An axiomatization of the paracomplete logic $L3A^{D}_{\rightarrow_1}$

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Abstract. In 2019 Hernández-Tello et al. introduce a more restricted concept of paracompleteness, namely the *genuine paracompleteness*. A *genuine paracomplete logic* is a logic rejecting $\vdash \varphi, \neg\varphi$ and $\neg(\psi \vee \neg\psi) \vdash$. This conditions are dual to those rejected by *genuine paraconsistent logic*: $\varphi, \neg\varphi \vdash \psi$ and $\vdash \neg(\varphi \wedge \neg\varphi)$, introduced by Béziau in 2016. Hernández-Tello et al. make a semantical analysis of the genuine paracompleteness in the context of three-valued logics and find two genuine paracomplete logics that conservatively extend the positive fragment of Classical Propositional logic, namely, $L3A^{D}_{\rightarrow_1}$ and $L3B^{D}_{\rightarrow_1}$.

We present here a deep analysis of the paracomplete logic $L3A^{D}_{\rightarrow_1}$. We provide a sound and complete Hilbert-type axiomatic system for $L3A^{D}_{\rightarrow_1}$ logic. The completeness proof is a direct proof using Kalmár's technique adapted for a three-valued logic, which means that a general technique to find the proof of any theorem in a systematic way is obtained.

Keywords: Paracomplete logic · Three-valued logic · Kalmár's completeness method

1 Introduction

The relation between mathematics, logic, and philosophy can be traced back to ancient times. Greek philosophers asserted that there are three basic principles of thinking that are fundamental to making correct reasoning. Classical reasoning is expected to conform to these logical principles, also called the laws of thought, namely:

1. The Law of Identity: *'A thing is what it is.'*
2. The Law of Excluded Middle: *'It is impossible to be and not to be the same thing.'*
3. The Law of Contradiction: *'It is impossible for any being to possess a quality, and at the same time not to possess it.'*

Nowadays logicians have shown that it is possible to define logical systems that do not obey some of these laws obtaining some interesting and useful non-classical logics. In 1908 Brouwer, in a paper entitled *'The untrustworthiness of the principles of logic'*, challenged the belief that the rules of the classical logic, which have come down to us essentially from Aristotle (384–322 B.C.) have an absolute validity, independent of the subject matter to which they are applied [10]. Brouwer and his school, mathematical intuitionism, did not admit the use of Law of Excluded Middle in mathematical proofs. On the other hand Vasil'év (1910), part of the Russian school of logic, proposed a modified Aristotelian syllogistic including statements of the form: S is both P and not P. In 1920 Jan Łukasiewicz, a leader of the Polish school of logic, trying to formalize Aristotle's future contingents, formulated a propositional calculus that had a third truth-value, neither truth nor false. His calculus rejected both the Law of Contradiction and the Law of Excluded Middle.

The problem when one wants to formalize the laws in terms of formal logic is that there are several ontological, doxastic and semantic versions for each of these laws, unfortunately in most of the cases they are not equivalent. We are interested in the semantic point of view, but even in that context, minimum changes in the interpretation of the laws can conduct to different formalizations. For instance the Law of Contradiction, whose intuitive meaning is *'it is not the case that φ and $\neg\varphi$ are simultaneously true'*, could be formalized in terms of (multiple-conclusion) consequence relations as either of the following:

$$(\mathbf{LC})\quad \varphi \wedge \neg\varphi \vdash \qquad \text{or} \qquad (\mathbf{LC}')\quad \vdash \neg(\varphi \wedge \neg\varphi)$$

On the other hand we have the Law of Excluded Middle, the intuitive meaning in this case is *'it is not the case that φ and $\neg\varphi$ are simultaneously false'*, and it could be formalized as either of the following:

$$(\mathbf{LEM})\quad \vdash \varphi \vee \neg\varphi \qquad \text{or} \qquad (\mathbf{LEM}')\quad \neg(\varphi \vee \neg\varphi) \vdash$$

In [11] Loparić and Da Costa define paraconsistent logics as non-trivial logics that contain a formula such that the formula and its negation are both true. They also define paracomplete logics as logics for which there exist a formula such that the formula and its negation are both false, which agree with the intuitive interpretation of the previous formalizations.

However as highlighted by Béziau in [6] (for the paraconsistent case) and by Hernandez-Tello et al. in [9] (for the paracomplete case) is that one can not use **LC** or **LC**$'$ indistinctly to formally define paraconsistent logics and one can not use **LEM** or **LEM**$'$ indistinctly for the case of paracomplete logics since the formulations are not equivalent, in fact they are independent. This motivates the definition of genuine paraconsistent logics, they are those logics that reject **LC** and **LC**$'$ at the same time. Analogously, genuine paracomplete logics are those logics rejecting both **LEM** and **LEM**$'$.

In general, the interest in paracomplete and in paraconsistent logic has grown in the last decades, we can find some interesting theoretical results about these families of logics in [11, 4, 7], we can find also very good papers highlighting its applications such as [1, 2].

In the present paper we study the genuine paracomplete logic $L3A^{D}_{\to_1}$. We provide a sound and complete Hilbert-type axiomatic system for $L3A^{D}_{\to_1}$ logic using Kalmár's technique adapted for a three-valued logic obtaining a general technique to find the proof of any theorem in a systematic way.

