When Laziness Leads to Stability: Approximate Equilibria in One-Dimensional Jurisdiction Formation Models

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Abstract

In this paper, we discuss the question of approximate stability in group partition (Alesina-Spolaore-like) models. Consider a world with a finite number of agents living along a line. The coordinate may represent a location in some geographical or virtual space. A subgroup of agents may build a facility at some point and enjoy benefits from it. Each facility costs some amount g, and every agent pays for her transportation. These costs may be somehow redistributed.

We seek for a stable division of the whole society into subgroups. Two main stability concepts are considered. Migrational stability means that no individual may decrease his cost by changing the group. Coalitional stability means that no new group may emerge and decrease the costs of all its members. We relax these concepts: an individual must substantively decrease her cost in order to break the structure, otherwise the change is not worth it.

It was shown by Bogomolnaia et al that for some rules there are no stable configurations. We try to maximize the approximation rate for which the counterexamples still exist. Specifically, we study three cases: coalitional stability without redistribution, migrational stability with the central median rule and migrational stability with Rawlsian redistribution. We found the worlds with instability 6.2%, 2.5% and 9.6% respectively. In the first case we prove that this is the maximal rate for all bipolar worlds.

Keywords

Facility location, group partition, coalitional stability, migrational stability, approximate equilibrium

1. Introduction

Every person in a society belongs to many groups and communities. For some specific relations, the groups must be disjoint. For instance, in the USA a citizen may be a registered supporter of only one political party. In many countries a person must have a unique address of residence, thus the municipalities or regions may be treated as non-intersecting communities. If a group of tourists choose where to go on a particular day, then everyone may choose only one excursion.

What is the rationale behind forming groups (communities, clubs, coalitions, jurisdictions etc.)? Why people cannot just live alone? The answer is that some valuable goods are either impossible or too costly to produce by a single agent. Thus, people unite themselves into clubs in order to provide such club goods. But organizing a club is costly, and the members may disagree with each other about the type and quantity of the provided club goods. Sometimes there could be congestion issues: the

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club good deteriorates when the number of users rises. This is why there could be no grand coalition, but some coalitional structure.

In this paper we study the question of *horizontal differentiation*. The club good has some features and all agents have preferences over them. The cost of organizing a club and providing the club good is constant and does not depend on the number of users. Thus, there are two opposite forces. On the one hand, large groups are better since they provide economy of scale. Each member pays less share of the total cost. On the other hand, small (and homogeneous) groups are better since they can tune the features of the good precisely under members' preferences. The main question is whether these forces can always equilibrate each other. And how do details of the model influence the answer?

Two main equilibrium concepts appear in the literature. *Migrational stability* means that no individual wishes to change her community unilaterally in order to reduce her costs. *Coalitional stability* suggests that no group of agents would like to secede from the club system, organize a new club and thus reduce the costs of all members. The distribution of agents may be described in different ways. Firstly, the underlying feature space may be of different dimensions. Secondly, the distribution itself may be of three types. It may be discrete, when a finite set of agents live at some specific points. It may be continuous, when a continuum of agents live according to some density. And it may be atomic, when a continuum of agents live at a finite number of points. The costs may be redistributed between the agents in some way: usually, less satisfied agents may be subsidized or pay a less share of the fixed cost. The combinations of these specifications lead to a family of models. In some of them an existence theorem may be proven. In some others an example without a stable partition may be constructed.

This study considers the frameworks without an equlibrium. We relax the notions of stability: suppose that a threatening individual or group must not only get better after the change, but decrease their costs by a considerable amount. Of course, if this amount is large enough, the stable configuration will always exist. Our research question is to find the maximal size of this amount when the configuration is still unstable. We consider three uni-dimensional frameworks:

- Coalitional stability without cost redistribution;
- Migrational stability without cost redistribution;
- Migrational stability with egalitarian redistribution.

In each case we find the most unstable world in some family. We do not know whether our examples are the furthest from stability globally, but we provide some evidence in favour of this conjecture.

1.1. Related literature

In this subsection we describe some previous studies in the areas of club goods, facility location and group partition.

The discussion about local public goods started in the 1950s. Samuelson [1] presented a model that demonstrated inefficiency of decentralized public good provision. The main idea was that the freerider problem is inevitable. If an agent buys a public good by himself, then he cares only about his own benefits, not about the others. This is why it will be always underfinanced. Tiebout [2] agreed that this is the case for the whole economy, but stated that the situation with local public goods is different. There is an option to change the jurisdiction, or to "vote by feet". This move simultaneously changes the tax level and the characteristics of the provided public good, so one may hope to obtain an equilibrium.

