat{F}_i(\cdot)$ such that, if $\hat{c}^i(\tau) = \hat{c}'^i(\tau)$ for $\tau \in [0, t]$, then $\Pi_\delta^i(\hat{c}^i)(\tau) = \Pi_\delta^i(\hat{c}'^i)(\tau)$ for $\tau \in [0, t + \delta T]$ if $(t + \delta T) \le T$ and $\Pi_\delta^i(\hat{c}^i)(\tau) = \Pi_\delta^i(\hat{c}'^i)(\tau)$ for $\tau \in [0, T]$ if $(t + \delta T) > T$. Moreover,*

$$\varepsilon^i(\delta) = \sup_{\hat{c}^i \in \widehat{F}_i(\cdot)} \|\hat{c}^i - \Pi_\delta^i(\hat{c}^i)\| \underset{\delta \to 0}{\longrightarrow} 0.$$

Let us fix the permutation $p = (i_1, \ldots, i_k, \ldots, i_n)$ and consider n-person multistep game $\Gamma_p^\delta(c_0, T)$ at every step which the agents $i_1, \ldots, i_n$ choose in sequence controls $u^{i_1}, \ldots, u^{i_k}, \ldots, u^{i_n}$.

Definition 2. *The strategy*

$$^{\delta}\varphi^{p}_{i_k} : \widehat{F}^{*}_{i_k}(\cdot) = \prod_{j \neq i_k} \widehat{F}_j(\cdot) \to \widehat{F}_{i_k}(\cdot),$$

of the agent i_k in the game $\Gamma^{\delta}_{p}(c_0, T)$ is a mapping such that if $\hat{c}^{j}(\tau) = \hat{c}'^{j}(\tau)$ for $j < i_k$, $\tau \in [0, l\delta T]$ and if $\hat{c}^{j}(\tau) = \hat{c}'^{j}(\tau)$ for $j > i_k$, $\tau \in [0, (l-1)\delta T]$, then $^{\delta}\varphi^{p}_{i_k}(\hat{c}^{*i_k}(\tau)) = {}^{\delta}\varphi^{p}_{i_k}(\hat{c}^{*' i_k}(\tau))$, $\tau \in [0, l\delta T]$. Here $\delta = 1/2^N$, $l = 1, 2, \ldots, 2^N$.

Let us denote by $^{\delta}\Phi^{p}_{i_k}$ the set of the strategies of the agent i_k in the game $\Gamma^{\delta}_{p}(c_0, T)$. In the game $\Gamma^{\delta}_{p}(c_0, T)$ the players $i_1, ..., i_n$ choose in sequence the strategies $^{\delta}\varphi^{p}_{i_1}, \ldots, {}^{\delta}\varphi^{p}_{i_n}$. The trajectory $\chi(^{\delta}\varphi^{p})$ is uniquely defined for every n-tuple $^{\delta}\varphi^{p} = (^{\delta}\varphi^{p}_{i_1}, \ldots, {}^{\delta}\varphi^{p}_{i_n})$ stepwise on successive intervals $[0, \delta T], \ldots, [T - \delta T, T]$. The payoff function of the agent $i \in I$ in the game $\Gamma^{\delta}_{p}(c_0, T)$ is defined as follows:

$$H^{\delta}_{i}(c_0, {}^{\delta}\varphi^{p}) = H_i(\chi^{\delta}(^{\delta}\varphi^{p})), \tag{14}$$

here $H_i(\cdot)$ is the functional of the same kind as (8).

So, the n-person differential game $\Gamma^{\delta}_{p}(c_0, T)$ with the prescribed duration T is defined in a normal form:

$$\Gamma^{\delta}_{p}(c_0, T) = \langle I, \{^{\delta}\Phi^{p}_{i}\}^{n}_{1}, \{H^{\delta}_{i}\}^{n}_{1} \rangle.$$

Using the Zermelo-Neumann theorem, the existence of ε-equilibrium for any $\varepsilon > 0$ in the multistep game $\Gamma^{\delta}_{p}(c_0, T)$ can be proved.

The previous Lemma 1 implies the following lemma.