2 Background

We introduce the syntax of the logical formulas considered in this paper, later the notions of consequence relation and logic, some definitions related to connectives as well as some concepts related to logic semantics.

We use a formal propositional language $\mathcal{L} = \langle atom(\mathcal{L}), \mathcal{C}, \mathcal{A}\rangle$, where $atom(\mathcal{L})$ is an enumerable set, whose elements are called atoms and are denoted by lowercase letters; $\mathcal{C} = \{\neg, \vee, \wedge, \to\}$ is the set of connectives, also known as signature, and $\mathcal{A}$ is the set of auxiliary symbols (comma and parenthesis in our case). Formulas are constructed as usual and will be denoted by lowercase Greek letters. Given a language $\mathcal{L}$ the set of all formulas of the language is denoted by $Form(\mathcal{L})$. Theories are sets of formulas and will be denoted by uppercase Greek letters.

Definition 1. *Given a formal propositional language, a* ***(tarskian) consequence relation*** $\vdash$ *between theories and formulas is a relation satisfying the following properties, for every theory* $\Gamma \cup \Delta \cup \{\varphi\}$*:*

*(**Reflexivity**) if* $\varphi \in \Gamma$*, then* $\Gamma \vdash \varphi$*;*
*(**Monotonicity**) if* $\Gamma \vdash \varphi$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash \varphi$*;*
*(**Transitivity**) if* $\Delta \vdash \varphi$ *and* $\Gamma \vdash \psi$ *for every* $\psi \in \Delta$*, then* $\Gamma \vdash \varphi$*.*

Other desirable properties of consequence relations are structurality (for every $\mathcal{L}$-substitution θ, it holds that $\Gamma \vdash \varphi$ implies $\theta(\Gamma) \vdash \theta(\varphi)$) and non-triviality (there exist some non-empty theory Γ and some φ such that $\Gamma \nvdash \varphi$).

A logic can be defined in terms of consequence relations (either semantical or proof-theoretical) as follows:

Definition 2. *Given a formal language* $\mathcal{L}$*, a* ***logic*** *is a pair* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}}\rangle$*, where* $\vdash_{\mathbf{L}}$ *is a structural and no trivial consequence relation, satisfying the rule known as Modus Ponens (MP), which means that for any formulas* φ *and* ψ *holds that* $\varphi \to \psi, \varphi \vdash_{\mathbf{L}} \psi$*.*

The notation $\Gamma \vdash_{\mathbf{L}} \varphi$ could be read as φ can be inferred from Γ in $\mathbf{L}$. Whenever the logic is clear the subscript will be dropped.

Definition 3. *[3] Let* **L** *be a logic in the language* $\mathcal{L}$ *with binary connectives* $\wedge$, $\vee$ *and* $\to$*, then:*

1. $\wedge$ *is a* ***conjunction*** *for L, when:* $\Gamma \vdash \varphi \wedge \psi$ *iff* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \psi$.
2. $\vee$ *is a* ***disjunction*** *for L, when:* $\Gamma, \varphi \vee \psi \vdash \sigma$ *iff* $\Gamma, \varphi \vdash \sigma$ *and* $\Gamma, \psi \vdash \sigma$.
3. $\to$ *is an* ***implication*** *for L, when:* $\Gamma, \varphi \vdash \psi$ *iff* $\Gamma \vdash \varphi \to \psi$.

In [8] the author define the concept of classical implication as follows.

Definition 4. *[8] Let* **L** *be a logic in the language* $\mathcal{L}$ *with a binary connective* $\to$*, it is a* ***classical implication*** *if:*

i) $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \varphi \to \psi$ *imply that* $\Gamma \vdash \psi$;
ii) $\Gamma \vdash \varphi \to (\psi \to \varphi)$;
iii) $\Gamma \vdash \Big(\varphi \to (\psi \to \sigma)\Big) \to \Big((\varphi \to \psi) \to (\varphi \to \sigma)\Big)$.

It is not difficult to prove in the context of tarskian consequence relations that the notions of implication in Definition 3 and Definition 4 agree. The usual manner to define many-valued logics is by means of a matrix.

Definition 5. *A* ***matrix*** *for a language* $\mathcal{L}$*, is a structure* $M = \langle V, D, F\rangle$*, where:*

V *is a non-empty set of truth values (domain);*
D *is a subset of* V *(set of designated values);*
$F := \{f_c | c \in \mathcal{C}\}$ *is a set of truth functions, with a function for each logical connective in* $\mathcal{L}$.

Definition 6. *Given a language* $\mathcal{L}$*, a function* $v : atom(\mathcal{L}) \longrightarrow V$ *that maps atoms into elements of the domain is a* ***valuation***.

It can be extended to all formulas $v : Form(\mathcal{L}) \longrightarrow V$ as usual, i.e. applying recursively the truth functions of logical connectives in F. Now we can define the notion of model.

Definition 7. *Given a matrix* M*, we say that* v *is a* ***model*** *of the formula* φ*, if* $v(\varphi) \in D$ *and we denote it by* $v \models_M \varphi$*. A formula* φ *is a* ***tautology*** *in* M *if every valuation is a model of* φ*, it is denoted by* $\models_M \varphi$.

Whenever the matrix is clear the subscript will be dropped. It is also possible to define a consequence relation by means of a matrix.

