On One Model of Information Distribution in a Spatially Distributed Environment

Andrey V. Nesterov [1[0000-0002-4702-4777]] and Alexander V. Zaborskiy [2[0000-0003-2480-7215]]

[1] Plekhanov Russian University of Economics, 36 Stremyanny lane, Moscow, 115998, Russia
andrenesterov@yandex.ru
[2] LLC RPE RADICO, 14a Marks Ave., Obninsk, 249035, Kaluga's region, Russia
alexander.zaborskiy@mail.ru

Abstract. We consider one of the models of information (or other substance) propagation in a spatially distributed multiphase environment, when the exchange rate between phases is much higher than the transfer rate, and the number of phases is large. The transfer is described by a singularly perturbed partial differential operator equation in the critical case. An asymptotic expansion for a small parameter of the initial problem solution is constructed. From the obtained formulas, it follows that in the first approximation, the initial perturbation propagates at a certain average speed with simultaneous diffusion spreading. Formulas for the average transfer rate and pseudodiffusion coefficient are obtained. The obtained formulas can be used both for qualitative analysis of problem solutions and for creating economical difference schemes that require significantly less (by orders of magnitude) computational resources.

Keywords: multiphase media, distributed systems, differential operator equations, small parameter, singular perturbations, asymptotic decomposition of the solution.

1 Introduction

In the description of a number of transport processes of various substances in a spatially-distributed multi-phase environments as mathematical models of phenomena described by systems of partial differential equations. In the case of one spatial variable x this system of equations has the form

$$(U_t + DU_x) = AU.$$

Here $U \in R_n$the vector-function, the dimension of which is determined by the number of phases, $D = \{D_1, \dots, D_n\}$ are the trasfer rates of each component of the solution, the matrix A describes the exchange processes between phases, the elements of the matrix a_{ij} make sense of the exchange rate between the phases i and j . Spatial variable x can make sense of the real spatial variable or to describe some characteristics of the environment, "along" which the transfer substance.

Proceedings of the 10th International Scientific and Practical Conference named after A. I. Kitov "Information Technologies and Mathematical Methods in Economics and Management (IT&MM-2020)", October 15-16, 2020, Moscow, Russia

In some cases, the exchange processes between phases have significantly higher speeds than the transfer processes (which, for example, may correspond to the processes of information transfer in a social environment, where the exchange "horizontally" (between phases) occurs much faster than the exchange "vertically" - by the variable x).Then when moving to dimensionless variables in the task matrix A takes the form $A=EA_1$,where $E >>1$ is a large positive parameter and A_1 has elements of order $O(1)$. It is convenient to rewrite A in $A=A_1/\varepsilon$ where $0<\varepsilon<<1$ is a small positive parameter. Below we denote a small positive parameter as ε^2 (second degree is introduced for convenience and to reduce the record below), and the lower index in the matrix A_1 down. In this case the system of equations takes the form

$$\varepsilon^2(U_t + DU_x) = AU.$$

The presence of a small parameter at higher derivative makes the problem singularly perturbed [16], numerical calculations solution of which is quite time consuming. However, to simplify numerical calculations of solutions and identify some of the hidden regularities of the behavior of the solution (and consequently, the simulated system), it is advisable to try to construct the asymptotic expansion (AE) of the solution in powers of the small parameterter ε. If the number of phases is very large or tends to infinity (in case $n>>1$), then the discrete index i, $1\le i\le n$, becomes a continuous parameter p, $p_1\le p\le p_2$. In this case the system of equations becomes one differential operator equation generalizing the above system to the case in which the matrix A is replaced by a linear operator acting on the parameter p . In this case, the vector function $U \in R_n$becomes a function $U(x,t,p)$, depending on the parameter p, and the system of equations is transformed into a differential-operator partial differential equation

$$\varepsilon^2(U_t + DU_x) = L_p U,$$

Similar equations (possible, without a small parameter) appear when modeling various processes, for example, coagulation processes [1]-[4], (Boltzmann and Smolukhovsky equations), turbulence processes [5]-[7], modeling social processes [8]-[12], and others [13]-[14]. Differential-operator equations have been studied in many works, for example [19]-[23]. To construct an asymptotic expansion of the solution, the technique developed in the works [16]-[18] is used.

