QRE in the Ising Game

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Abstract
Static and dynamic equilibria in a noisy binary choice game with Ising externalities on complete and random graphs with topology corresponding to configuration model are considered. It is shown that static equilibria realise Quantal Response Equilibria (QRE). Corresponding regime switching (phase transitions) at critical values of parameters is discussed. Myopic dynamics having the discussed static equilibria as stationary configurations is described.

Keywords
Nosy binary choice game, Quantal Response Equilibrium, Master equation

Let us consider a noisy binary choice game played by N agents placed at vertices of a graph $\mathcal{G}$ with an adjacency matrix g_{ij} equipped with a set of strategies parametrized by $s_i = \pm 1$, $i = 1, \cdots N$ in which a utillity of each agent i contains a strategy-dependent additive random contribution ϵ_{s_i} known to i but unknown to his neighbours characterised by a common distribution function $\phi(\epsilon_{s_i})$, its form assumed to be common knowledge. Games of this type were considered in a variety os socioeconomic settings, see e.g. [1, 2, 3, 4]. Let us note that a presence of random contributions to utility means that a description of this game (equilibria, evolution, et.) is necessarily probabilistic, i.e. its equilibrium is characterised by probabilities of playing certain strategies

$$\{s_i = \pm 1\} \quad \longrightarrow \quad \{p_i^{\pm}\} \tag{1}$$

In what follows we shall restrict our consideration to the case of independent random contributions to utility for different agents corresponding to Nash equilibrium in mixed strategies.

A choice of an agent i is determined by his expectations concerning the choices of his neighbours. The *Ising game* is defined by the corresponding expected utility of the following form

$$\langle U_i(s_i) \rangle = \left[H_i + \sum_{j \neq i} g_{ij} J_{ij} \langle s_j \rangle_{(i)} \right] s_i + \epsilon_{s_i}$$

where $\langle s_j \rangle_{(i)}$ stands for $i's$ expectation with respect of the choice of the neighbour j Strategy s_i is preferred to $-s_i$ if

$$\langle U_i(s_i) \rangle > \langle U_i(-s_i) \rangle$$

Probability p_{s_i} of choosing the strategy s_i by an agent i is

$$p_{s_i} = F_<^{(i)} \left(\left[2H_i + 2 \sum_j g_{ij} J_{ij} \langle s_j \rangle_{(i)} \right] s_i \right)$$

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where $F_<^{(i)}(z)$ is a distribution function for the difference $\epsilon_{-s_i} - \epsilon_{s_i}$

$$F_<^{(i)}(z) = \int^z dz_1 \int dz_2 \; \phi^{(i)}(z_2)\phi^{(i)}(z_2 + z_1)$$

In particular, the probability of choosing $s_i = 1$ is equal to

$$p_i^+ = F_<^{(i)} \left(2H_i + 2 \sum_j g_{ij} J_{ij} \langle s_j \rangle_{(i)} \right)$$

Quantal response equilibrium (QRE) [5, 6, 7] is a particular form of mixed strategies in which the vector of expected payoffs for agent's alternative actions is mapped into a probability distribution of choices over these strategies. The QRE does arise from realization of beliefs. A player's payoffs are computed based on beliefs about other players' probability distribution over strategies. An equilibrium is a set of probabilities such that player's beliefs are correct.

Quantal response equilibrium (QRE) of the Ising game under consideration is thus a set of probabilities $(p_1^+, \cdots, p_N^+)$ such that

$$\langle s_j \rangle_{(i)} = 2p_j^+ - 1$$

so that mixed strategies chosen by all agents are consistent with the corresponding expectations of other agents with respect to this choice. From (2) we obtain the following system of equations defining the QRE probabilties $(p_1^+, \cdots, p_N^+)$ of the Ising game:

$$p_i^+ = F_<^{(i)} \left(2H_i + 2 \sum_j g_{ij} J_{ij} (2p_j^+ - 1) \right), \quad \forall i$$

In terms of equilibrium averages $m_i = 2p_i^+ - 1$ these read

$$m_i = 2F_<^{(i)} \left(2H_i + 2 \sum_j g_{ij} J_{ij} m_j \right) - 1 \, , \quad \forall i$$

Let us now consider the Ising game on the complete graph. In this case it is customary to rescale the coupling $J_{ij} = J/N$. Besides that, we assume for simplicity that $H_i = H$. For the complete graph G all local averages are the same and QRE is defined by the single equation

$$m = 2F_< \left[2H + 2Jm \right] - 1$$

coinciding with the equation obtained in [8, 1]. For $H = 0$ the space of solutions of (2) undergoes restructuring (a phase transition) at $4Jf(0) = 1$ so that for $4Jf(0) < 1$ the only solution is $m = 0$, and for $4Jf(0) > 1$ there appears a pair of additional solutions $\pm m \neq 0$.