Definition 8. *[3] Given a matrix* M*, its* ***induced consequence relation****, denoted by* $\vdash_M$*, is defined by:* $\Gamma \vdash_M \varphi$ *if every model of* Γ *is a model of* φ.

Definition 9. *[3] Given a matrix* M *over a language* $\mathcal{L}$*, the* ***induced logic****, is the logic* $\langle \mathcal{L}, \vdash_M\rangle$*, i.e. the logic obtained with the consequence relation induced by the matrix.*

There are more restrictive conditions than those on Definition 3 for connectives such as the following definition.

Definition 10. *[8] Let $M = \langle V, D, F\rangle$ be a matrix, and let $\overline{D}$ denote the set of non-designated values, i.e $\overline{D} = V \setminus D$ and v any valuation, then:*

1. *$\neg$ is a* ***Neoclassical negation****, if it holds that:*
$$v(\neg\varphi) \in \overline{D} \text{ iff } v(\varphi) \in D.$$
2. *$\wedge$ is a* ***Neoclassical conjunction****, if it holds that:*
$$v(\varphi \wedge \psi) \in D \text{ iff } v(\varphi) \in D \text{ and } v(\psi) \in D.$$
3. *$\vee$ is a* ***Neoclassical disjunction****, if it holds that:*
$$v(\varphi \vee \psi) \in \overline{D} \text{ iff } v(\varphi) \in \overline{D} \text{ and } v(\psi) \in \overline{D}.$$
4. *$\rightarrow$ is a* ***Neoclassical implication****, if it holds that:*
$$v(\varphi \rightarrow \psi) \in D \text{ iff } v(\varphi) \in \overline{D} \text{ or } v(\psi) \in D.$$

Specifically, we have that items 2 and 3 on Definition 10 imply items 2 and 3 on Definition 3. Moreover, item 1 on Definition 10 is equivalent to item 1 on Definition 3.

Definition 11. *A n-valued function $\circledast$ of arity k ($\circledast : V^k \longrightarrow V$) is a:*

Conservative extension *of an m-valued function $\odot : V_1^k \longrightarrow V_1$ where $V_1 \subsetneq V$ and $|V_1| = m$, if the restriction of $\circledast$ to V_1 coincide with $\odot$ (i.e. $\circledast|_{V_1} = \odot$).*
Molecular *if the range of it is a proper subset of V. [6]*

3 Genuine paracomplete logics

The notion of genuine paracomplete logic, is presented in [9]. Genuine paracomplete logics reject the dual principles defining genuine paraconsistent logic.

Definition 12. *A logic* **L** *with negation and disjunction is said to be a* ***genuine paracomplete logic*** *(or a* strong paracomplete logic*) if neither (***LEM***) nor (***LEM′***) is valid, that is: for some formulas φ and ψ,*

$$(\mathbf{GP}1_D) \quad \nvdash \varphi \vee \neg\varphi \qquad \textit{and} \qquad (\mathbf{GP}2_D) \quad \neg(\psi \vee \neg\psi) \nvdash .$$

Examples:

1. Intuitionistic Propositional Logic **IPL** is paracomplete, but it is not genuine paracomplete: the formula $\neg(\varphi \vee \neg\varphi)$ is unsatisfiable.
2. The Belnap-Dunn logic $FOUR$ (with the truth ordering) and Nelson logic **N4** are both genuine paraconsistent and genuine paracomplete.
3. The 3-valued logic **MH**, introduced in [7], is genuine paracomplete.

Authors in [9] develop a study among three-valued logics in order to find all connectives defining genuine paracomplete logics, they proceed in similar way to the analysis done in [6]. As a result they found two pairs of connectives of negation and a disjunction that works accordingly to the previous definition. Later a conjunction is added obtaining the logics $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$. Finally, proceeding analogously to [8], they found a neoclassical and non molecular implication for $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$, leading to the logics, $\mathbf{L3A^D_{\rightarrow_1}}$ and $\mathbf{L3B^D_{\rightarrow_1}}$.

In that paper the authors perform a complete semantical analysis of the concept of genuine paraconsistency. In this paper we start a study of the concept from the proof-theoretical point of view. Particularly we focus on the logic $\mathbf{L3A^D_{\rightarrow_1}}$ in order to find a Hilbert-Type axiomatic system for it.

4 The logic $L3A^D_{\rightarrow_1}$

In this section we present the logic $L3A^D_{\rightarrow_1}$, a genuine paracomplete three-valued logic and some remarks about it.

Definition 13. *The logic $L3A^D_{\rightarrow_1}$ is the three-valued logic induced by the matrix $\mathcal{M} = \langle \{0,1,2\}, \{2\}, \mathcal{O}\rangle$ over the signature $\mathcal{C} = \{\neg, \vee, \wedge, \rightarrow\}$ and whose truth tables are:*

	$\neg$
0	2
1	0
2	0

$\vee$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\rightarrow$	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

Fig. 1. Truth tables for $L3A^D_{\rightarrow_1}$

Remark 1. Some of the results about $L3A^D_{\rightarrow_1}$ presented in [9] are the following:

- The connective $\neg$ corresponds to the negation of logic G_3, the three-valued logic of Gödel.
- The connective $\vee$ is a disjunction, it is a neoclassical disjunction and it is a conservative extension of the classical disjunction and it is not molecular.
- The connective $\wedge$ is a conjunction, it is a neoclassical conjunction and it is a conservative extension of the classical conjunction and it is not molecular and corresponds to the minimum function in the natural order.
- The connective $\rightarrow$ is an implication, is a neoclassical implication and it is a conservative extension of the classical implication and it is not molecular.
- The logic satisfies the positive fragment of classical logic.
- The non implicative fragment of $L3A^D_{\rightarrow_1}$ is a logic dual to the genuine paracomplete logic $L3A$ defined in [6].
- The constants $\bot$ and $\top$ are definable as $\bot := \varphi \wedge \neg\varphi$ and $\top := \neg(\varphi \wedge \neg\varphi)$ for any formula φ.