Tiebout's paper lacked a formal model. This is why the successive studies tried to formalize Tiebout's intuition. In particular, the researchers tried to justify the existence and efficiency of the suggested equilibrium. The early endeavours included the papers by Westhoff [3] and Bewley [4]. A thorough analysis was done by Greenberg and Weber [5] who introduced the notions of migrational and coailitional stability and justified both of them by quotes from the Tiebout's paper. All these efforts were done in the framework of vertical differentiation: the good was homogeneous, but the agents differed in willingness to pay for it and were self-selecting in jurisdictions with different levels of public good.

The notion of horizontal differentiation was brought to the field by the seminal paper of Alesina and Spolaore [6]. They introduced a one-dimensional parameter that describes both the local public good and the agents' preferences. They interpreted the model in the spirit of political science: the parameter describes a geographical location, the coalitions are nations and the facilities are their capitals. Each citizen pays taxes to finance the capital and needs to visit it sometimes. The paper also considered the concepts of migrational and coalitional stability.

In the next decades many variations and generalizations of the model were considered in various papers. In particular, the issues of nonuniform density or discrete configuration, various cost redistribution rules and multiple dimensions were studied. In some frameworks it was shown that an equilibrium does always exist, in other cases very involved counterexamples were constructed. Haimanko et al [7] show that if any redistribution may be applied, then a coalitionary stable partition does always exist. Weber et al [8] present an example where no migrational equilibrium may exist under the egalitarian redistribution rule. Bogomolnaia et al [9], [10] and Savvateev [11] studied discrete models with a finite number of agents and presented several examples in different frameworks where no equilibrium exists. On the other hand, Musatov et al [12], Savvateev et al [13] and Marakulin [14] presented rather general existence theorems in the continuous framework.

The concept of approximate stability was employed in the two-dimensional analysis by Drèze et al [15]. It was later elaborated by Golman and Musatov [16]. This paper expands the results obtained there.

1.2. Roadmap

The remaining part of the paper is organized as follows. In Sect. 2 a detailed formal model is presented, including the definitions of approximate stability. In Sect. 3 we extensively analyze the case of coalitional stability of a bipolar world. In Sect. 4 and Sect. 5 we present a preliminary analysis of migrational stability in two settings: egalitarian redistribution and the central median rule. Finally, Sect. 6 presents a conclusion.

2. The model

2.1. General setting

Here we formulate a model of jurisdiction formation in a linear world with a finite number of agents. This model is similar to the models in [9] and [8]. A continuous model may be found, for instance, in [12]. In our model we have an agent set $A = \{1, \dots, N\}$. Every agents i "lives" at location x_i, where $x_1 \le \dots \le x_N$. A community S is a nonempty subset of A. It may build a facility at some point m. We consider two possible rules of determining m:

- *The median rule.* The point m is a median of S, i.e., the sets $\{i \mid x_i \le m\}$ and $\{i \mid x_i \ge m\}$ must have equal sizes. If S contains an even number of agents and there is a segment of medians, then any median will fit.

- *The central median rule.* The same as before, but if there is a segment of medians, than the middle point of this segment is chosen.

Any median point has two features. Firstly, if the transportation cost function is linear, then a median location minimizes the total cost. Secondly, it wins over any other option by majority voting. The central median rule, moreover, always returns a unique point. In the sequel we denote by $\text{med}(S)$ the set of suitable medians under the specified rule. It will be either a segment or a singleton.

If a jurisdiction is established, then all its members may use its facility. There is no congestion, so the cost of maintenance does not depend neither on the number of users, nor on the location of the facility. Denote this cost by g. Apart of this cost, all members should be transported to the location. The transportation cost of an agent at point x to the facility at point m equals $t \cdot |x - m|$. We consider two rules of distributing the total cost:

- *The private cost rule.* Every agent pays for himself, so the total cost of agent i within coalition S with facility at m equals
$$C_P(i, S, m) = \frac{g}{|S|} + t \cdot |x_i - m|,$$
where $|S|$ is the cardinality of S.

- *The Rawlsian (egalitarian) rule.* The total cost is redistributed such that every agent pays an equal share. Here we have the following cost function:
$$C_R(i, S, m) = \frac{1}{|S|}(g + t \sum_{j \in S} |x_j - m|).$$

Note that under the central median rule the facility is uniquely located, so we can omit m in the notation.

Suppose that the whole society is divided into communities $S_1, \dots, S_k$ with facilities at medians $m_1, \dots, m_k$. We analyze the following notions of stability:

- *Migrational (or Nash) stability.* No individual wishes to change his community. Formally, for every agent i belonging to community S_j and any other community S_l it holds that
$$C(i, S_j) \leq C(i, S_l \cup \{i\}).$$
Note that agent i anticipates that she would change the community she joins and the distribution of costs within it. The existing members of S_l may become better off or worse off, it does not matter. The median of S_j is predetermined, and the median of $S_l \cup \{i\}$ is either uniquely defined, or can be chosen arbitrarily among possible variants.