2 Statement of the problem

Consider the initial problem for a singularly perturbed differential operator equation

$$\varepsilon^2(U(x,t,p)_t + D(p)U(x,t,p)_x) = L_p U(x,t,p)(1)$$

$$U(x,o,p) = w\left(\frac{x}{\varepsilon},p\right) \qquad (2)$$

here: $U(x,t,p)$– solution , $\{x,t,p\} \in H = \{|x| < \infty;\ 0 < t < \infty;\ p_1 \leq p \leq p_2\}$,$0 < \varepsilon \ll 1$ is a small positive parameter; $D(p)$ continuous function on $[p_1, p_2]$; $L_p: A_p \to A_p$ is a linear operator acting in the spaceA_p of continuous by p functions $U(x,t,p)$with the scalar product (h_1, h_2) ; the initial condition $w(z,p)$ satisfies the inequality $|w(z,p)| \leq Ce^{-\beta z^2}$ together with their derivatives at z up to order *N+3* ,here *N* - some natural number .

Let the operator L_p has a single eigenvalue $\lambda_1 = 0$, h_1- corresponding eigenfunction, h_1^* a eigenfunction of the adjoint operator L_p^*, corresponding $\lambda_1^* = 0$.
.

It follows from the condition $\lambda_1 = 0$ that in the evolution of the "generalized quantity" of a substance does not change. Indeed, multiplying (1) on h_1^* scalar and integrating the result on *x* from -∞ to +∞, we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(U, h_1^*)dx = 0.$$

I. We require that the remaining eigenvalues λ of the operator L_p have negative real parts $Re\lambda \leq -k, k > 0$.

II. $(h_1, h_1^*) \neq 0$. In this condition it is possible to choose these functions so that $(h_1, h_1^*) = 1$.

3 Algorithm for constructing an AE solution

AE of solutions is constructed as the sum of the functions of the surge S , concentrated in the neighborhood of a line $\{l: \zeta = 0\}$ – "Pseudocharacteristic" of equations and boundary layer functions P concentrated in the neighborhood of the boundary $t = 0$ and the remainder term R:

$$U(x,t,p) = S(\zeta,t,p) + P(\xi,\tau,p) + R =$$

$$= \sum_{i=0}^{N} \varepsilon^i (s_i(\zeta,t,p) + p_i(\xi,\tau,p)) + R \quad (3)$$

here $\zeta = \frac{x-Vt}{\varepsilon}$; $V \stackrel{\text{def}}{=} (Dh_1, h_1^*)$ is a variable, which describes the function of the surg S; $\xi = \frac{x}{\varepsilon}$; $\tau = \frac{t}{\varepsilon^2}$ the stretched variables, which describe the boundary layer function P. The algorithm for constructing a AE similar to the algorithm described in [3].

3.1 Building a surge function

Function S must satisfy the original equation (1):

$$\varepsilon^2(S_t + DS_x) = L_p S + O(\varepsilon^{N+2}) \qquad (4)$$

Move on to equation (1) from variables(x,t) to new variables (ζ, t)

$$\varepsilon^2 S_t + \varepsilon\Psi(p)S_\zeta = L_p S \quad (5)$$

where

$$\Psi(p) = D(p) - (D(p)h_1, h_1^*) \quad (6)$$

Function S is searched in the form:

$$S(\zeta, t, p) = \sum_{i=0}^{N} \varepsilon^i s_i(\zeta, t, p) \quad (7)$$

Substituting (7) into (5), in a standard way [4] we get the system of equations for the terms of the expansion :

$$\varepsilon^0: \ L_p s_0 = 0,$$

$$\varepsilon^1: \ L_p s_1 = \Psi s_{0,\zeta},$$

$$\varepsilon^2: \ L_p s_2 = \Psi s_{1,\zeta} + s_{0,t},$$

$$\dots$$

$$\varepsilon^i: \ L_p s_i = \Psi s_{i-1,\zeta} + s_{i-2,t},$$

s_0 is:

$$s_0 = \varphi_0(\zeta, t)h_1(p) \quad (8)$$

where φ_0 - as yet unknown function. Write conditions for the solvability of the equations for $s_1$and s_2 [4]:

$$\left(\Psi s_{0,\zeta}, h_1^*\right) = 0 \quad (9)$$

$$\left(s_{0,t} + \Psi s_{1,\zeta}, h_1^*\right) = 0 \quad (10)$$

The condition (9) is true due to the choice of the variable ζ, therefore, s_1 can be written as:

$$s_1 = \varphi_1 h_1 + G\Psi\varphi_{0,\zeta} h_1 \quad (11)$$

where G – pseudo-inverse to operator L_p . Substituting (8), (11) in (10), and eliminating φ_1 derived equation to determine φ_0:

$$(\varphi_{0,t} + M\varphi_{0,\zeta\zeta} = 0, M = (\Psi G\Psi h_1, h_1^*). \quad (12)$$

The equation for finding the following approximations $s_n, n \geq 1$ are obtained similarly. Omitting the calculations, we give only the result.