5 A Hilbert Calculus for $L3A^{D}_{\to_1}$

Up to this point we have defined the logic $L3A^{D}_{\to_1}$ and we have revisited some properties from the semantic point of view. It is time to switch to the proof theoretical approach. In this section the Hilbert Calculus $\mathbb{L}_{L3A^{D}_{\to_1}}$ for the logic $L3A^{D}_{\to_1}$ is presented, some basic results are stated and finally the adequacy of the calculus is proved.

5.1 The calculus $\mathbb{L}_{L3A^{D}_{\to_1}}$

In order to define an axiomatic theory for $L3A^{D}_{\to_1}$ over the signature $\mathcal{C} = \{\neg, \vee, \wedge, \to\}$ we are going to introduce the following abbreviations for the sake of the simplicity:

Definition 14. *Let $\mathcal{C}$ be the signature $\{\neg, \vee, \wedge, \to\}$ then:*

- $\sim \varphi := \varphi \to (\varphi \wedge \neg\varphi)$
- $G'(\varphi) := \neg\varphi$
- $N'(\varphi) := \neg(\varphi \vee \neg\varphi)$
- $D'(\varphi) := \neg\big(\varphi \to (\varphi \wedge \neg\varphi)\big) = \neg \sim \varphi$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$

Considering these new connectives we can proceed to define the axiomatic theory for the logic $L3A^{D}_{\to_1}$ as follows.

Definition 15. *Let $\mathbb{L}_{L3A^{D}_{\to_1}}$ be an axiomatic theory whose axioms are the schemes listed below and whose rule of inference is Modus Ponens (**MP**).*

$$\begin{aligned}
\textbf{Pos1} &: \varphi \to (\psi \to \varphi)\\
\textbf{Pos2} &: \Big(\varphi \to (\psi \to \sigma)\Big) \to \Big((\varphi \to \psi) \to (\varphi \to \sigma)\Big)\\
\textbf{Pos3} &: (\varphi \wedge \psi) \to \varphi\\
\textbf{Pos4} &: (\varphi \wedge \psi) \to \psi\\
\textbf{Pos5} &: \varphi \to \Big(\psi \to (\varphi \wedge \psi)\Big)\\
\textbf{Pos6} &: \varphi \to (\varphi \vee \psi)\\
\textbf{Pos7} &: \psi \to (\varphi \vee \psi)\\
\textbf{Pos8} &: \Big(\varphi \to \sigma\Big) \to \Big((\psi \to \sigma) \to \big((\varphi \vee \psi) \to \sigma\big)\Big)\\
\textbf{Pos9} &: (\varphi \to \psi) \vee \varphi\\
\textbf{REM} &: \neg\varphi \vee \neg\neg\varphi\\
\textbf{EXP} &: \varphi \to (\neg\varphi \to \psi)\\
\textbf{ENN} &: \neg\neg\varphi \leftrightarrow \sim \neg\varphi\\
\textbf{ENC} &: \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)\\
\textbf{END} &: \neg(\varphi \vee \psi) \leftrightarrow \Big((\sim \varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \sim \psi)\Big)\\
\textbf{ENI} &: \neg(\varphi \to \psi) \leftrightarrow (\varphi \wedge \neg\psi)
\end{aligned}$$

$$\textit{Modus Ponens} \quad \frac{\varphi, \varphi \to \psi}{\psi}\,\boldsymbol{MP}$$

With $\Lambda \vdash \lambda$ we denote a deduction of λ with Λ as the set of hypotheses, when $\Lambda = \emptyset$ we write $\vdash \lambda$ or simply λ, which means that it is possible to prove λ without assumptions. As usual $\Lambda, \varphi \vdash \psi$ denotes $\Lambda \cup \{\varphi\} \vdash \psi$.

Some general properties of $\mathbb{L}_{L3A^D_{\rightarrow_1}}$ are the following.

Theorem 1. *Let Γ, Δ be theories and φ, ψ be formulas, the following properties hold in $\mathbb{L}_{L3A^D_{\rightarrow_1}}$:*

i) ***Monotonicity (Mon)***	If $\Gamma \vdash \varphi$, then $\Gamma, \Delta \vdash \varphi$.
ii) ***Deduction Theorem*** **(DT)**	$\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \rightarrow \psi$.
iii) ***Cut***	If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$.
iv) ***AND-Rules(R-AND)***	$\Gamma \vdash \varphi \wedge \psi$ iff $\Gamma \vdash \varphi$ y $\Gamma \vdash \psi$.
v) ***Weak Proof by Cases*** **(WPC)**	$\Gamma, \neg\varphi \vdash \psi$ and $\Gamma, \neg\neg\varphi \vdash \psi$ iff $\Gamma \vdash \psi$.