Lemma 2. *If $i_k > i_1$, $^{\delta}\varphi^{p}_{i_k} \in {}^{\delta}\Phi^{p}_{i_k}$, then $\Pi^{i_k}_{\delta} \cdot {}^{\delta}\varphi^{p}_{i_k} \in {}^{\delta}\Phi^{p_{i_k}}_{i_k}$, where $p_{i_k} = (i_k, \widetilde{p})$, $\widetilde{p}$ is a permutation of the set $I \setminus i_k$; moreover, for $\widehat{c}^{*i_k} \in \widehat{F}^{*}_{i_k}(\cdot)$*

$$\|^{\delta}\varphi^{p}_{i_k}(\widehat{c}^{*i_k}) - (\Pi^{i_k}_{\delta} \cdot {}^{\delta}\varphi^{p}_{i_k})(\widehat{c}^{*i_k})\| \leq \varepsilon(\delta) \underset{\delta \to 0}{\longrightarrow} 0.$$

The following lemma from [8] is valid.

Lemma 3. *Let the game $\Gamma_{H'} = \langle I, \{X'_i\}^{n}_{1}, \{H'_i\}^{n}_{1} \rangle$ be obtained from the game $\Gamma_H = \langle I, \{X_i\}^{n}_{1}, \{H_i\}^{n}_{1} \rangle$ by the epimorphic mapping $\alpha_i : X_i \to X'_i$, $i = 1, \ldots, n$, with*

$$\|H(x) - H'(\alpha x)\| \leq \varepsilon, \qquad \alpha x = (\alpha_1(x_1), \ldots, \alpha_n(x_n)).$$

Then, if x is an ε-equilibrium of the game Γ_H, then αx is the 3ε-equilibrium of the game $\Gamma_{H'}$.

Let us define the main game $\Gamma(c_0, T)$.

Definition 3. *The pair $(\delta_i, \{^{\delta}\varphi^{p_i}_{i}\}_{\delta = 1/2^N})$ is called the strategy of the agent i. Here $N \in Z$, δ_i is a range of dyadic partition of the time interval $[0, T]$ and $^{\delta}\varphi^{p_i}_{i}$ is the strategy of the agent i in the game $\Gamma^{\delta}_{p_i}(c_0, T)$ for the permutation $p_i = (i, \widetilde{p})$ and $\widetilde{p}$ is the permutation of the set $I \setminus i$.*

For n-tuple $\varphi = (\varphi_1, \ldots, \varphi_n)$ the game $\Gamma(c_0, T)$ is played as follows. The smallest $\delta_i = \delta$ is chosen and the trajectory $\chi(\cdot)$ is constructed for n-tuple ${}^{\delta}\varphi = ({}^{\delta}\varphi_1^{p_1}, \ldots, {}^{\delta}\varphi_n^{p_n})$. This trajectory is unique.

The game $\Gamma(c_0, T)$ is obtained from the game $\Gamma_p^{\delta}(c_0, T)$ by the epimorphic mapping which is defined in Lemma 2. Since in the game $\Gamma_p^{\delta}(c_0, T)$ there exists ε-equilibrium, then the existence of the 3ε-equilibrium in the game $\Gamma(c_0, T)$ follows from Lemma 2 and Lemma 3.

Thus, the following theorem is valid.

Theorem 1. *There exists ε-equilibrium in the noncooperative n-person differential game $\Gamma(c_0, T)$ for all $\varepsilon > 0$.*

4 Numerical Method

The numerical method based on the dynamic programming method [2] and the finite difference scheme [13] is proposed for the numerical solving of the auxuliary multistep game $\Gamma_p^{\delta}(c_0, T)$.

To construct a difference scheme for our problem, we use an approach that was developed in [13] for constructing a homogeneous difference scheme for the nonstationary heat conduction problem with a one-point heat source that is defined by a Dirac delta function expression.

On the rectangle $\overline{\Omega} = [0, d_1] \times [0, d_2]$, we construct the uniform grid with the step h_1 on x_1 and the step h_2 on x_2

$$\begin{aligned} \overline{\omega}_h = \{ & x_{1l} = lh_1,\ l = 0, \ldots, N_1;\ x_{10} = 0,\ x_{1N_1} = d_1; \\ & x_{2k} = kh_2;\ k = 0, \ldots, N_2;\ x_{20} = 0,\ x_{2N_2} = d_2 \}, \end{aligned} \tag{15}$$

For simplicity, the points of the agents' dumps are assumed to be grid nodes, namely $(x_{1l}, x_{2k}) = (\bar{x}_1^i, \bar{x}_2^i)$ is a location of the agent i, $i = 1, 2$.