The function s_n is:

$$s_n = \varphi_n h_1 + G(s_{n-2,t} + \Psi s_{n-1,\zeta}).$$

A function φ_n defined by the equation:

$$\varphi_{n,t} + M\varphi_{n,\zeta\zeta} = \Phi_{ef,n} \qquad (13)$$

where $\Phi_{ef,n}$ is a linear combination of the functions $s_i, 0 \leq i \leq n-1$, and their derivatives.

Thus, the obtained expression for finding and equations to determine the members in these expressions for all functions $\varphi_i, 0 \leq i \leq N+2$.

Apply the condition of parabolicity on equations (12), (13):

III. $M = (\Psi G \Psi h_1, h_1^*) < 0$

3.2 The construction of boundary layer functions

A function $S(\zeta, t)$ under any initial conditions for equations (12), (13) does not satisfy initial conditions (2). To meet these conditions is constructed, the boundary layer function $P(\xi, \tau), \xi = x/\varepsilon, \tau = {t}/{\varepsilon^2}$ [4]. The boundary layer function P needs together with the function S to satisfy the initial conditions (2):

$$S(\zeta, 0, p) + P(\xi, 0, p) = w\left(\frac{x}{\varepsilon}, p\right) \qquad (14)$$

the original equation (1):

$$(P_t + DP_x) = L_p P \qquad (15)$$

and the condition:

$$\lim_{\tau\to\infty} P = 0 \qquad (16)$$

The function P is constructed as:

$$P(\xi, \tau, p) = \sum_{i=0}^{N} \varepsilon^i p_i(\xi, \tau, p) \qquad (17)$$

Substituting (17) into (15) [4], we get the equations for determining p_i

$$\varepsilon^0: p_{0,\tau} = L_p p_0, \qquad (18)$$

$$\varepsilon^i: p_{i,\tau} = L_p p_i - Dp_{i-1,\xi}, \ \ i \geq 1.$$

Substituting the series (8), (17) in condition (14), given that $\frac{x}{\varepsilon} = \xi$, we get the resulting equations for determining the initial conditions:

$$\varepsilon^0: s_0(\zeta, 0, p) + p_0(\xi, 0, p) = w(\xi, p) \ (19)$$

$$\varepsilon^i: s_i(\zeta, 0, p) + p_i(\xi, 0, p) = 0; \ i \geq 1 \ (20)$$

Imposing some additional conditions on the eigenvalues of the operator L_p (IV-VI,[3]) and omitting intermediate calculations, we give the result. The function p_0 has the form

$$p_0(\xi,\tau,p) = \sum_{i=2}^{\infty} C_{oi}(\xi) h_i(p) e^{\lambda_i \tau}, \quad (21)$$

Substituting (8), (21) in (19) we obtain the equation for determining the initial conditions $\varphi_0(\zeta,0)$ for the equations (13) and functions $C_{oi}(\xi)$:

$$\varphi_0(\zeta,0)h_1(p) + \sum_{i=2}^{\infty} C_{oi}(\xi) h_i(p) = w(\xi,p) \; (22)$$

From (22) in the IV-VI [3] we get:

$$\varphi_0(\zeta,0) = \big(h_1(p), w(\xi,p)\big), C_{oi}(\xi) = \big(h_i(h), w(\xi,p)\big); i = 2,3,\ldots$$

Thus, the obtained initial condition for equation (13) from which φ_0 is determined , as well as the function p_0 itself .

The construction of the subsequent functions p_i is similar [4].

Thus, all members of the far solution (3) - functions $S(\zeta,t,p)$ and (ξ,τ,p), clearly defined.

3.3 The function evaluation of splash and border functions

If the condition III *(M<0)* is met, for any $T_1 > 0$ all $\varphi_i(\zeta,t)$ exist, are unique and for a any fixed N and are valid estimates

$$|\varphi_i(\zeta,t)| < Ce^{-k\zeta^2}, C > 0, k > 0, i \leq N.$$

When the conditions are met I-VI all p_i exist, are unique and satisfy the estimate:

$$|\mathrm{p_i}| \leq \mathrm{Ce^{-k\tau}}; \mathrm{C} > 0, k > 0, i \geq 2.$$

3.4 Evaluation of the residual term

Write the solution of the original problem (1)-(2) in the form:

$$U = U_N + R = \sum_{i=0}^{N} \varepsilon^i (s(\zeta,t,p)_i + p(\xi,\tau,p)_i) + R \; (233)$$

where $U_N = S + P$ built above the AE of the solution, R - the residual term.