- *Coalitional (or core) stability.* No group of agents wish to establish a new community such that all members become better off. Formally, there is no set S with a median m such that for all communities S_j and all agents $i \in S \cap S_j$ it holds that
$$C(i, S, m) < C(i, S_j, m_j),$$
where m_j is the median of S_j.

Note that migrational and coalitional notions of stability are incomparable. Namely, if an individual changes the jurisdiction, then the old members of the new jurisdiction may become worse off due to the change of the median. Thus, they would not like to admit the new member. Thus, this change would be a threat for migrational stability but not for coalitional stability. Vice versa, it may occur that no individual move may decrease the cost, but a collective change may decrease the costs of all participants.

2.2. Approximate notions of stability

Suppose that changing the jurisdiction or establishing a new one is costly: for instance, it takes some effort for negotiation. In this case a structure would be unstable only if some change may substantively decrease the cost of all involved agents. We consider two measures of instability:

- *Absolute, or additive, instability.* A threat is credible only if all agents performing the move decrease their costs by some absolute value α. Of course, the scale should be fixed somehow in order to compare different unstable societies with each other.

- *Relative, or multiplicative, instability.* A threat is credible only if all agents performing the move decrease their costs by some fraction δ. This definition does not depend on the scale.

Now let us put it formally.

Definition 1. Consider a society structure $\mathcal{P}$, i.e. a partition $X = S_1 \sqcup \cdots \sqcup S_k$ with facilities at points $m_1, \ldots, m_k$, respectively. Denote by $S(i)$ the community that contains agent i and by $m(i)$ the respective median. Denote by $C(i, \mathcal{P})$ the cost of i in $\mathcal{P}$, i.e. $C(i, S(i), m(i))$. (Where $C(\cdot)$ denotes either $C_P(\cdot)$ or $C_R(\cdot)$). Now define two measures of absolute instability of a group structure:

- The *absolute migrational instability* of the structure is the value

$$\Delta^M_{abs}(\mathcal{P}) = \max_{i \in A} (C(i, \mathcal{P}) - \min_{\substack{j=1,\ldots,k \\ m \in \mathrm{med}(S_j \cup \{i\})}} C(i, S_j \cup \{i\}, m));$$

- The *absolute coalitional instability* of the structure is the value

$$\Delta^C_{abs}(\mathcal{P}) = \max_{T \subset A, T \neq \emptyset, m \in \mathrm{med}(T)} \min_{i \in T} \big(C(i, \mathcal{P}) - C(i, T, m) \big).$$

Note that all instability measures are always non-negative: the status quo costs lie in the range of minimization, so the zero difference is achievable. It should be also clear that the instability is zero if and only if the structure is stable. We proceed by defining the relative measures.

Definition 2. Keep all the notation from the previous definition. Define the following measures of relative instability:

- The *relative migrational instability* of the structure is the value

$$\Delta^M_{rel}(\mathcal{P}) = \max_{i \in A} \frac{C(i, \mathcal{P}) - \min_{\substack{j=1,\ldots,k \\ m \in \mathrm{med}(S_j \cup \{i\})}} C(i, S_j \cup \{i\}, m)}{C(i, \mathcal{P})};$$

- The *relative coalitional instability* of the structure is the value

$$\Delta^C_{rel}(\mathcal{P}) = \max_{T \subset A, T \neq \emptyset, m \in \mathrm{med}(T)} \min_{i \in T} \frac{C(i, \mathcal{P}) - C(i, T, m)}{C(i, \mathcal{P})}.$$

We call the structure δ-stable in some sense, if its instability in this sense is not greater then δ.

Note that instead of taking the maximin of $\frac{C(i,\mathcal{P})-C(i,T,m)}{C(i,\mathcal{P})}$ we may take the minimax of $\frac{C(i,T,m)}{C(i,\mathcal{P})}$. Later we refer to this value as to the *new-to-old* ratio.

Our general question is the following:

Problem 1. Find the least upper bounds on $\Delta(x_1, \dots, x_n)$ in different variations. Which worlds are the least stable under which rules? And which rules produce the least stable worlds?

Partial solutions to this problem include computing the values of $\Delta(x_1, \dots, x_n)$ for particular worlds, finding parameters with large instability, establishing some upper bounds and exact solution for some families of worlds (for instance, parametric).