On every interval $[t_s, t_{s+1}], s = \overline{0, N_\sigma - 1}$, we construct the uniform net with step τ

$$\overline{\omega}_{\tau,s} = \{\bar{t}_j = j\tau,\ j = \overline{0, N_3};\ \bar{t}_0 = t_s,\ \bar{t}_{N_3} = t_{s+1}\}.$$

Here $t_s \in \sigma$, where σ is the time interval partition

$$\sigma = \{t_0 = 0 < t_1 < \ldots < t_{N_\sigma} = T\}.$$

For numerical simulations, we will consider the admissible control parameters set $U_i = [\overline{U}_i^1, \overline{U}_i^2]$, where $\overline{U}_i^1 = \text{const}$, $\overline{U}_i^2 = \text{const}$, $i = 1, 2$, and construct the following partition on the set U_i:

$$\Delta_i = \{u_{i,0} = \overline{U}_i^1 < u_{i,1} < \ldots < u_{i,N_4} = \overline{U}_i^2\},\ i = 1, 2.$$

On the time interval $[t_s, t_{s+1}]$ we will consider grid functions ${}^{i}y^{s}(x_{1l}, x_{2k}, \bar{t}_j)$ instead of functions $z^i(x_1, x_2, t)$ of continuous arguments $(x_1, x_2, t) \in \overline{\Omega} \times [t_s, t_{s+1}]$, where $i \in I$ – the agent's number, $s = \overline{0, N_\sigma - 1}$. Here the argument of the grid function $(x_{1l}, x_{2k}, \bar{t}_j)$ is a node of the grid $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_{\tau,s}$.

Let us denote by ${}^i y_{l,k}^{j,s} = {}^i y^s(x_{1l}, x_{2k}, \bar{t}_j)$ the grid function defined on the net $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_{\tau,s}$.

We construct for the problem (2)–(7) the following purely implicit difference schemes [13] on the net $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_{\tau,s}$ for any pair of admissible controls $(u_{1,\xi_1}, u_{2,\xi_2}) \in \Delta_1 \times \Delta_2$, $\xi_i \in \overline{0, N_4}$, $i = 1, 2$:

$$\frac{{}^i y_{0,k}^{j+1/2,s} - {}^i y_{0,k}^{j,s}}{\tau} = 2D\,\frac{{}^i y_{1,k}^{j+1/2,s} - {}^i y_{0,k}^{j+1/2,s}}{h_1^2} - {}^i y_{0,k}^{j,s} r, \tag{16}$$

$$\begin{aligned}\frac{{}^i y_{l,k}^{j+1/2,s} - {}^i y_{l,k}^{j,s}}{\tau} &= D\frac{{}^i y_{l+1,k}^{j+1/2,s} - 2\,{}^i y_{l,k}^{j+1/2,s} + {}^i y_{l-1,k}^{j+1/2,s}}{h_1^2} - \\ &- {}^i y_{l,k}^{j,s} r + u_{i,\xi_i}^s \overline{\delta}_{l,k}^i, \qquad l = \overline{1, N_1 - 1},\end{aligned} \tag{17}$$

$$\frac{{}^i y_{N_1,k}^{j+1/2,s} - {}^i y_{N_1,k}^{j,s}}{\tau} = -2D\frac{{}^i y_{N_1,k}^{j+1/2,s} - {}^i y_{N_1-1,k}^{j+1/2,s}}{h^2} - {}^i y_{N_1,k}^{j,s} r, \tag{18}$$

$$k = \overline{0, N_2},$$

$$\frac{{}^i y_{l,0}^{j+1,s} - {}^i y_{l,0}^{j+1/2,s}}{\tau} = 2D\,\frac{{}^i y_{l,1}^{j+1,s} - {}^i y_{l,0}^{j+1,s}}{h_2^2} - {}^i y_{l,0}^{j+1/2,s} r, \tag{19}$$