Fair

Theorem. *Let the conditions I-IX [3].*

Then the solution of problem (1)-(2)can be represented in the form $U = U_N + R$ where built above the AE U_N of the solution, the residual term R satisfies the asymptotic bound on the discrepancy:

$$\begin{cases} \varepsilon^2(R_t + DR_x) = L_p R + O(\varepsilon^{N+1}) \\ R|_{t=0} = 0. \end{cases}$$

A full proof is given in [3].

4 Discussion

1. Built AE solutions of singularly perturbed differential-operator equation (1) for $t>t_0$ where $t_0>0$ is any positive, independent of the ε, taking into account the estimate (23), has the form

$$U = \varphi_0\zeta, t)h_0(p) + O(\varepsilon) \qquad (244)$$

Functions φ_0 is the solution of the initial problem for a parabolic equation:

$$\varphi_{0,t}(\zeta, t) + M\varphi_{0,\zeta\zeta}(\zeta, t) = 0, t > 0, |x| < \infty, \qquad (25)$$

$$\varphi_0(\zeta, 0) = k_1 w(\zeta), \ (25)$$

which is neither a small parameter nor a parameter p.

2. The above result can be interpreted in terms of a qualitative description of the evolution of the solution. The main term in AE has the form $s_0 = \varphi_0(\zeta, t)h_1(p)$, there $\varphi_0(\zeta, t)$ is a solution to the parabolic equation (26) where $\zeta = \frac{x-Vt}{\varepsilon}$; $V \stackrel{\text{def}}{=} (Dh_1, h_1^*)$. This suggests that the initial perturbation is transferred from the effective (average) speed, and the transfer is accompanied by a "pseudodiffusion" blur. Averaging the rate and "pseudodiffusion" the blur is due to the fact that there is a "rapid mixing solution" for the variable p, and the speed of migration is different for different p.

3. For technical systems with well-defined inputs you can calculate an approximate solution for $t>t_0$ using the problem (26)-(27). For processes with poorly defined input data (social, economic, informational) in equation (26) can give a qualitative description of the process of moving a heterogeneous interactive information (the dependence on a parameter *p)* along social strata (variable *x*) with some "effective speed" while blur, that describes the slow the spread and diffusion of information thanks to the intensive exchange.

4. The results obtained can be generalized to equations with a large number of spatial variables, to equations with variable coefficients.

5. The most interesting results are obtained if a weak nonlinearity is added to the right side of the equation

$$\varepsilon^2(U_t + DUS_x) = L_pU + \varepsilon^2 f(U, p) \ (26)$$

The AE of the solution of the equation (26) with the initial condition (2) has the same form (3), but the equation for the determining φ_0 becomes nonlinear

$$\varphi_{0,t}(\zeta,t) + M\varphi_{0,\zeta\zeta}(\zeta,t) = F_{eff}(\varphi_0), t > 0, |x| < \infty, \quad (267)$$

$$\varphi_0(\zeta,0) = k_1 w(\zeta),$$

where $F_{eff}(\varphi_0)$ is determined through $f(U,p)$ and the problem data. For a different form of weak nonlinearity on the right side, the equation (27) can take the form of a generalized Burgers equation.

6. The obtained asymptotic formulas make it possible to significantly (up to several orders of magnitude) reduce the computational resources required for numerical calculation of the solution, since the solution of a singularly perturbed differential operator equation reduces to the solution of a parabolic equation (25) without a small parameter.

5 Conclusion

1. An asymptotic expansion of the solution of the initial problem for a singularly perturbed differential operator transfer equation is obtained. Under the conditions imposed on the problem, the main term of the asymptotics is described by a parabolic equation, linear or nonlinear, depending on the presence of a small nonlinearity in the original problem. The resulting formulas can be used to calculate the solution and for qualitative analysis of the solution behavior.

2. We can figuratively say that "strong mixing generates irreversibility", since, despite the reversibility of time in the original problem, the solution quickly begins to evolve as a solution of the parabolic equation, which is characterized by irreversibility.