3. Approximate coalitional stability

This question was previously studied by the last two authors in [16]. That paper was focused on bipolar worlds, where the agents live in just two points. It was found out that the least absolutely stable configuration is the one with 3 agents at one point and 4 agents at another point. For cost parameters $g = t = 1$ the distance is $\frac{5}{24} \approx 0.208$. The analysis of relative instability was done numerically. It was shown that the maximal possible relative instability is somewhere between 0.061 and 0.063. In this paper we provide an exact solution and prove that the maximal instability in a bipolar world is (asymptotically) $\sqrt{65} - 8 \approx 0.0622$. Our conjecture is that this bound holds for any world.

3.1. Group structures in a bipolar world

Consider a bipolar world with L agents at point 0 and R agents at point d. W.l.o.g. we postulate that $g = t = 1$ and $L \leq R$. It is also instrumental to think that the weight of a single agent is $\frac{1}{L}$, so the total weight of the left agents is 1 and the total weight of the right agents is $r = \frac{R}{L} \geq 1$. We consider several specific structures.

Definition 3.

- *Union* is the structure with one grand coalition and the facility at point d.
- *Federation* is the structure with two coalitions: the left coalition with all agents at 0 and the right coalition with all agents at d.
- *Mixed structure* is a structure where one coalition includes equal number of agents from both poles and has a facility somewhere in between (we call such a coalition *ambiguous*). Among those structures we consider the *MaxAmbig*s, where one coalition includes a unit weight of agents from each pole and the other contains agents of weight $r - 1$ from the right. The former coalition will be also referred to as the MaxAmbig. These atructures are parameterized by the facility position $\mu \in [0, d]$.
- *Pseudofederation* is a structure where one coalition includes all agents from one pole and some agents from the other, and the second coalition is formed by the remaining agents of the other pole. We distinguish *right pseudofederations* where the first coalition includes all agents from the right and *left pseudofederations* where it includes all agents from the left. All pseudofederations may be parameterized by a single parameter $k \in (-1, r)$. If $k < 0$, then the dispersed coalition consists of $|k|$ agents from the left and r agents from the right. If $k > 0$, then it consists of mass-1 agents from the left and k agents from the right. In case $k = 0$ the structure turns into the Federation. In case $k \in \{-1, r\}$ the structure becomes the Union.

Of course, this list is not exhaustive. In particular, some structures may contain three or more coalitions. We claim that the most stable configuration may be of one of three types: Union, MaxAmbig or Pseudofederation(k) where $k \in [0, r-1]$. In the sequel we prove this assertion and find the world where the relative instability of these structures is maximal. In fact, we proceed in the opposite way. Firstly, we maximize the instability of the three types of partition over all possible worlds. Then we show that the instabilities of all other structures of this specific world are even greater.

3.2. Analysis of the three main structures

It was shown by Savvateev [17] that in an unstable structure it must hold that $d > 0.5$, $r < 1.62$, $r < d + 1$ and $dr < 1$ (the exact bounds are tighter, but these are enough for us). In the sequel we take these bounds as granted. Now consider all credible threats for each structure and calculate the new-to-old ratios for them.

Lemma 1. *If the Union is unstable, then the set of all left agents wins the most from the secession.*

Proof. Indeed, all right agents have the smallest possible cost in the Union. So only a coalition of left agents may want to secede. But the cost is the least possible in the coalition of all left agents. So this set has the maximal willingness to secede. □

Note that the new-to-old ratio is $1/\left(\frac{1}{1+r} + d\right)$.

Lemma 2. *Let Pseudofederation(k), where $k \in [0, r-1]$, be unstable. If the Union is also unstable, then the group that wins the most from the secession is either the MaxAmbig group or the set of all right agents.*

Proof. Recall that $k > 0$ means that there is a group with all agents from the left and some agents from the right. Suppose that some group with the center on the left wins from the secession. Since the left agents are better off, the group must be larger then the initial left group. If all left agents join the group, they are even better off. Some agents from the right must also be in that group and be better off. Thus, the right agents will join the seceding group until the center is still at 0. So, the limit case is the MaxAmbig group.

Similarly, if some ambiguous group is a credible threat, then it must be larger than the initial left group. Thus it must include agents from the right group and the MaxAmbig group is a greater threat. If some group with the center on the right is a credible threat, then it will be more credible when expanded to the whole right pole. If there were some agents from the left, then the grand coalition will be a greater threat again. But we assume that the Union is unstable, so the set of all left agents reduces the costs after seceding from the Union. By transitivity, it means that the left pole becomes better off when seceding from the Pseudofederation, which is a contradiction. □

The analysis of the new-to-old ratio is more complex here. We should maximize the ratio over all possible threats and all agents participating in a threat. Consider two possibilities:

- MaxAmbig threat with median μ. Here the agents from the left compare the new cost $\frac{1}{2}+\mu$ to the old cost $\frac{1}{1+k}$. The agents from the right compare $\frac{1}{2}+d-\mu$ to either $\frac{1}{r-k}$ or $\frac{1}{1+k}+d$. It can be shown that with our parameters the latter is greater than the former, so we should maximizes the ratios $\frac{\frac{1}{2}+\mu}{\frac{1}{1+k}}$ and $\frac{\frac{1}{2}+d-\mu}{\frac{1}{r-k}}$. Since the first ratio grows with μ and the second one decreases, the maximum of them is the largest when they are equal. Solving the equation, we get $\mu = \frac{1}{2} + d - \frac{(1+k)(1+d)}{1+r}$. Plugging this back to the ratios, we get the value $\frac{(1+d)(1+k)(r-k)}{1+r}$.