Proof. The proof of properties $i) - iv)$ is straightforward. Let us check the last property.

Suppose $\Gamma, \neg\varphi \vdash \psi$ and $\Gamma, \neg\neg\varphi \vdash \psi$. By **DT** we have that $\Gamma \vdash \neg\varphi \rightarrow \psi$ and $\Gamma \vdash \neg\neg\varphi \rightarrow \psi$. Using **Pos8** and **Mon** we obtain $\Gamma \vdash (\neg\varphi \rightarrow \psi) \rightarrow \Big((\neg\neg\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \vee \neg\neg\varphi) \rightarrow \psi)\Big)$ y applying *Modus Ponens* with $\Gamma \vdash \neg\varphi \rightarrow \psi$, we conclude that $\Gamma \vdash (\neg\neg\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \vee \neg\neg\varphi) \rightarrow \psi)$. Once again by *Modus Ponens* between the last step and $\Gamma \vdash \neg\neg\varphi \rightarrow \psi$, it follows that $\Gamma \vdash (\neg\varphi \vee \neg\neg\varphi) \rightarrow \psi$. Separately, by **Mon** and **REM**, $\Gamma \vdash \neg\varphi \vee \neg\neg\varphi$. Finally we obtain that $\Gamma \vdash \psi$ by means of *Modus Ponens*. On the other hand, if we suppose that $\Gamma \vdash \psi$, by **Mon** $\Gamma, \neg\varphi \vdash \psi$ and $\Gamma, \neg\neg\varphi \vdash \psi$.

The Lemma 1 encloses a list of properties of $\mathbb{L}_{L3A^D_{\rightarrow_1}}$. Particularly items $f)$, $h)$, $i)$ and axiom **Pos9** suggest that $\sim$ behaves as classical negation. Just take ψ in **Pos9** as $\varphi \wedge \neg\varphi$ to obtain $\sim \varphi \vee \varphi$ recovering somehow the excluded middle principle. As shown in Table 1 the connective $\sim$ is a neoclassical negation, see Definition 10.

Note 1. From the semantical point of view, connectives G', N' and D' act as identifiers of the truth values 0, 1 and 2 respectively, see Table 1. The reading from the semantical point of view of property $s)$ as well as the results in Lemmas 1 and 2 is very intuitive.

φ	$\sim \varphi$	$G'(\varphi)$	$N'(\varphi)$	$D'(\varphi)$
0	2	2	0	0
1	2	0	2	0
2	0	0	0	2

Table 1. Truth tables of connectives $\sim$, G', N' y D' in $L3A^D_{\rightarrow_1}$

Lemma 1. *If φ, ψ, σ, ξ are formulas in $\mathbb{L}_{L3A^D_{\to_1}}$ the following properties hold:*

a) $\vdash \varphi \to \neg\neg\varphi$
b) $\vdash (\neg\neg\psi \to \neg\neg\varphi) \leftrightarrow (\neg\varphi \to \neg\psi)$
c) $\vdash \neg\varphi \leftrightarrow \neg\neg\neg\varphi$
d) $\vdash (\neg\varphi \to \psi) \to (\neg\psi \to \neg\neg\varphi)$
e) $\neg\neg\varphi, \neg\neg\psi \vdash \neg\neg(\varphi \vee \psi)$
f) $\sim\varphi, \varphi \vdash \psi$
g) $\neg\varphi \vdash \sim\varphi$
h) $\vdash \varphi \leftrightarrow \sim\sim\varphi$
i) $\vdash (\varphi \to \psi) \leftrightarrow (\sim\psi \to \sim\varphi)$
j) $\varphi \vdash \sim\neg\varphi$
k) $\sim\varphi,\ \sim\psi \vdash\ \sim(\varphi \vee \psi)$
l) $\sim\varphi, \neg\neg\varphi \vdash N'(\varphi)$
m) $N'(\varphi) \vdash \sim\varphi$
n) $N'(\varphi), \varphi \vdash \psi$
o) $N'(\varphi) \vdash \neg\neg\varphi$
p) $\vdash D'(\varphi) \leftrightarrow \varphi$
q) $D'(\varphi) \vdash \neg\neg\varphi$
r) *If* $\sim\varphi \vdash \psi$ *and* $\neg\sim\varphi \vdash \psi$*, then* $\vdash \psi$*.*
s) $G'(\varphi) \vdash \psi$*,* $N'(\varphi) \vdash \psi$*,* $D'(\varphi) \vdash \psi$ *imply* $\vdash \psi$*.*

Thanks to the identifiers G', N' and D' Lemma 2 reflects the semantical behavior of the four primitive connectives of the logic $L3A^D_{\to_1}$ into the proof theory. For instance, given a formula φ and a valuation v, if $v(\varphi) = 0$ then we have that v is a model of $G'(\varphi)$. On the other hand if $v(\varphi) = 1$, then v models $N'(\varphi)$, and if $v(\varphi) = 2$, then v models $D'(\varphi)$ (see Definition 16). Particularly, **N1** states that: if φ takes the value 0 identified as $G'(\varphi)$, then its negation $\neg\varphi$ must take the value 2, i.e. $D'(\neg\varphi)$. Similarly, **N2** states that: if φ takes the value 1, $N'(\varphi)$, then its negation must take the value 0, $G'(\neg\varphi)$. Finally **N3** asserts that if a formula takes the value 2 its negation must take the value 0. Therefore, by showing that **N1**, **N2** and **N3** are theorems in $\mathbb{L}_{L3A^D_{\to_1}}$, we prove that the negation connective $\neg$ matches with the truth tables of the connective in $L3A^D_{\to_1}$. Analogously, **D1-D6** do the job for disjunction, **C1-C6** check the case of conjunction and **I1-I5** model the behavior of implication.