References

1. Galkin, V.A.: Metric Theory of Functional Solutions of the Cauchy Problem for a System of Conservation Laws. Doklady Mathematics. **81**(2), 219-221 (2010). https://doi.org/10.1134/S1064562410020158
2. Galkin, V.A., Bykovskikh, D.A., Gavrilenko, T.V., Stulov, P.A.: A Filtration Model of Ideal Gas Motion in Porous Medium. Proceedings in Cybernetics. 4(24), 50-57 (2016). https://www.elibrary.ru/item.asp?id=28376388 (in Russian)
3. Galkin, V.A.: Analysis of mathematical models: systems of conservation laws, Boltzmann and Smolukhovsky equations. BINOM.Laboratoryscale, Moscow (2011). (in Russian)
4. Foias, S., Nicolaenko, B., Sell, G.R., Temam, R.: Inertial Manifolds for the Kuramoto-Sivashinsky Equation and an Estimate of their Lowest Dimension. Journal de Mathématiques Pures et Appliquées. **67**, 197-226 (1988).
5. Kuramoto, Yu.: Chemical Oscillations, Waves, and Turbulence. SSSYN, vol. 19. Berlin. Springer (1984). https://doi.org/10.1007/978-3-642-69689-3

6. Larkin, N.A.: Korteweg–de Vries and Kuramoto–Sivashinsky equations in bounded domains. Journal of Mathematical Analysis and Applications. **297**(1), 169-185 (2004). https://doi.org/10.1016/j.jmaa.2004.04.053
7. Sivashinsky, G.I.: Weak turbulence in periodic flows. Physica D: Nonlinear Phenomena. **17**(2), 243-255 (1985). https://doi.org/10.1016/0167-2789(85)90009-0
8. Mikhailov, A.P.: Mathematical modeling of power distribution dynamics in hierarchical structures. Mat. modeling. **6**(6), 108-138 (1994). (in Russian)
9. Mikhailov, A.P., Petrov A.P., Marevtseva, N.A., Tretiakova, I.V.: Development of a model of information dissemination in society. Mathematical Models and Computer Simulations. **6**(5), 535-541 (2014). https://doi.org/10.1134/S2070048214050093
10. Pronchev, G.B., Mikhailov, A.P., Petrov, A.P., Proncheva, O.G.: Modeling the decline of public attention to a past one-time political event. Doklady Akademii Nauk. **480**(4), 397-400 (2018). https://doi.org/10.7868/S0869565218160028 (in Russian)
11. Mikhailov, A.P., Lankin, D.F.: The structures of power hierarchies. Mathematical Models and Computer Simulations. **2**(2), 200-210 (2010). https://doi.org/10.1134/S2070048210020079
12. Mikhailov, A.P.: Simulation of the System "Power-Society". Nauka-Fizmatlit, Moscow (2006). (in Russian)
13. Bradley, R.M., Hofsass, N.: Nanoscale patterns produced by self-sputtering of solid surfaces: The effect of ion implantation. Journal of Applied Physics. **120**(7), 074302 (2016). https://doi.org/10.1063/1.4960807
14. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. AMS, vol. 68. Springer-Verlag New York (1997). https://doi.org/10.1007/978-1-4612-0645-3
15. Zaborskiy, A.V., Nesterov, A.V.: The asymptotic decomposition of solution of singularly perturbed differential and operational equation in the critical case. Matem. Mod. **26**(4), 65-79 (2014). (in Russian)
16. Vasilieva, A.B., Butuzov, V.F.: Singularly perturbed equations in critical cases. Moscow Publishing house MSU (1978). (in Russian)
17. Butuzov, V.F.: Asymptotics of the solution of a system of singularly perturbed equations in the case of a multiple root of the degenerate equation. Differential equations. **50**(2), 177-188 (2014). https://doi.org/10.1134/S0012266114020050
18. Butuzov, V.F., Nefedov, N.N., Recke, L., Schneider, K.R.: On a singularly perturbed initial value problem in the case of a double root of the degenerate equation. Nonlinear Analysis: Theory, Methods & Applications. **83**, 1-11 (2013). https://doi.org/10.1016/j.na.2013.01.013
19. Arnold, V. I.: Geometrical Methods in the Theory of Ordinary Differential Equations. Berlin-Heidelberg-New York, Springer-Verlag (1983).
20. Krein, S.G.: Linear Differential Equations in Banach Space. American Mathematical Society (1972).
21. Neymark, M.A.: Linear differential operators. Nauka Publ., Moscow (1969). (in Russian)
22. Daleckii, J.L., Krein, M.G.: Stability of Solutions of Differential Equations in Banach Space. American Mathematical Society (1974).
23. Kolmanovskii, V., Myshkis, A.: Introduction to the Theory and Applications of Functional Differential Equations. MAIA, vol. 46. Springer Science & Business Media B.V. (1999). https://doi.org/10.1007/978-94-017-1965-0