- The right pole threat. Here the agents from the undivided coalition compare $\frac{1}{r}$ to $\frac{1}{r-k}$ and the agents from the right part of the dispersed coalition compare $\frac{1}{r}$ to $\frac{1}{1+k} + d$. Thus the maximal new-to-old ratio is $\max\left\{1 - \frac{k}{r}, \frac{1+k}{r(1+d+dk)}\right\}$.

Minimization of the maximal new-to-old ratio yields the following result:

Lemma 3. *The minimax of the new-to-old ratio in a Pseudofederation(k) is achieved when* $k = \frac{1-rd}{r(1+d)}$. *The minimum is reached simultaneously for the right pole threat and for the MaxAmbig threat with* $\mu = \frac{1}{2} + d - \frac{(1+k)(1+d)}{1+r}$. *The corresponding new-to-old ratio is* $1 - \frac{1-rd}{r^2(1+d)}$.

Proof sketch. The proof is rather straightforward, but technical, so we present only the general direction. It can be shown that MaxAmbig is the most credible threat for k close to 0 and for k close to $r-1$ and the right pole is the most credible for the intermediate values of k. Thus, the value of the most credible threat has two local peaks. It can be checked that the left one is higher and corresponds to the claimed values of k and μ. □

Finally, we consider the last case.

Lemma 4. *Let MaxAmbig(μ) be unstable. If the Union and the Federation are also unstable, then the group that wins the most from secession is either the right pole, or the left pole united with the part of the right pole not in the ambiguous coalition.*

Proof. It is clear that an ambiguous coalition cannot be a credible threat. Indeed, it cannot be larger than MaxAmbig, and the facility becomes further from one half of the agents. The Union cannot be credible if it is unstable, analogously to lemma 2. If a credible threat has the facility on the right, then it must contain agents from both the ambiguous coalition and the remaining right coalition. Thus, the whole right pole must be more credible. Similarly, if it has the facility on the left, it must contain some agents from the right. Otherwise, the Federation would be stable. Thus, the coalition containing the left part of the ambiguous coalition and the whole right coalition must be the most credible. □

Minimization of the maximal new-to-old ratio yields the following result:

Lemma 5. *The maximal new-to-old ratio is minimal in a MaxAmbig(μ) when* $\mu = \frac{d}{2}$.

Proof. If the right pole is a credible threat, then the agents from the right part of the ambiguous coalition compare $\frac{1}{r}$ to $\frac{1}{2} + d - \mu$ and the agents from the right coalition compare $\frac{1}{r}$ to $\frac{1}{r-1}$. Similarly, if the left pole plus the right coalition is a credible threat, then the agents from the left compare $\frac{1}{r}$ to $\frac{1}{2} + \mu$, and the agents from the right compare $\frac{1}{r} + d$ to $\frac{1}{r-1}$. Note that

$$\frac{1}{2} + d - \mu < \frac{1}{2} + d < \frac{1}{2} + \frac{1}{r} < \frac{1}{r-1},$$

where the second inequality arises from $dr < 1$ and the last one from $r < 2$. Thus, in the first case the right half of the ambiguous coalition is relatively less satisfied than the right coalition. To determine the instability, we should now compare three ratios: $\frac{1}{r}/\left(\frac{1}{2} + d - \mu\right)$, $\frac{1}{r}/\left(\frac{1}{2} + \mu\right)$ and $\left(\frac{1}{r} + d\right)/\frac{1}{r-1}$. It is clear that the maximum of the first two values is minimal when $\mu = \frac{d}{2}$. Then it can be shown that $\frac{1}{r}/\left(\frac{1}{2} + \frac{d}{2}\right) > \left(\frac{1}{r} + d\right)/\frac{1}{r-1}$. Indeed, it is equivalent to $2 > (1 + dr)(1 + d)(r - 1)$. Since $dr < 1$, it follows from $(1 + d)(r - 1) < 1$, which follows from $dr < 1$ and $r < d + 1$. □

Now we compare the new-to-old ratios of all three structures and prove the following result:

Theorem 1. *The maximal new-to-old ratio is minimal among Union, Pseudofederation(k) and MaxAmbig(μ) for* $d = \frac{\sqrt{65}-3}{8} \approx 0.6328$ *and* $r = \frac{1+\sqrt{13/5}}{2} \approx 1.3062$. *The relative instability is* $\epsilon = \sqrt{65} - 8 \approx 0.06226$ *and the parameter of Pseudofederation is* $k = \frac{1-rd}{r(1+d)} \approx 0.0813$.