For the Gumbel noise $\phi(\epsilon) = \beta \exp(-\beta\epsilon - \exp(-\beta\epsilon))$ we get the Curie-Weiss equation well known in the physics of magnetics, see e.g. [9]:

$$m = \tanh\left(\beta(H + Jm)\right) \tag{2}$$

In physics the Curie-Weiss equation (2) defines *minima* for the "noisy potential" - the free energy $\beta F = \beta E - S$ at temperature $T = 1/\beta$:

$$\beta J m \quad = \quad \text{atanh}(m) \;\leftrightarrow\; \frac{d}{dm} \left[\frac{1}{2} \beta J m^2 - S(m) \right] = 0$$

$$S(m) \quad = \quad -\frac{1-m}{2}\ln\frac{1-m}{2}-\frac{1+m}{2}\ln\frac{1+m}{2}$$

Let us note that it is possible to define an optimisation problem in which the equation (2) arises from finding a minimum of a certain function [1] which, however, does not possess properties of a game-theoretic potential.

Let us now turn to the analysis of the so-called annealed approximation, see e.g. [10], which is in fact equivalent to a random graph topology as generated by the so-called configuration model, see e.g. [11], in which the matrix elements g_{ij} are replaced by the probabilities of forming a link i, j between the nodes with degrees k_i, k_j

$$g_{ij} \simeq \frac{k_i k_j}{N\langle k\rangle}$$

In this approximation all vertices having the same degree are equivalent so QRE is now defined in terms of averages for the nodes with the same degree:

$$m_k = \langle s_i\rangle|_{\forall i:\ \mathrm{deg}(i)=k} \tag{3}$$

The corresponding system of equations defining QRE reads

$$\begin{aligned} m_k &= 2F_{<}\left[2H + 2Jk\sum_{k'}\frac{k' p_{k'}}{\langle k\rangle} m_{k'}\right] - 1 \\ &= 2F_{<}\left[2H + 2Jk m_w\right] - 1 \end{aligned} \tag{4}$$

where we have defined the weighted average

$$m_w = \sum_k \frac{k p_k}{\langle k\rangle} m_k$$

The corresponding generalised Curie-Weiss equation for m_w reads

$$m_w = \sum_k \frac{k p_k}{\langle k\rangle} 2F_{<}\left[2H + 2Jk m_w\right] - 1$$

The phase transition takes place at

$$4J\langle k^2\rangle f(0) = \langle k\rangle$$

differing from the result for the complete graph by the factor $\langle k\rangle/\langle k^2\rangle$ which routinely appears in the analysis of random graph related problems (appearance of giant cluster, epidemic threshold, etc. [11]).

Is it possible to recover the above-found QRE equilibrium as a stationary point of a dynamical game? Generically stochastic dynamics operates with a probability distribution of observing a certain configuration of strategies $\{s\}$ at time t:

$$P\left(\{s\}(t)\right) = P\left[s_1(t), s_2(t), \cdots s_N(t)\right]$$

Conventional independent choice assumption corresponding to nixed state static Nash equilibria corresponds to a factorised distribution

$$P\left(\{s\}(t)\right) = P\left[s_1(t), s_2(t), \cdots s_N(t)\right] = \prod_{i=1}^{N} p(s_i(t))$$

Temporal evolution is driven by strategy flips $s_i(t) \longrightarrow -s_i(t)$

Let us assume that within an infinitesimally small time interval one can have only one strategy flip. Then the process of dynamical evolution is fully described by a time-dependent strategy flip probability $w(s_i \longrightarrow -s_i)$. The local strategy change can in principle take into account past realized (memory) and future expected (forward-looking) configurations of strategies. In the simplest myopic response approximation agents base their decisions on the configuration of neighbour's strategies immediately preceding the decision time. The evolution equation for $P(\{s\}(t))$ then reads

$$\begin{aligned} \frac{dP(\{s\}(t))}{dt} &= \sum_i [w(-s_i \longrightarrow s_i) P(s_1, \cdots, -s_i, \cdots s_N) \\ &\quad - w(s_i \longrightarrow -s_i) P(s_1, \cdots, s_i, \cdots s_N)] \end{aligned}$$

The corresponding evolution equation of the local average strategies $m_i(t) = \langle s_i \rangle_{P(\{s\}(t))}$ then reads

$$\frac{dm_i(t)}{dt} = -2\langle s_i(t) w(s_i \longrightarrow -s_i) \rangle_{P(\{s\}(t))}$$

A natural choice for $w(s_i \longrightarrow -s_i)$ is [1]

$$w(s_i \longrightarrow -s_i) = \lambda p_{-s_i} = \lambda F_<^{(i)} \left(- \left[2H_i + 2 \sum_j g_{ij} J_{ij} s_j(t) \right] s_i(t) \right)$$

To derive the evolution equation for $m_i(t)$ it is convenient to use the following identity:

$$p_{-s_i} = \frac{1}{2} \left[1 - s_i(s_i p_{s_i} - s_i p_{-s_i}) \right] \equiv \frac{1}{2} \left[1 - s_i \langle s_i \rangle \right]$$

The resulting evolution equation for $m_i(t)$ reads

$$\frac{dm_i(t)}{dt} = -\lambda \left\{ m_i(t) - \left(2F_<^{(i)} \left[2H_i + 2 \sum_j g_{ij} J_{ij} m_j(t) \right] - 1 \right) \right\}$$

We see that stationary points of this system of evolution equations are exactly the above-described QRE equilibria described by the equations (2).

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