$$\begin{aligned}\frac{{}^i y_{l,k}^{j+1,s} - {}^i y_{l,k}^{j+1/2,s}}{\tau} &= D\frac{{}^i y_{l,k+1}^{j+1,s} - 2\,{}^i y_{l,k}^{j+1,s} + {}^i y_{l,k-1}^{j+1,s}}{h_2^2} - \\ &- {}^i y_{l,k}^{j+1/2,s} r + u_{i,\xi_i}^s \overline{\delta}_{l,k}^i, \qquad k = \overline{1, N_2 - 1},\end{aligned} \tag{20}$$

$$\frac{{}^i y_{l,N_2}^{j+1,s} - {}^i y_{l,N_2}^{j+1/2,s}}{\tau} = -2D\frac{{}^i y_{l,N_2}^{j+1,s} - {}^i y_{l,N_2-1}^{j+1,s}}{h_2^2} - {}^i y_{l,N_2}^{j+1/2,s} r, \tag{21}$$

$$l = \overline{0, N_1}, \quad j = \overline{0, N_3 - 1},$$

$${}^i y_{l,k}^{0,s} = {}^i y_{l,k}^{N_3,s-1}, \quad l = \overline{0, N_1}, \ k = \overline{0, N_2}, \quad \xi_i = \overline{0, N_4}, \quad s = \overline{1, N_\sigma - 1}, \tag{22}$$

$${}^i y_{l,k}^{0,0} = c_0^i(x_{1l}, x_{2k}), \quad l = \overline{0, N_1}, \ k = \overline{0, N_2}, \tag{23}$$

Here

$$\overline{\delta}_{l,k}^i = \begin{cases} 0, & \text{if } (x_{1l}, x_{2k}) \neq (\bar{x}_1^i, \bar{x}_2^i), \\ \dfrac{1}{h_1 h_2}, & \text{if } (x_{1l}, x_{2k}) = (\bar{x}_1^i, \bar{x}_2^i), \end{cases}$$

The constructed absolutely stable difference scheme (16)–(23) is solved by the elimination method [13].

The payoff function of the agent $i,\ i=1,2$ in the game $\Gamma_p^\delta(c_0,T)$ is approximated as follows:

$$\underline{H}_i(u_1^0,\dots,u_1^{N_\sigma-1},u_2^0,\dots,u_2^{N_\sigma-1}) = \tau\sum_{s=0}^{N_\sigma-1}\sum_{j=0}^{N_2-1} P_i u_i^s(1-pf_i^j) - \\ - \tau\sum_{s=0}^{N_\sigma-1}\sum_{j=0}^{N_2-1} M_i u_i^s, \tag{24}$$

Let $\underline{V}_i^\delta(\cdot)$ be the value of the payoff function of the agent $i,\ i=1,2$ at the equilibrium point. The following recurrence equations are valid:

$$\underline{V}_i^\delta({}^1y^{N_\sigma-1},{}^2y^{N_\sigma-1},t_{N_\sigma-1}) = \max_{u_{i,\xi_i}^{N_\sigma-1}\in\Delta_i}\{\underline{H}_i(u_{i,\xi_i}^{N_\sigma-1},\bar{u}_{\{I\setminus i\},\xi_{\{I\setminus i\}}}^{N_\sigma-1})\}, \tag{25}$$

$$\underline{V}_i^\delta({}^1y^s,{}^2y^s,t_s) = \max_{u_{i,\xi_i}^s\in\Delta_i}\{\underline{H}_i(u_{i,\xi_i}^s,\bar{u}_{\{I\setminus i\},\xi_{\{I\setminus i\}}}^s) + \underline{V}_i^\delta({}^1y_{\xi_1}^{s+1},{}^2y_{\xi_2}^{s+1},t_{s+1})\}, \\ s=\overline{N_\sigma-2,0}, \tag{26}$$

$$\underline{V}_i^\delta({}^1y^0,{}^2y^0,t_{N_\sigma}) = 0, \tag{27}$$

Here

$$\underline{H}_i(u_{i,\xi_i}^s,\bar{u}_{\{I\setminus i\},\xi_{\{I\setminus i\}}}^s) = \tau\sum_{j=0}^{N_2-1} P_i u_{i,\xi_i}^s(1-pf_i^j) - \tau\sum_{j=0}^{N_2-1} M_i u_{i,\xi_i}^s. \tag{28}$$