Lemma 2. *In the formal axiomatic theory $\mathbb{L}_{L3A^D_{\to_1}}$ the following formulas are theorems.*

N1: $G'(\varphi) \to D'(\neg\varphi)$
N2: $N'(\varphi) \to G'(\neg\varphi)$
N3: $D'(\varphi) \to G'(\neg\varphi)$

D1: $\big(G'(\varphi) \wedge G'(\psi)\big) \to G'(\varphi \vee \psi)$
D2: $\big(G'(\varphi) \wedge N'(\psi)\big) \to G'(\varphi \vee \psi)$
D3: $\big(N'(\varphi) \wedge G'(\psi)\big) \to G'(\varphi \vee \psi)$
D4: $\big(N'(\varphi) \wedge N'(\psi)\big) \to N'(\varphi \vee \psi)$
D5: $D'(\varphi) \to D'(\varphi \vee \psi)$
D6: $D'(\psi) \to D'(\varphi \vee \psi)$

C1: $G'(\varphi) \to G'(\varphi \wedge \psi)$
C2: $G'(\psi) \to G'(\varphi \wedge \psi)$
C3: $\big(N'(\varphi) \wedge N'(\psi)\big) \to N'(\varphi \wedge \psi)$
C4: $\big(N'(\varphi) \wedge D'(\psi)\big) \to N'(\varphi \wedge \psi)$
C5: $\big(D'(\varphi) \wedge N'(\psi)\big) \to N'(\varphi \wedge \psi)$
C6: $\big(D'(\varphi) \wedge D'(\psi)\big) \to D'(\varphi \wedge \psi)$

I1: $G'(\varphi) \to D'(\varphi \to \psi)$
I2: $N'(\varphi) \to D'(\varphi \to \psi)$
I3: $D'(\psi) \to D'(\varphi \to \psi)$
I4: $\big(D'(\varphi) \wedge G'(\psi)\big) \to G'(\varphi \to \psi)$
I5: $\big(D'(\varphi) \wedge N'(\psi)\big) \to N'(\varphi \to \psi)$

5.2 Soundness and completeness

Now that we have the calculus $\mathbb{L}_{L3A^D_{\to_1}}$ and some results we can proceed with the soundness and completeness proof.

Theorem 2 (Soundness). *Let φ be a formula. If φ is a theorem in $\mathbb{L}_{L3A^D_{\to_1}}$, then φ is a tautology $L3A^D_{\to_1}$.*

Proof. The proof is straightforward by checking that all axioms are tautologies and that **MP** preserves tautologies.

To simplify the completeness proof, Definition 16 formally introduce a transformation over formulas of $L3A^D_{\to_1}$ using G', N' y D'. This transformation generalizes the transformation proposed in the well known Kalmár's lemma used to proof completeness of Classical Propositional Logic [12].

Definition 16. *Given a valuation v and φ a formula in the language of $L3A^D_{\to_1}$:*

$$\varphi_v = \begin{cases} G'(\varphi) & \text{if } v(\varphi) = 0, \\ N'(\varphi) & \text{if } v(\varphi) = 1, \\ D'(\varphi) & \text{if } v(\varphi) = 2. \end{cases}$$

For a set of formulas Γ we have that $\Gamma_v = \{\varphi_v | \varphi \in \Gamma\}$.

Using the previous definition, Lemma 3 states that given a valuation, the set of the transformed atoms of a formula derives the transformed formula. This lemma is a generalization of the Kalmár's lemma [12], adapted to the logic $L3A^D_{\to_1}$ with the correct truth value identifiers.

Lemma 3. *Let φ be a formula and v be a valuation in $L3A^D_{\to_1}$. Then in $\mathbb{L}_{L3A^D_{\to_1}}$ it holds that $Atoms(\varphi)_v \vdash \varphi_v$.*

Proof. The proof is by induction over the complexity of φ.

Basis: φ is an atom, e.g. $\varphi = p$. In this case $Atoms(\varphi)_v = \{\varphi_v\} = \{p_v\}$, so it is enough to prove that $\varphi_v \vdash \varphi_v$. But it follows directly by monotonicity.

Inductive hypothesis: For any formula ψ of $L3A^D_{\to_1}$ with lower complexity than φ it holds that $Atoms(\psi)_v \vdash \psi_v$.

Induction step: It is divided into four sub-cases, namely, $\varphi = \neg\psi$, $\varphi = \psi \vee \sigma$, $\varphi = \psi \wedge \sigma$ and $\varphi = \psi \to \sigma$ such that the complexity of ψ and σ are lower that those of φ. We present here only the last case, namely, $\varphi = \psi \to \sigma$, the remaining ones are proved similarly.