Proof sketch. We should minimize the maximum of $U(d,r) = 1/\left(\frac{1}{1+r} + d\right)$, $P(d,r) = 1 - \frac{1-rd}{r^2(1+d)}$ and $M(d,r) = \frac{1}{r}/\left(\frac{1}{2} + \frac{d}{2}\right)$. It can be shown that M is decreasing by both variables, P is increasing by both and U is decreasing by d and increasing by r. Thus, if the maximum is reached by only one function, then it can be decreased by changing some variable. If it is reached by M and U simultaneously, then it can be decreased by rising d. If it is reached by P and U, then it can be decreased by lowering r. And if it is reached by M and P, then it equals $\frac{2(r+1)}{3r+1}$ and thus can be decreased by lowering r. Thus, the maximum is minimal only if all three maximands are equal. Solving the equations, one can obtain the claimed values. □

3.3. Analysis of the remaining structures

Now we show that in any setting the least unstable configuration is either the Union, or a Pseudofederation, or a MaxAmbig. We start from the following theorem proven in [16].

Theorem 2. *In any bipolar world, the most unstable configuration consists of at most three jurisdictions, of which:*

- *At most one locates the facility at* 0*, at most one locates it at* d *and at most one locates it somewhere else;*
- *Among coalitions with the facilities at* 0 *and* d *at most one contains agents from the other pole.*

Now we should exclude all possibilities except the three considered types. The idea is to prove that in any other configuration some coalition wishes to break out with willingness greater than $\sqrt{65} - 8$. The whole proof is rather technical and tedious, so we present only the general idea.

Theorem 3. *In any bipolar world, the most unstable configuration is either the Union, or a Pseudofederation(k) for* $k \in [0, r-1]$*, or a MaxAmbig(μ).*

Proof idea. Theorem 2 leaves the following possible configurations: two coalitions where one is ambiguous but not maximal; a Pseudofederation(k) with $k \notin [0, r-1]$, or a configuration with 3 coalitions. The following facts may be demonstrated and imply the statement:

- If one coalition is ambiguous but not maximal, then some threat is more credible than the most credible one in the MaxAmbig with the same median. Thus, MaxAmbig is more stable.
- The relative instability of Pseudofederation(k) increases when $k > r - 1$. If $k \in [0, 1]$, then the relative instability of Pseudofederation($-k$) is higher than the one of Pseudofederation(k). Thus, the minimal instability of a Pseudofederation is reached for $k \in [0, r-1]$.
- In the case of three coalitions several subcases are considered. If the ambiguous coalition is rather small, it will be much better off when joining some other coalition. If it is large, then it can be shown that MaxAmbig is more stable. For intermediate sizes it can be directly shown that the instability is larger than $\sqrt{65} - 8$.

□

4. Approximate migrational stability under egalitarian redistribution

Here we consider the model described by Weber et al [8]. All costs inside a jurisdiction are redistributed equally among all its members. The authors show that there exists a world without a stable configuration. We analyze their construction and tune the parameters in order to obtain the maximal possible instability. We believe that this value is close to maximum.

The example is the following: there are 6 agents living at 3 points. Agents 1 lives at point 0, agents 2 and 3 live at point a and agents 4, 5 and 6 live at point $a+b$. In [8] the parameters are the following: $a = 1.9$, $b = 2.2$. The idea behind the result is the following. The grand coalition is not stable because agent 1 wishes to secede and not to subsidize the others' transportation. Then agent 2 would also like to secede for the same reason (the configuration is $(\{1\}, \{2\}, \{3, 4, 5, 6\})$). Then agent 3 would like to join agent 2: $(\{1\}, \{2, 3\}, \{4, 5, 6\})$. Then agent 1 would also like to join: $(\{1, 2, 3\}, \{4, 5, 6\})$ (Note that agents 2 and 3 may become worse off, but in the migrational setting it does not matter.) But now agent 3 would like not to subsidize agent 1's transportation and would better join the right coalition: $(\{1, 2\}, \{3, 4, 5, 6\})$. Now agent 1 would like to be alone, and we return to $(\{1\}, \{2\}, \{3, 4, 5, 6\})$.

Now we calculate the values compared by the switching agents. Every row of the following table presents the initial configuration, the final configuration, the agent who wishes to move, the values she compare (the initial cost reduced by δ and the final cost) and the resulted bound on δ.