5 Numerical Results

The numerical experiments were carried out for the following input data: $D=4.4$, $d_1=d_2=30$, $h=0.5$, $T=320$, $\tau=1$, $N_\sigma=8$, $r=0.005$, $p=3.1$, $M_1=4.5$, $M_2=5.5$, $P_{1(2)}=15$, $U_1^i=\{1,10,20,30\}$, $U_2^j=\{1,10,20,30\}$, $(\bar{x}_1^1,\bar{x}_2^1)=(5,5)$, $(\bar{x}_1^2,\bar{x}_2^2)=(25,25)$, $(\bar{x}_{1w},\bar{x}_{2w})=(15,5)$, $C_w=3$, $c_0(x)=0$.

The results of numerical experiments are presented in Figures 1–5.

Figure 1 presents the computed distribution of the pollutant concentration of the first and second agents. The realizations of their optimal strategies are shown in Fig. 2. The agents reduce the intensity of dumping the pollutant (stop production) to minimize fines. Then the intensity of production is resumed to the maximum values for obtaining the maximum income by the time T. The time development of the payoff functions for both agents is given in Fig. 3. The dynamics of changing the pollutant concentration at the intake point is presented in the Fig. 4. The agent pays the penalty (Fig. 5) in the case of exceeding the maximum permissible value C_w.

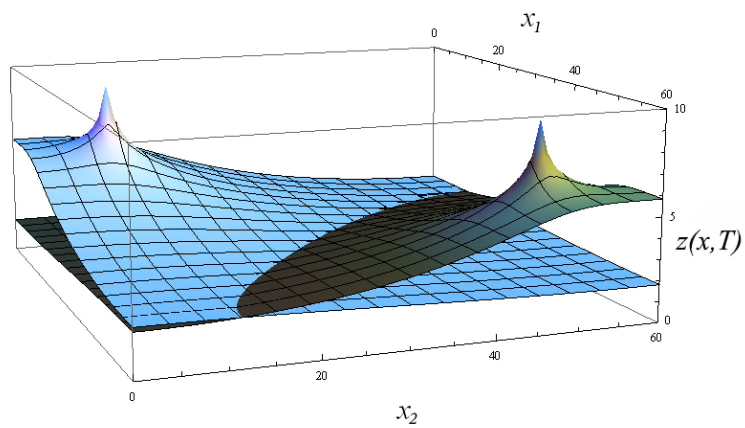

Fig. 1. Distribution of the pollutant concentration of agents at the moment T: 1 – light color, 2 – dark.

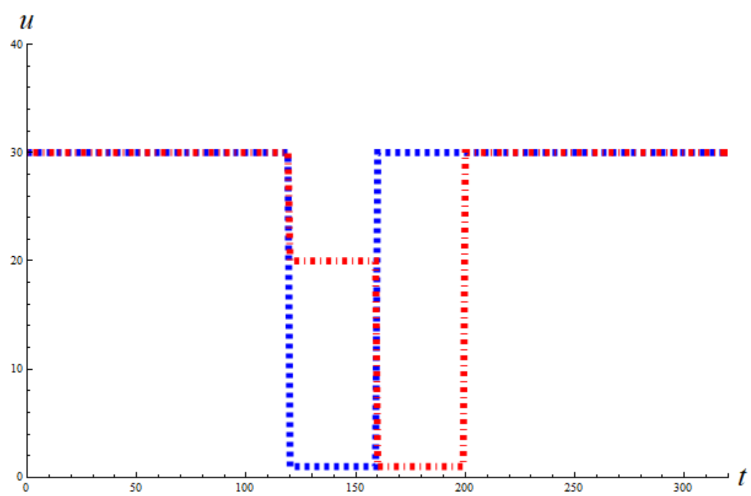

Fig. 2. Optimal strategies of agents: 1 – blue dashed line, 2 – red dot-dashed line.