By inductive hypothesis it holds that:

$$Atoms(\varphi)_v \vdash \psi_v \ (1) \quad \text{and} \quad Atoms(\varphi)_v \vdash \sigma_v \ (2)$$

According to the truth values of assigned by v to ψ and to σ we have the following cases.

Case 1: $v(\psi) \in \overline{\mathcal{D}}$. Regardless the value of σ we have that $v(\varphi) \in \mathcal{D}$, i.e. $v(\varphi) = 2$, then $\varphi_v = D'(\varphi) = D'(\psi \to \sigma)$. Considering the values of ψ we have the following sub cases.

Sub-case 1a: If $v(\psi) = 0$, then $\psi_v = G'(\psi)$. By **I1** and **DT**, $G'(\psi) \vdash D'(\psi \to \sigma)$, equivalently, $\psi_v \vdash \varphi_v$. Using **Cut** between (1) and the last formula we obtain $Atoms(\varphi)_v \vdash \varphi_v$.

Sub-case 1b: If $v(\psi) = 1$, then $\psi_v = N'(\psi)$. By **I2** and **DT** it follows that $N'(\psi) \vdash D'(\psi \to \sigma)$. Analogously to the previous sub-case, by applying **Cut** between (1) and the last formula we obtain $Atoms(\varphi)_v \vdash \varphi_v$.

Case 2: $v(\sigma) \in \mathcal{D}$. Now regardless the value of ψ, we have $v(\varphi) \in \mathcal{D}$. As a result $\varphi_v = D'(\varphi) = D'(\psi \to \sigma)$ and $\sigma_v = D'(\sigma)$. Now by **I3** and **DT**, $D'(\sigma) \vdash D'(\psi \to \sigma)$, equivalently $\sigma_v \vdash \varphi_v$; by **Cut** with (2), we can conclude that $Atoms(\varphi)_v \vdash \varphi_v$.

Case 3: $v(\psi) \in \mathcal{D}$ and $v(\sigma) \in \overline{\mathcal{D}}$. Then $\psi_v = D'(\psi)$ and $v(\varphi) \in \overline{D}$, also $v(\varphi) = v(\sigma)$ and there are two options for the truth value of σ.

Sub-case 3a: If $v(\sigma) = 0$, then $v(\varphi) = 0$, $\varphi_v = G'(\varphi) = G'(\psi \to \sigma)$ and $\sigma_v = G'(\sigma)$. On one hand, since $Atoms(\varphi)_v \vdash \psi_v$ and $Atoms(\sigma)_v \vdash \sigma_v$ by **R-AND** we obtain that $Atoms(\varphi)_v \vdash \psi_v \wedge \sigma_v$. On the other hand, $D'(\psi) \wedge G'(\sigma) \vdash G'(\psi \to \sigma)$ (**DT(I4)**), or equivalently $\psi_v \wedge \sigma_v \vdash \varphi_v$. Finally by **Cut** between $Atoms(\varphi)_v \vdash \psi_v \wedge \sigma_v$ and $\psi_v \wedge \sigma_v \vdash \varphi_v$ we conclude that $Atoms(\varphi)_v \vdash \varphi_v$.

Sub-case 3b: If $v(\sigma) = 1$, then $v(\varphi) = 1$, $\varphi_v = N'(\varphi) = N'(\psi \to \sigma)$ and $\sigma_v = N'(\sigma)$. As in the previous sub-case we have that $Atoms(\varphi)_v \vdash \psi_v \wedge \sigma_v$ and $D'(\psi) \wedge N'(\sigma) \vdash N'(\psi \to \sigma)$ (**DT(I5)**). One application of **Cut** lead us to, $Atoms(\varphi)_v \vdash \varphi_v$.

Therefore if φ is a formula and v a valuation in $L3A^D_{\to_1}$, then $Atoms(\varphi)_v \vdash \varphi_v$.

Theorem 3 (Completeness). *If φ is a tautology in $L3A^D_{\to_1}$, then φ is a theorem in $\mathbb{L}_{L3A^D_{\to_1}}$.*

Proof. Suppose that φ is a tautology in $L3A^D_{\to_1}$. Let $\Phi = Atoms(\varphi)$. For any valuation $v(\varphi) = 2$, therefore $\varphi_v = D'(\varphi)$. By Lemma 3 it holds that $\Phi_v \vdash D'(\varphi)$ and by the item p) of Lemma 1 we have that $\vdash D'(\varphi) \to \varphi$. Applying **MP** to the last results we obtain that for any valuation v it holds that $\Phi_v \vdash \varphi$. Let p be an atom in Φ and $\Gamma := \Phi \backslash \{p\}$. For any valuation v, $\Phi_v \vdash \varphi$, equivalently $\Gamma_v, p_v \vdash \varphi$. Since we have three truth values, there will be three different values for p and we will have that $\Gamma_v, G'(p) \vdash \varphi$, $\Gamma_v, N'(p) \vdash \varphi$ and $\Gamma_v, D'(p) \vdash \varphi$. By item s) of Lemma 1 we can conclude that $\Gamma_v \vdash \varphi$. One can repeat the previous technique to eliminate another atom in Γ and after a finite number of steps all atoms in Φ will be eliminated. As a result we have that $\vdash \varphi$, i.e. that φ is a theorem in $\mathbb{L}_{L3A^D_{\to_1}}$.