Old config.	New config.	Agent	Ineq. on util.	Ineq. on δ
$\{1, 2, 3, 4, 5, 6\}$	$\{1\}, \{2, 3, 4, 5, 6\}$	1	$\frac{a+3b+1}{6}(1-\delta) > 1$	$\delta < \frac{a+3b-5}{a+3b+1}$
$\{1\}, \{2, 3, 4, 5, 6\}$	$\{1\}, \{2\}, \{3, 4, 5, 6\}$	2	$\frac{2b+1}{5}(1-\delta) > 1$	$\delta < \frac{2b-4}{2b+1}$
$\{1\}, \{2\}, \{3, 4, 5, 6\}$	$\{1\}, \{2, 3\}, \{4, 5, 6\}$	3	$\frac{b+1}{4}(1-\delta) > \frac{1}{2}$	$\delta < \frac{b-1}{b+1}$
$\{1\}, \{2, 3\}, \{4, 5, 6\}$	$\{1, 2, 3\}, \{4, 5, 6\}$	1	$1-\delta > \frac{a+1}{3}$	$\delta < \frac{2-a}{3}$
$\{1, 2, 3\}, \{4, 5, 6\}$	$\{1, 2\}, \{3, 4, 5, 6\}$	3	$\frac{a+1}{3}(1-\delta) > \frac{1+b}{4}$	$\delta < \frac{4a+1-3b}{4a+4}$
$\{1, 2\}, \{3, 4, 5, 6\}$	$\{1\}, \{2\}, \{3, 4, 5, 6\}$	1	$\frac{a+1}{2}(1-\delta) > 1$	$\delta < \frac{a-1}{a+1}$

The analysis of the inequalities lead to the following result:

Theorem 4. *In every world of the described type, the relative instability is at most $\delta \approx 0.0963$ (the exact value is the real root of the equation $8\delta^3 - 24\delta^2 + 23\delta - 2 = 0$). The corresponding values of the distances are $a = 2-3\delta \approx 1.711$ and $b = \frac{4+\delta}{2-2\delta} \approx 2.2665$. Every value of instability below this threshold is achievable.*

Proof idea. We should find the maximal δ for which all inequalities are satisfiable. It may be shown that the binding inequalities are $\delta < \frac{2b-4}{2b+1}$, $\delta < \frac{2-a}{3}$ and $\delta < \frac{4a+1-3b}{4a+4}$. Solving the system of corresponding equalities yields the claimed values. □

5. Approximate migrational stability under the central median rule

Here we analyze a model described by Bogomolnaia et al [10]. Namely, we analyze migrational stability under no-redistribution and central median rules. Recall that the latter means that if a group has an even number of members, then the facility is located at the middle of the segment of medians.

The considered example is the following: agent 1 lives at point 0, agents 2 and 3 live at point a, agent 4 lives at point $a+b$ and agent 5 lives at point $a+b+c$. In [10] the values are $a = \frac{23}{30}$, $b = \frac{1}{5}$ and $c = \frac{5}{8}$. The idea behind the instability result is the following: agents 2, 3 and 4 are always in the same coalition. But agent 5 prefers to be in the main coalition iff agent 1 does not belong to it, and agent 1 prefers to be in the main coalition iff agent 5 belongs to it. Thus we have the following cycle of configurations:

$(\{1\},\{2,3,4\},\{5\}) \longrightarrow (\{1\},\{2,3,4,5\}) \longrightarrow (\{1,2,3,4,5\}) \longrightarrow (\{1,2,3,4\},\{5\}) \longrightarrow (\{1\},\{2,3,4\},\{5\})$.
We analyze the following table:

Old config.	New config.	Agent	Ineq. on util.	Ineq. on δ
$\{1\},\{2,3,4\},\{5\}$	$\{1\},\{2,3,4,5\}$	5	$1-\delta > \frac{1}{4}+c+\frac{b}{2}$	$\delta < \frac{3}{4}-c-\frac{b}{2}$
$\{1\},\{2,3,4,5\}$	$\{1,2,3,4,5\}$	1	$1-\delta > \frac{1}{5}$	$\delta < \frac{4}{5}-a$
$\{1,2,3,4,5\}$	$\{1,2,3,4\},\{5\}$	5	$\left(\frac{1}{5}+b+c\right)(1-\delta) > 1$	$\delta < \frac{5b+5c-4}{5b+5c+1}$
$\{1,2,3,4\},\{5\}$	$\{1\},\{2,3,4\},\{5\}$	1	$\left(\frac{1}{4}+a\right)(1-\delta) > 1$	$\delta < \frac{4a-3}{4a+1}$

The analysis of the inequalitites shows the following:

Theorem 5. *In every world of the described type, the relative instability is at most* $\delta = \frac{41-\sqrt{1601}}{40} \approx 0.0247$. *The corresponding values of the distances are* $a = \frac{\sqrt{1601}-9}{40} \approx 0.775$ *and a range of* b *and* c*: for instance,* $b = \frac{1}{4}$ *and* $c = \frac{47}{80}$ *would fit. Every value of instability below this threshold is achievable.*

Proof idea. It turns out that only the conditions on a are binding. For the claimed value of a the values are equal. It can be checked that for the specified values of b and c both other inequalities are satisfied. □

6. Conclusion

This work provides an analysis of approximate stability of jurisdiction partitions in several frameworks. We believe that the considered examples are either the most unstable worlds or close to them. The future research should clarify this issue and give the provable least unstable constructions. Another possible approach is the algorithmic one. Suppose that an algorithm gets the total description of a world and a number δ. Can we efficiently compute whether this world is δ-stable? Or is this problem hard for some computational complexity class?

What can we learn about the real world from our constructions? It depends on two things. Firstly, how typical are our examples? It seems that a bipolar world may occur in different situations, like twin cities or cat and dog lovers. And the situation that one pole is about 30% larger than the other is also realistic. For instance, it is how much Minneapolis is greater by population than Saint Paul. Secondly, what are the real values of δ? It seems that in long-existing structures it can be much larger than several percents, so the participants may want to make a new coalition but then see the cost and "all of the sudden we're scared to change a thing". A study of such real cases would be an interesting research topic.

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References

[1] P. A. Samuelson, The pure theory of public expenditure, The Review of Economics and Statistics (1954) 387–389.

[2] C. M. Tiebout, A pure theory of local expenditures, Journal of political economy 64 (1956) 416–424.

[3] F. Westhoff, Existence of equilibria in economies with a local public good, Journal of Economic Theory 14 (1977) 84–112.

[4] T. F. Bewley, A critique of Tiebout's theory of local public expenditures, Econometrica: Journal of the Econometric Society (1981) 713–740.

[5] J. Greenberg, S. Weber, Strong Tiebout equilibrium under restricted preferences domain, Journal of Economic Theory 38 (1986) 101–117.

[6] A. Alesina, E. Spolaore, On the number and size of nations, The Quarterly Journal of Economics 112 (1997) 1027–1056.

[7] O. Haimanko, M. Le Breton, S. Weber, Voluntary formation of communities for the provision of public projects, Journal of Economic Theory 115 (2004) 1–34.

[8] S. Weber, A. Savvateev, M. Le Breton, A. Bogomolnaia, The Egalitarian sharing rule in provision of public goods, Economics Bulletin 8 (2005) 1–5. URL: https://ideas.repec.org/a/ebl/ecbull/eb-05h40002.html.

[9] A. Bogomolnaia, M. Le Breton, A. Savvateev, S. Weber, Stability under unanimous consent, free mobility and core, International Journal of Game Theory 35 (2007) 185–204.

[10] A. Bogomolnaia, M. Le Breton, A. Savvateev, S. Weber, Stability of jurisdiction structures under the equal share and median rules, Economic Theory 34 (2008) 525–543.

[11] A. Savvateev, Uni-dimensional models of coalition formation: non-existence of stable partitions, Moscow Journal of Combinatorics and Number Theory 2 (2012) 49–62.

[12] D. V. Musatov, A. V. Savvateev, S. Weber, Gale–Nikaido–Debreu and Milgrom–Shannon: Communal interactions with endogenous community structures, Journal of Economic Theory 166 (2016) 282–303.

[13] A. Savvateev, C. Sorokin, S. Weber, Multidimensional free-mobility equilibrium: Tiebout revisited, 2018. URL: https://arxiv.org/abs/1805.11871.

[14] V. Marakulin, Spatial equilibrium in a multidimensional space: an immigration-consistent division into countries centered at barycenter, in: International Conference on Mathematical Optimization Theory and Operations Research, Springer, 2019, pp. 651–672.

[15] J. Drèze, M. Le Breton, A. Savvateev, S. Weber, "Almost" subsidy-free spatial pricing in a multidimensional setting, Journal of Economic Theory 143 (2008) 275–291.

[16] A. Golman, D. Musatov, Approximate coalitional equilibria in the bipolar world, in: Y. Evtushenko, M. Jaćimović, M. Khachay, Y. Kochetov, V. Malkova, M. Posypkin (Eds.), Optimization and Applications. OPTIMA 2018, volume 974, Springer, 2019, pp. 512–526.

[17] A. Savvateev, An analysis of coalitional stability in a bipolar world, Journal of New Economic Association 17 (2013) 10–44. In Russian.