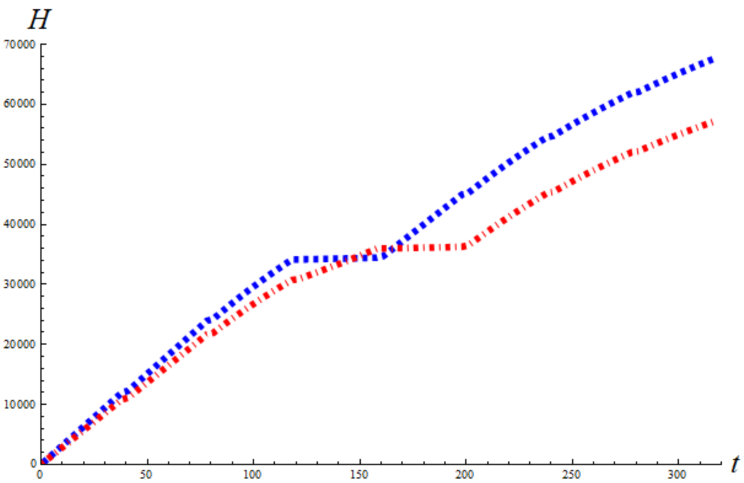

Fig. 3. Payoff functions of agents: 1 – blue dashed line, 2 – red dot-dashed line.

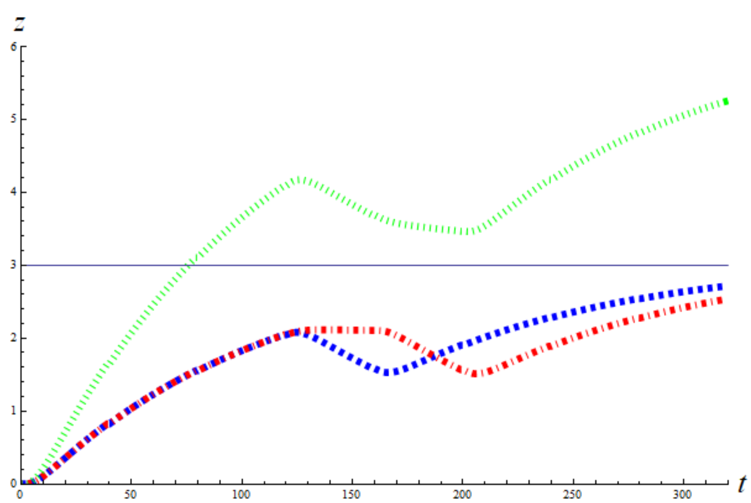

Fig. 4. The pollutant concentrations at the point of intake: total – green dotted line, 1 – blue dashed line, 2 – red dot-dashed line.

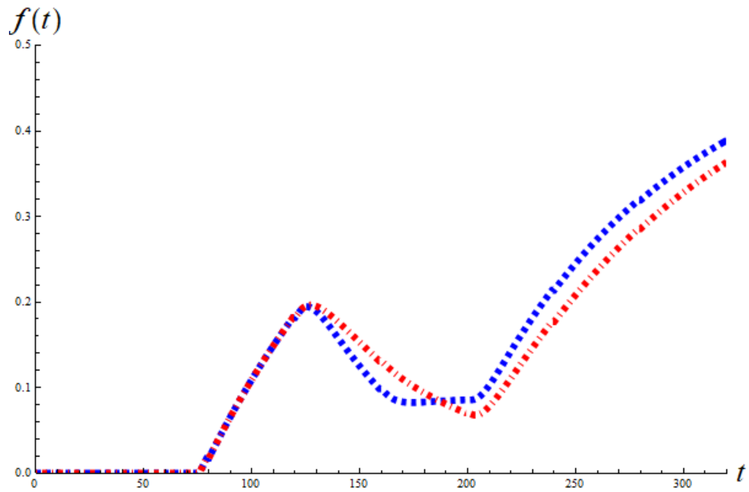

Fig. 5. The penalty functions $f(t)$: 1 – blue dashed line, 2 – red dot-dashed line.

6 Conclusion

We investigated the noncooperative two-person differential game in Banach space which models a conflict-controlled process of the contaminating a closed water reservoir. The dynamics of the agents is described by the initial boundary value problem for the two-dimensional diffusion equation with a point source.

The proposed numerical algorithm for solving the considered differential game is based on the dynamic programming method and the finite difference scheme. It has been applied to compute the auxiliary multistep game.

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