Thanks to Theorem 2 and Theorem 3 we have that the logic $L3A^D_{\rightarrow_1}$ is sound and complete with respect to the calculus $\mathbb{L}_{L3A^D_{\rightarrow_1}}$ which is the main contribution of this paper.

6 Proof construction

Making a demonstration of completeness in a direct way consists in taking a tautology and constructing its formal proof. When doing so, not only the theorem of completeness is obtained, but a general technique for finding the proof of any theorem systematically is derived too. In [13] the Kalmár's meta-proof is revisited and a recursive algorithm to construct any formal proof is proposed. Thanks to the generalization of the technique of Kalmár proposed here and the detailed proof of Lemma 3 which analyzes all possible valuations and cases as well as the proofs of all theorems in Lemma 2, it is possible to apply the same idea here. So there is an algorithm to construct the proof of any theorem in $L3A^D_{\rightarrow_1}$ by following the meta-proof. The crucial step is to construct the proof of facts like $Atoms(\varphi)_v \vdash \varphi_v$, let us see a brief example.

Examples: Let be $\varphi = \neg p \rightarrow \neg q$ and v a valuation such that $v(p) = 2$ and $v(q) = 1$. Then we have that $Atoms(\varphi)_v = \{D'(p), N'(q)\} = \{\neg \sim p, \neg(q \vee \neg q)\}$ and $\varphi_v = \neg \sim \varphi = \neg \sim (\neg p \rightarrow \neg q)$. By Lemma 3 it holds that $\neg \sim p, \neg(q \vee \neg q) \vdash \neg \sim (\neg p \rightarrow \neg q)$. Let us construct the proof following the meta-proof suggested by Lemma 3.

Let ψ and σ be the formulas surrounded by the horizontal brackets

$$\neg \sim p, \neg(q \vee \neg q) \vdash \overbrace{\neg \sim (\underbrace{\neg p}_{\psi} \rightarrow \underbrace{\neg q}_{\sigma})}^{\varphi_v}$$

Then $v(\psi) = v(\neg p) = 0$ and $v(\sigma) = v(\neg q) = 0$. This corresponds to the Case 1, Sub-case 1a of Lemma 3 and the proof becomes:

1. $Atoms(\varphi)_v \vdash \neg\neg p$	(*) Induction hypothesis
2. $\neg\neg p \vdash \neg \sim (\neg p \rightarrow \neg q)$	**I1** and **DT**
3. $Atoms(\varphi)_v \vdash \neg \sim (\neg p \rightarrow \neg q)$	**Cut (1,2)**

Now lets prove the Induction hypothesis (*). Let ψ and σ be the formulas surrounded by the horizontal brackets:

$$\neg \sim p, \neg(q \vee \neg q) \vdash \underbrace{\overbrace{\neg\neg \underbrace{p}_{\psi}}^{\varphi_v}}_{\varphi}$$

In this case $v(\psi) = v(p) = 2$ which corresponds to one of the cases of negation in which **N3** is used.

1. $Atoms(\varphi)_v \vdash \neg \sim p$ (**)Induction Hypothesis
2. $\neg \sim p \vdash \neg\neg p$ **N3** and **DT**
3. $Atoms(\varphi)_v \vdash \neg\neg p$ **Cut (1,2)**

The proof of (**) is direct since $\neg \sim p \in Atoms(\varphi)_v$, and the complete proof can be rewritten as:

1. $\vdash \neg \sim p$ Hypothesis
2. $\vdash \neg(q \vee \neg q)$ Hypothesis
3. $\neg \sim p \vdash \neg\neg p$ **N3** and **DT**
4. $\vdash \neg\neg p$ **Cut (1,3)**
5. $\neg\neg p \vdash \neg \sim (\neg p \to \neg q)$ **I1** and **DT**
6. $\vdash \neg \sim (\neg p \to \neg q)$ **Cut (4,5)**
7. $\neg \sim p, \neg(q \vee \neg q) \vdash \neg \sim (\neg p \to \neg q)$ 1-6

7 Conclusions

The logic $L3A^D_{\to_1}$ is a genuine paracomplete logic that conservatively extends the positive fragment of classical propositional logic. It was defined by Hernández-Tello et al. in [9] by means of many-valued semantics. The non implicative fragment of the logic $L3A^D_{\to_1}$ is the logic $L3A^D$, which is dual of the genuine paraconsistent logic $L3A$ defined by Béziau in [5]. In this paper a Hilbert-type axiomatization for $L3A^D_{\to_1}$ is presented using the Kalmár's technique. The completeness proof presented here allows us to construct the proof of any theorem in a recursive way. Constructing and adequate formal theory for a logic expressed by semantical terms by means of the Kalmár's technique, require to select a particular set of axiom schemes that allows to assert all the requirements in Lemma 2 are fulfilled. This is one of the major problems one has to face for obtain a formal theory with this method.

It is important to find the axiomatization of other genuine paracomplete logics in order to have a better picture of the concept and its relation with other paracomplete logics, we have considered it as future work. We are interested in exploring as many properties as possible to compare the whole family of three-valued paracomplete logics, where $L3A^D_{\to_1}$ is just the tip of the iceberg. However, we consider the results presented in this paper relevant for settle down a starting point in the study of genuine paracomplete logics.

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